LIFTINGS OF DISTRIBUTIVE LATTICES BY LOCALLY MATRICIAL ALGEBRAS WITH RESPECT TO THE Id_c FUNCTOR

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The paper is dedicated to Walter Taylor

ABSTRACT. We study representations of distributive $\langle 0,1\rangle$-lattices, considered as join-semilattices, by semilattices of finitely generated two-sided ideals of locally matricial algebras over a field k, aiming to find a functorial solution of the problem. We find simple examples of a finite subcategory of the category $\mathbf{L_d}$ of distributive $\langle 0,1\rangle$-lattices and of a subcategory of $\mathbf{L_d}$ corresponding to a partially ordered class which cannot be lifted with respect to the Id_c functor. On the other hand, we prove that there is such a lifting of every diagram in $\mathbf{L_d}$ or of a subcategory $\mathbf{L_{d1}}$ of $\mathbf{L_d}$ whose objects are all distributive $\langle 0,1\rangle$-lattices and whose morphisms are $\langle \vee,\wedge,0,1\rangle$-embeddings.

INTRODUCTION

This paper is a continuation of [6], where we have proved that every distributive $\langle 0,1\rangle$-lattice is, as a join-semilattice, isomorphic to the semilattice of finitely generated ideals of a locally matricial algebra. Having discussed this result with Friedrich Wehrung in a Summer School in Košická Belá, Slovakia, in 2003, we dealt with the question whether it can be solved functorially, that is, whether there is a functor from a category $\mathbf{L_d}$ of distributive lattices to the category of locally matricial algebras such that its composition with the functor Id_c, which assigns to a locally matricial algebra the lattice of its finitely generated ideals (see Basic concepts), is equivalent to the identity functor. It is easily rejected for the category of all distributive $\langle 0,1\rangle$-lattices, however, it still can be true if we restrict ourselves to its suitable subcategory. One such restriction was made in [12], where F. Wehrung asked the following:

[12, Problem 3]**.** Let k be a field. Does there exist a functor Φ, from distributive $\langle 0,1\rangle$-lattices with $\langle \vee,\wedge,0,1\rangle$-embeddings to locally matricial algebras over k with (unital) ring k-linear homomorphisms such that $\mathrm{Id}_c\Phi$ is equivalent to the identity?

We are going to prove that such a functor Φ exists. Moreover, we prove that every diagram of the category of distributive lattices can be lifted with respect to the Id_c functor and we illustrate on simple examples that these results cannot be much improved. Our proofs are based on the result that a functor to $\mathbf{L_d}$ can be

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lifted with respect to the Id_c functor if and only if it can be lifted with respect to the functor Θ_∞ from a category $\mathbf{D_\infty}$ to the category $\mathbf{L_d}$; objects of $\mathbf{D_\infty}$ are projections $P\colon X \to L$ from a set X on a distributive $\langle 0,1\rangle$-lattice L such that the pre-image of every element of L is infinite and morphisms are commutative squares

$$F:\quad \begin{tikzcd} X_1 \arrow[r, "f"] \arrow[d, "p_1"'] & X_2 \arrow[d, "p_2"] \\ L_1 \arrow[r, "f'"'] & L_2 \end{tikzcd}\ ,$$

where f is $\langle \vee,\wedge,0,1\rangle$-homomorphism, and $f'\colon X_1 \to X_2$ is a map satisfying the property (3.1) below, and Θ_∞ is a functor which assigns to an object $P : X \to L$ the distributive $\langle 0,1\rangle$-lattice L and to a morphism $F = (f, f')$ the $\langle \vee,\wedge,0,1\rangle$-homomorphism f (Corollary 4.3). Proving the existence of a lifting of a given functor to the category $\mathbf{L_d}$ with respect to the functor Θ_∞ is much easier than proving the existence of its lifting with respect to the functor Id_c.

There has already appeared a number of papers related to the problem of the representation of distributive $\langle \vee,0,1\rangle$-semilattices as the semilattices of finitely generated ideals of a von Neumann ring, in particular, of a locally matricial algebra. Thus, G. M. Bergman [1] has proved that every distributive $\langle \vee,0,1\rangle$-semilattice which either is countable or corresponds to the semilattice of all compact hereditary subsets of a partially ordered set is isomorphic to the semilattice of locally matricial algebra. F. Wehrung proved that every distributive $\langle \vee,0,1\rangle$-semilattice is isomorphic to the semilattice of finitely generated ideals of some von Neumann regular ring [10] but it follows from his results in [11] that we cannot require the ring to be unit regular, so not even locally matricial. Finally, the results in [8, 9] give an example of a distributive $\langle \vee,0,1\rangle$-semilattice which is not isomorphic to the semilattice of finitely generated ideals of any von Neumann regular ring. In [6], we have proved that a distributive $\langle 0,1\rangle$-lattice is isomorphic to the semilattice of finitely generated ideals of a locally matricial algebra. A different proof, based on similar methods as the Bergman's constructions in [1], is given by M. Ploščica in [5].

Basic concepts

Categories. Given a category $\mathbf{C}$ and objects $a,\ b \in \mathbf{C}$, we denote by $\mathbf{C}(a,b)$ the collection of all morphisms from $a \to b$. The identity morphism at an object $a \in \mathbf{C}$ is denoted by $\mathbf{1}_a$. Recall that given a pair of functors $\Phi,\ \Psi$ from a category $\mathbf{C}$ to a category $\mathbf{D}$, a *natural transformation* $\eta : \Phi \to \Psi$ is a family $\eta = \{\eta_a : \Phi(a) \to \Psi(a) \mid a \in \mathbf{C}\}$ of morphisms in $\mathbf{D}$ such that $\eta_b \circ \Phi(f) = \Psi(f) \circ \eta_a$, for every morphism $f : a \to b \in \mathbf{C}$. A *natural equivalence* is a natural transformation η such that η_a is an isomorphism for every object $a \in \mathbf{C}$. If there is a natural equivalence $\eta : \Phi \to \Psi$, we say that the functors $\Phi,\ \Psi$ are *equivalent* (via η). We say that a functor $\Phi : \mathbf{C} \to \mathbf{D}$ *respects direct limits*, if it maps the direct limit of any directed system in $\mathbf{C}$ to the direct limit of the image of the directed system in $\mathbf{D}$ (see [2, Definition 7.8.1]).

We are going to meet with the following obstacle resulting from the fact that direct limits in abstract categories are not defined uniquely but uniquely *up to isomorphism.* We will have defined a functor $\Phi : \mathbf{C}' \to \mathbf{D}$ from a full subcategory $\mathbf{C}'$ of a category $\mathbf{C}$, moreover, such that every object in $\mathbf{C}$ will be a direct limit of objects in $\mathbf{C}'$. We will look for an expansion of the functor Φ to the whole category $\mathbf{C}$. The most natural way to expand the functor will be to represent every object $a \in \mathbf{C}$ as a suitable direct limit of a directed system in $\mathbf{C}'$ and then to define its image under Φ as the direct limit of the image of this directed system. But this is the trouble since the direct limit is not defined as a single object but rather as an isomorphism class of objects (see [2, Definition 7.5.2]). In all our cases the category $\mathbf{D}$ will be a category of algebras of a finitary type, and for those, we have a specific construction of direct limits as in [2, Lemma 8.1.10]: Given an upwards directed partially ordered set P and a directed system $\langle A_p, f_{p,q}\rangle_{p<q \text{ in } P}$ in $\mathbf{D}$, we denote by A' the disjoint union of the underlying sets of A_p-s, for $a \in A_p$, $b \in A_q$ we set $a \sim b$ provided that for some $r \geq p, q$, their images in A_r coincide, and we let A denote the set of all equivalence classes of A' with respect to $\sim$, and $[a]$ the class containing an element $a \in A'$. For each $p \in P$, the correspondence $a \mapsto [a]$ defines a map $f_p : A_p \to A$. Then the set A together with the collection of the maps $\{f_p, p \in P\}$ form a set-theoretic direct limit of the directed system above. Since we deal with algebras of a finitary type, we can define operations on A so that the maps f_p are homomorphisms in $\mathbf{D}$, and $\langle A, f_p\rangle_{p\in P}$ is a direct limit of the directed system $\langle A_p, f_{p,q}\rangle_{p<q \text{ in } P}$ in $\mathbf{D}$ (see the proof of [2, Lemma 8.1.10]). We will use the notation $\varinjlim$ for this particular direct limit, while the abstract direct limit in the categorical sense is denoted by $\varinjlim$.

Definition. Let $\Psi : \mathbf{C}_1 \to \mathbf{C}_2$, $\Theta : \mathbf{C}_1 \to \mathbf{C}_3$, and $\Phi : \mathbf{C}_2 \to \mathbf{C}_3$ be functors as in Figure 1. We say that Ψ *lifts* Θ with respect to Φ provided that the composition $\Phi\Psi$ is equivalent to the functor Θ. In particular, if Θ is an inclusion functor, we say that Ψ *lifts* $\mathbf{C}_1$ with respect to Φ.

$$\begin{tikzcd}
\mathbf{C}_1 \arrow[rr, "\Psi"] \arrow[dr, "\Theta"'] & \arrow[d, phantom, "\overset{\eta}{\underset{\simeq}{\longleftarrow}}"] & \mathbf{C}_2 \arrow[dl, "\Phi"] \\
& \mathbf{C}_3 &
\end{tikzcd}$$

FIGURE 1

As in [7, Section 5], given a category $\mathbf{C}$, a *diagram* of $\mathbf{C}$ is a functor $\mathcal{D} : I \to \mathbf{C}$, where I is a partially ordered set. Our definition of the "lifting of functors" corresponds to the definition of the lifting of diagrams in [7, page 455].

Lattices and semilattices. For definitions and basic properties of algebraic lattices, distributive lattices, and distributive semilattices we refer to [3, Section II]. Given a finite distributive join-semilattice L, we denote by $J(L)$ the partially ordered set of its nonzero join-irreducible elements. For a partially ordered set P, we let $H(P)$ denote the lattice of its hereditary subsets (i.e., the subsets which contain with every element all the elements below it). We denote by

– $\mathbf{L_d}$ the category of all distributive $\langle 0, 1\rangle$-lattices (with $\langle \vee, \wedge, 0, 1\rangle$-homomorphisms),

– $\mathbf{S_d}$ the category of all distributive $\langle \vee, 0, 1\rangle$-semilattices (with $\langle \vee, 0, 1\rangle$-homomorphisms),

– $\mathbf{S_{fd}}$ the full subcategory of $\mathbf{S_d}$ of all finite distributive $\langle \vee, 0, 1\rangle$-semilattices,

– $\mathbf{S_{fb}}$ the full subcategory of $\mathbf{S_d}$ of all finite Boolean $\langle \vee, 0, 1\rangle$-semilattices.

Algebras over a commutative field. We will make use of associative algebras over a commutative field k (called k-algebras). We will consider only algebras with an identity element. There is a functor Id_c from the category of these algebras to the category of $\langle\vee,0,1\rangle$-semilattices which assigns to a k-algebra A the semilattice $\mathrm{Id}_c(A)$ of its finitely generated ideals and to a k-linear ring homomorphism $\varphi : A \to B$ the $\langle\vee,0,1\rangle$-homomorphism $\mathrm{Id}_c(\varphi) : \mathrm{Id}_c(A) \to \mathrm{Id}_c(B)$ defined by the correspondence $I \mapsto B\varphi(I)B$. It's easy to verify that the functor Id_c respects direct limits.

A matricial k-algebra is a k-algebra of the form $\mathbb{M}_{t_1}(k) \times \cdots \times \mathbb{M}_{t_n}(k)$, where $t_1, \ldots, t_n$ are natural numbers and $\mathbb{M}_t(k)$ denotes the ring of all matrices of type $t \times t$ over a commutative field k. A locally matricial k-algebra is a direct limit (i.e., a directed union) of matricial k-algebras. We denote by $\mathbf{M}_k$ the category of locally matricial k-algebras (with unital k-linear ring homomorphisms), and by $\mathbf{m}_k$ its full subcategory of matricial k-algebras.

Some set-theoretic notation. We denote by On the class of all ordinal numbers. For a set X let $\mathcal{P}(X)$ denote the set of all its subsets. Given a map $f : X_1 \to X_2$, we denote by $f_* : \mathcal{P}(X_1) \to \mathcal{P}(X_2)$ the map sending $Y \mapsto \{f(y) \mid y \in Y\}$, for every $Y \subseteq X_1$. Similarly, we denote by $f^* : \mathcal{P}(X_2) \to \mathcal{P}(X_1)$ the map sending $Y \mapsto \{x \in X_1 \mid f(x) \in Y\}$, for every $Y \subseteq X_2$.

1. The category $\mathbf{c}$ revised

Objects of $\mathbf{c}$ are finite families $\mathbf{B} = \{B^i \mid i \in I\}$ of finite nonempty pairwise disjoint sets. Given objects $\mathbf{B}_1 = \{B_1^i \mid i \in I_1\}$ and $\mathbf{B}_2 = \{B_2^j \mid j \in I_2\}$ a *premorphism* from $\mathbf{B}_1$ to $\mathbf{B}_2$ is a family $\mathbf{h} = \{h^j \mid j \in I_2\}$ of bijections

$$h^j : \bigcup_{i \in I_1} \left(C^{i,j} \times B_1^i\right) \to B_2^j,$$

where $\mathbf{C} = \{C^{i,j} \mid i \in I_1,\, j \in I_2\}$ is a family of (possibly) empty finite sets. The collection of all premorphisms from $\mathbf{B}_1$ to $\mathbf{B}_2$ is denoted by $\mathbf{c}'(\mathbf{B}_1, \mathbf{B}_2)$. Premorphisms $\mathbf{h}$ and $\widetilde{\mathbf{h}}$ from $\mathbf{B}_1$ to $\mathbf{B}_2$ are *equivalent*, which we denote by $\mathbf{h} \sim \widetilde{\mathbf{h}}$, if there exist maps $g^{i,j} : C^{i,j} \to \widetilde{C}^{i,j}$ such that

$$h^j(c, b) = \widetilde{h}^j(g^{i,j}(c), b)$$

for every $c \in C^{i,j}$ and $b \in B_1^i$, as Figure 2 displays.

$$\begin{tikzcd}
\bigcup_{i\in I_1}(C^{i,j} \times B_1^i) \arrow[d, "\bigcup_{i\in I_1}(g^{i,j}\times \mathbf{1}_{B_1^i})"'] \arrow[dr, "h_j"] & \\
\bigcup_{i\in I_1}(\widetilde{C}^{i,j} \times B_1^i) \arrow[r, "\widetilde{h}_j"'] & B_2^j
\end{tikzcd}$$

Figure 2

It is easy to see that the relation $\sim$ is an equivalence on $\mathbf{c}'(\mathbf{B}_1, \mathbf{B}_2)$ and the morphisms in $\mathbf{c}$ are its equivalence classes. The symbol $[\mathbf{h}]$ denotes the class represented by $\mathbf{h}$. Given premorphisms $\mathbf{h}_1 \in \mathbf{c}'(\mathbf{B}_1, \mathbf{B}_2)$ and $\mathbf{h}_2 \in \mathbf{c}'(\mathbf{B}_2, \mathbf{B}_3)$, we put

$$C^{i,k} = \bigcup_{j \in I_2} (C_2^{i,k} \times C_1^{i,j})$$

and

$$h^k((c_2, c_1), b) = h_2^k(c_2, h_1^j(c_1, b))$$

for all $b \in B_1^i$, $c_1 \in C_1^{i,j}$, and $c_2 \in C_2^{j,k}$. The family $\mathbf{h} = \{h^k \mid k \in I_3\}$ forms a premorphism $\mathbf{h}$ from $\mathbf{B}_1$ to $\mathbf{B}_3$ which we denote by $\mathbf{h}_2 \circ \mathbf{h}_1$ and call the *composition of premorphisms* $\mathbf{h}_2$ and $\mathbf{h}_1$. It is proved [6, Lemma 2.2.] that the equivalence class $\mathbf{h}$ does not depend on the choice on the representatives of the classes $\mathbf{h}_2$ and $\mathbf{h}_1$ and so we can define the composition of morphisms in $\mathbf{c}$ by $\mathbf{h}_2 \circ \mathbf{h}_1 = [\mathbf{h}_2 \circ \mathbf{h}_1]$. The composition of premorphisms is depicted in Figure 3. In [6, Section 2] we have verified that $\mathbf{c}$ is a category. Recall, that the identity morphism at an object $\mathbf{B} = \{B^i \mid i \in I\}$ in $\mathbf{c}$ corresponds to the equivalence class of the collection of maps $h^i : i \times B^i \to B^i$, $(i, b) \mapsto b$.

$$\begin{tikzcd}
\bigcup_{j\in I_2}\left(C_2^{j,k}\times\left(\bigcup_{i\in I_1}(C_1^{i,j}\times B_1^i)\right)\right) \arrow[r, "\simeq"] \arrow[d, "\bigcup_{j\in I_2}\left(\mathbf{1}_{C_2^{j,k}}\times h_1^j\right)"'] & \bigcup_{i\in I_1}\Big(\underbrace{\bigcup_{j\in I_2}(C_2^{j,k}\times C_1^{i,j})}_{C^{i,k}}\times B_1^i\Big) \arrow[ddl, "h_k"'] \\
\bigcup_{j\in I_2}(C_2^{j,k}\times B_2^j) \arrow[d, "h_2^k"] & \\
B_3^k &
\end{tikzcd}$$

FIGURE 3

To every object $\mathbf{B} = \{B^i \mid i \in I\}$ of $\mathbf{c}$, we have assigned the Boolean semilattice $(\mathcal{P}(I), \cup)$ and given a morphism $[\mathbf{h}] \in \mathbf{c}(B_1, B_2)$, the correspondence

$$J \mapsto \Big\{j \in I_2 \mid \bigcup_{i \in J} C^{i,j} \neq \emptyset\Big\},$$

where J runs over all subsets of I_1, determines a $\langle \vee, 0, 1\rangle$-homomorphism $\Lambda[\mathbf{h}] : \Lambda(\mathbf{B}_1) \to \Lambda(\mathbf{B}_2)$. Thus we have defined a functor Λ from the category $\mathbf{c}$ to the category $\mathbf{S_{fb}}$ of finite Boolean join-semilattices. Further, given a commutative field k, we have defined a functor A from $\mathbf{c}$ to the category $\mathbf{m}_k$ so that there is a natural equivalence $\eta : \mathrm{Id}_c A \to \Lambda$. As the consequence of [6, Lemma 2.9], we get the following proposition.

Proposition 1.1. *The functor $A : \boldsymbol{c} \to \boldsymbol{m}_k$ lifts, via the natural equivalence $\eta : \mathrm{Id}_c A \to \Lambda$, the functor Λ with respect to Id_c (see Figure 4).*

2. THE CORRESPONDENCE $\mathrm{Bo} : \mathbf{S_{fd}} \to \mathbf{S_{fb}}$ REVISED

In [6, Section 1], we have defined a correspondence $\mathrm{Bo} : \mathbf{S_{fd}} \to \mathbf{S_{fb}}$ as follows. For $S \in \mathbf{S_{fd}}$ we define $\mathrm{Bo}(S)$ to be the finite Boolean $\langle \emptyset, \cup\rangle$-semilattice $\mathcal{P}(J)$, where J denotes the set of join-irreducible elements of S. Given a homomorphism $f : S_1 \to$

$$\begin{tikzcd} \boldsymbol{c} \arrow[rr, "A"] \arrow[dr, "\Lambda"'] & \arrow[d, phantom, "\overset{\simeq}{\underset{\eta}{\Leftarrow}}"] & \boldsymbol{m}_k \arrow[dl, "\mathrm{Id}_c"] \\ & \boldsymbol{S_{fb}} & \end{tikzcd}$$

FIGURE 4

S_2 in $\mathbf{S_{fd}}$, we define $\mathrm{Bo}(f) : \mathrm{Bo}(S_1) \to \mathrm{Bo}(S_2)$ to be a map sending $X \subseteq J(S_1)$ to $\{j \in J(S_2) \mid j \leq f(\bigvee X)\}$. The correspondence Bo preserves the composition of morphisms but the image of an identity morphism at S is an identity morphism at $\mathrm{Bo}(S)$ iff S is Boolean.

For every $S \in \mathbf{S_{fd}}$ denote by u_S and v_S the $\langle \vee, 0, 1\rangle$-homomorphisms defined by

$$(2.1) \qquad \begin{aligned} u_S &: \mathrm{Bo}(S) \to S & \qquad v_S &: S \to \mathrm{Bo}(S) \\ X &\mapsto \bigvee X & x &\mapsto \{j \in J(S) \mid j \leq x\}. \end{aligned}$$

Observe that for every $S \in \mathbf{S_{fd}}$, $u_S \circ v_S = \mathbf{1}_S$, and for every homomorphism $f : S_1 \to S_2$ in $\mathbf{S_{fd}}$, $v_{S_2} \circ f \circ u_{S_1} = \mathrm{Bo}(f)$, that is, the following two diagrams commute:

$$(2.2) \qquad \begin{tikzcd} S \arrow[rr, "\mathbf{1}_S"] \arrow[dr, "v_S"'] & & S \\ & \mathrm{Bo}(S) \arrow[ur, "u_S"'] & \end{tikzcd} \qquad \begin{tikzcd} S_1 \arrow[r, "f"] & S_2 \arrow[d, "v_{S_2}"] \\ \mathrm{Bo}(S_1) \arrow[u, "u_{S_1}"] \arrow[r, "\mathrm{Bo}(f)"'] & \mathrm{Bo}(S_2) \end{tikzcd}$$

Lemma 2.1. *Let P be an upwards directed partially ordered set without maximal elements and let $\langle S_p, f_{p,q}\rangle_{p<q \text{ in } P}$ be a directed system in $\boldsymbol{S_{fd}}$. If*

$$\langle S_p, f_p\rangle_{p\in P} = \varinjlim \langle S_p, f_{p,q}\rangle_{p<q \text{ in } P}, \text{ then}$$

$$\langle S_p, f_p \circ u_{S_p}\rangle_{p\in P} = \varinjlim \langle \mathrm{Bo}(S_p), \mathrm{Bo}(f_{p,q})\rangle_{p<q \text{ in } P}.$$

Proof. Denote by Q the set $P \times \{0,1\}$ ordered by $(p,i) < (q,j)$ iff $p < q$. For every $p \in P$ set $S_{(p,0)} = S_p$ and $S_{(p,1)} = \mathrm{Bo}(S_p)$ and for every $p < q$ in P define $f_{(p,1),(q,0)} = f_{p,q} \circ u_p$, $f_{(p,0),(q,1)} = v_q \circ f_{p,q}$, $f_{(p,1),(q,1)} = \mathrm{Bo}(f_{p,q})$, and $f_{(p,0),(q,0)} = f_{p,q}$. Finally, for every $p \in P$ set $f_{(p,0)} = f_p$ and $f_{(p,1)} = f_p \circ u_p$. It follows directly from the commutativity of diagrams (2.2) that $\langle S_{(p,i)}, f_{(p,i),(q,j)}\rangle_{(p,i)<(q,j) \text{ in } P\times\{0,1\}}$ is a directed system in $\mathbf{S_{fd}}$, and since, again due to (2.2), for every $p < q$ in P,

$$\begin{aligned} f_{(p,1)} &= f_{(q,0)} \circ f_{(p,1),(q,0)} = f_p \circ u_p = f_q \circ f_{p,q} \circ u_p = f_{(q,0)} \circ f_{(p,1),(q,0)}, \text{ and} \\ f_{(p,1)} &= f_q \circ f_{p,q} \circ u_p = f_q \circ u_q \circ v_q \circ f_{p,q} \circ u_p = f_q \circ u_q \circ \mathrm{Bo}(f_{p,q}) = \\ &= f_{(q,1)} \circ f_{(p,1),(q,1)}, \end{aligned}$$

$\langle S_{(p,i)}, f_{(p,i)}\rangle_{(p,i)\in P\times\{0,1\}}$ is its direct limit. Since both $P \times \{0\}$ and $P \times \{1\}$ are cofinal in $P \times \{0,1\}$, it concludes the proof. □

Lemma 2.1 coincides with [6, Proposition 1.1]. Its proof is straightforward but it requires a number of tedious verifications. Therefore we present another shorter proof here. The proof is based only on the commutativity of diagrams (2.2), and so it can be generalized for a similar situation in an arbitrary category. Anyway, we shall need it only in the presented form.

The proof of the following lemma is simple and we leave it to the reader.

Lemma 2.2. *Let $\boldsymbol{C}$ be a category with direct limits. Let P and Q be upwards directed partially ordered sets, let $\langle A_p, f_{p,q}\rangle_{p<q \text{ in } P}$, and $\langle B_p, g_{p,q}\rangle_{p<q \text{ in } Q}$ be directed systems in $\boldsymbol{C}$, and let $\langle A, f_p\rangle_{p\in P}$, $\langle B, g_q\rangle_{q\in Q}$ be their direct limits, respectively. Suppose that for every $p \in P$, there exists $p^\star \in Q$ and a homomorphism $h_p : A_p \to B_{p^\star}$ such that if $p < q$ in P and $p^\star$, $q^\star < r$ in Q, then $q_{p^\star,r} \circ h_p = q_{q^\star,r} \circ h_q \circ f_{p,q}$. Then there exists a unique homomorphism $h : A \to B$ such that for every $p \in P$, $h \circ f_p = g_{p^\star} \circ h_p$.*

3. Representation of distributive lattices revised

The category D. Denote by $\mathbf{D}$ the category whose objects are projections $p : X \to L$ of a set X on a distributive $\langle 0, 1\rangle$-lattice L, and whose morphisms are commutative diagrams

$$F: \quad \begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ L_1 & \xrightarrow[f']{} & L_2 \end{array} \quad ,$$

where $p_1 : X_1 \to L_1$ and $p_2 : X_2 \to L_2$ are objects of the category $\mathbf{D}$, f' is the $\langle\vee, \wedge, 0, 1\rangle$-homomorphism, and f is a map satisfying

$$f(x) = f(y) \text{ for some } x \neq y \text{ in } X_1 \implies p_2(f(x)) = p_2(f(y)) = 0, \tag{3.1}$$

with obvious composition of morphisms and identities.

Denote by Θ the functor $\mathbf{D} \to \mathbf{S_d}$ which assigns to an object $p : X \to L$ the lattice L and to a morphism $F = (f, f')$ the $\langle\vee, \wedge, 0, 1\rangle$-homomorphism f'.

Denote by $\mathbf{D_f}$ the full subcategory of $\mathbf{D}$ whose objects are projections of a finite set on a finite distributive lattice, and let Θ_f denote the restriction $\Theta \restriction \mathbf{D_f}$. We shall now define a functor $[\Phi]$ from the category $\mathbf{D_f}$ to the category $\mathbf{c}$.

Given an object $p : X \to L$ in $\mathbf{D_f}$ and an element $a \in L$, set

$$a^p = \{x \in X \mid p(x) \geq a\},$$

and given a morphism $F = (f, f')$ in $\mathbf{D_f}(p_1, p_2)$ and an element $a \in L_1$, define $a^F = f_*(a^{p_1})$.

For a set X denote by $TO(X)$ the set of all total orders on X, and for each $\alpha \in X$ denote by $H(\alpha)$ the set of all hereditary subsets (including the empty set) of X with respect to the total order α. Let X be a finite set and Y a subset of X. Denote by $\alpha \restriction Y$ the restriction of a $\alpha \in TO(X)$ to Y, and given $\alpha : \alpha_0 < \cdots < \alpha_{n-1}$ and $\beta : \beta_0 < \cdots < \beta_{n-1}$, write $\alpha \sim_Y \beta$ provided that for every $i < n$, $\alpha_i \neq \beta_i$ implies that α_i, $\beta_i \in Y$. Thus we have defined an equivalence relation on the set $TO(X)$, and we denote by $[\alpha]_Y$ the equivalence class represented by α.

Let $f : X_1 \to X_2$ be an embedding of a finite set X_1 to a finite set X_2. For $\alpha : \alpha_0 < \cdots < \alpha_{n-1} \in TO(X_1)$ set $f\alpha : f(\alpha_0) < \cdots < f(\alpha_{n-1}) \in TO(f(X_1))$.

Definition 3.1. Let $p : X \to L$ be an object of $\mathbf{D_f}$. For every $a \in J(L)$ denote by $\Phi(p)^a$ the set of all $\alpha \in TO(a^p)$ satisfying $a'^p \notin H(\alpha)$ for every $a' \in J(L)$ with $a < a'$, and set $[\Phi](p) = \{\Phi(p)^a \mid a \in J(L)\}$.

Let

$$\begin{tikzcd} F: & X_1 \arrow[r, "f"] \arrow[d, "p_1"'] & X_2 \arrow[d, "p_2"] \\ & L_1 \arrow[r, "f'"'] & L_2 \end{tikzcd}$$

be a morphism in $\mathbf{D_f}$, and let $a \in J(L_1)$, $b \in J(L_2)$. If $f'(a) \ngeq b$, define $\operatorname{dom} \Phi(F)^{a,b} = \emptyset$. If $f'(a) \geq b$, then $a^F \subseteq b^{p_2}$, and we denote by $\operatorname{dom} \Phi(F)^{a,b}$ the set of all classes $[\beta']_{a^F}$, where $\beta' \in TO(b^{p_2})$, satisfying the following properties:

(Φ_1) $a^F \in H(\beta' \restriction (b^{p_2} \cap f_*(X_1)))$;

(Φ_2) $b'^{p_2} \notin H(\beta')$, for every $b' \in J(L_2)$ with $b < b' \leq f'(a)$.

Observe that the validity of (Φ_1), (Φ_2) does not depend on the choice of the representative of the class $[\beta']_{a^F}$.

As in [6], our construction makes use of the following well-known property of lattice homomorphisms [4, Exercise 2.63.10].

Lemma 3.2. *Let L_1, L_2 be finite distributive lattices and let $f' : L_1 \to L_2$ be a $\langle \vee, \wedge, 0, 1 \rangle$-homomorphism and let $b \in J(L_2)$. Then ${f'}^{-1}([b)_{L_2}) = [c)_{L_1}$ for some $c \in J(L_1)$.*

Corollary 3.3. *Let $F = (f, f') : p_1 \to p_2$ be a morphism in $\boldsymbol{D_f}$ and let $b \in J(L_2)$. Then $f^*(b^{p_2}) = c^{p_1}$ for some $c \in J(L_1)$.*

Lemma 3.4. *Let*

$$\begin{tikzcd} F: & X_1 \arrow[r, "f"] \arrow[d, "p_1"'] & X_2 \arrow[d, "p_2"] \\ & L_1 \arrow[r, "f'"'] & L_2 \end{tikzcd}$$

be a morphism in $\boldsymbol{D_f}$ and let $b \in J(L_2)$. Then the correspondence

$$([\beta']_{a^F}, \alpha) \mapsto \beta,$$

where $\beta \in TO(b^{p_2})$ satisfies $\beta' \sim_{a^F} \beta$ and $\beta \restriction a^F = f\alpha$ defines a map

$$\Phi(F)^b : \bigcup_{a \in J(L_1)} \big(\operatorname{dom} \Phi(F)^{a,b} \times \Phi(p_1)^a\big) \to \Phi(p_2)^b.$$

Proof. Let $a \in J(L_1)$ and $b \in J(L_2)$. If $f'(a) \ngeq b$, then $\operatorname{dom} \Phi(F)^{a,b} = \emptyset$. Suppose that $f'(a) \geq b$, and let $[\beta']_{a^F} \in \operatorname{dom} \Phi(F)^{a,b}$ and $\alpha \in \Phi(p_1)^a$. It follows from (3.1) that $f \restriction a^{p_1}$ is one-to-one, and so we can define $f\alpha$. According to [6, Lemma 4.1], there is a unique $\beta \in TO(b^{p_2})$ such that $\beta' \sim_{a^F} \beta$ and $\beta \restriction a^F = f\alpha$. We prove that $\beta \in \Phi(p_2)^b$. Toward the contradiction suppose that there is $b' \in J(L_2)$ such that $b < b'$ and $b'^{p_2} \in H(\beta)$. By Corollary 3.3, there is $c \in J(L_1)$ such that $f^*(b'^{p_2}) = c^{p_1}$. It follows that $c^F = b'^{p_2} \cap f_*(X_1)$, and so $c^F \in H(\beta \restriction (b^{p_2} \cap f_*(X_1)))$. Since, by ($\Phi_1$), $a^F \in H(\beta' \restriction (b^{p_2} \cap f_*(X_1)))$, whence $a^F \in H(\beta \restriction (b^{p_2} \cap f_*(X_1)))$, either $a^F \subseteq c^F$ or $c^F \subsetneq a^F$, that is, either $c \leq a$, or $a < c$. If $a < c$, then $\beta \restriction a^F = f\alpha$ and $c^F \in H(\beta \restriction (b^{p_2} \cap f_*(X_1)))$ implies that $c^{p_1} \in H(\alpha)$, which is in a contradiction with $\alpha \in \Phi(p_1)^a$. If $c \leq a$, then $b' \leq f'(a)$, that is, $a^F \subseteq b'^{p_2}$. By our assumption $b'^{p_2} \in H(\beta)$, and since $a^F \in H(\beta' \restriction (b^{p_2} \cap f_*(X_1)))$ and $\beta \restriction_{a^F} \beta'$, it implies that $b'^{p_2} \in H(\beta')$. This contradicts (Φ_2), since then $b < b' \leq f'(a)$ and $b'^{p_2} \in H(\beta')$. $\square$

Lemma 3.5. *The map $\Phi(F)^b$ is a bijection.*

Proof. First we prove that $\Phi(F)^b$ is onto. Let β be an arbitrary element of $\Phi(p_2)^b$. By Corollary 3.3, there exists $c \in J(L_1)$ with $f^*(b^{p_2}) = c^{p_1}$, that is, $b^{p_2} \cap f_*(X_1) = c^F$. The set

$$\mathcal{A} = \{a' \in J(L_1) \mid {a'}^F \in H(\beta \restriction (b^{p_2} \cap f_*(X_1)))\}$$

is nonempty since it contains c. Observe that the set $\{{a'}^F \mid a' \in \mathcal{A}\}$ is totally ordered by inclusion and denote by a the element of $\mathcal{A}$ corresponding to the minimal a^F with respect to this order. Now denote by α the total order of a^{p_1} such that $f\alpha = \beta \restriction a^F$. Observe that $\alpha \in \Phi(p_1)^a$.

We prove that $[\beta]_{a^F} \in \operatorname{dom} \Phi(F)^{a,b}$. Since $a \in \mathcal{A}$, $a^F \in H(\beta \restriction (b^{p_2} \cap f_*(X_1)))$. Let $b' \in J(L_2)$ satisfy $b < b' < f'(a)$. Then, since $\beta \in \Phi(p_2)^b$, ${b'}^{p_2} \notin H(\beta)$.

Finally we prove that the map $\Phi(F)^b$ is one-to-one. Let $\beta \in \operatorname{dom} \Phi(p_2)^b$ and let $a \in J(L_1)$ and $\alpha \in \Phi(p_1)^a$ be elements constructed above. Suppose that

$$\Phi(F)^b([\bar{\beta}]_{\bar{a}^F}, \bar{\alpha}) = \beta,$$

for some $\bar{a} \in J(L_1)$, $\bar{\alpha} \in \Phi(p_1)^{\bar{a}}$, and $[\bar{\beta}]_{\bar{a}^F} \in \operatorname{dom} \Phi(F)^{\bar{a},b}$. By (Φ_1), $\bar{a}^F \in H(\bar{\beta} \restriction (b^{p_2} \cap f_*(X_1)))$, and since $\bar{\beta} \sim_{\bar{a}^F} \beta$, $\bar{a}^F \in H(\beta \restriction (b^{p_2} \cap f_*(X_1)))$, hence $\bar{a} \in \mathcal{A}$. By the definition $f\bar{\alpha} = \beta \restriction \bar{a}^F$, and so it follows from the properties of $\Phi(F)^{\bar{a}}$ that $\bar{a}^F$ is a minimal element of the set $\{{a'}^F \mid a' \in \mathcal{A}\}$, ordered by inclusion. This set is totally ordered, and so $\bar{a} = a$. Now it is easy to see that $\bar{\alpha} = \alpha$ and $[\bar{\beta}]_{\bar{a}^F} = [\bar{\beta}]_{a^F} = [\beta]_{a^F}$. $\square$

Corollary 3.6. *Let F be a morphism in the category $\boldsymbol{D_f}$. Then $\Phi(F)$ is a premorphism in the category $\boldsymbol{c}$.*

Definition 3.7. We say that a morphism

$$F:\quad \begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ L_1 & \xrightarrow[f']{} & L_2 \end{array}$$

in $\mathbf{D_f}$ is *efficient* if for every $b \in J(L_2)$, there exists $x \in X_2 \setminus f_*(X_1)$ with $p_2(x) = b$.

Lemma 3.8. *Let*

$$F:\quad \begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ L_1 & \xrightarrow[f']{} & L_2 \end{array}$$

be an efficient morphism in $\boldsymbol{D_f}$. Then $\operatorname{dom} \Phi(F)^{a,b} \neq \emptyset$ *iff* $b \leq f(a)$, *for every* $a \in J(L_1)$, $b \in J(L_2)$.

Proof. The implication "$\Leftarrow$" follows directly from the definition. In order to prove the opposite one, let $a \in J(L_1)$, $b \in J(L_2)$, and suppose that $b \leq f(a)$. Since the morphism F is efficient, there is $x \in X_2 \setminus f_*(X_1)$ with $p_2(x) = b$. Let $\alpha \in \Phi(p_1)^a$. Choose $\beta : \beta_0 < \cdots < \beta_n \in TO(b^{p_2})$ such that $x = \beta_0$ and $\alpha^F \in H(\beta \restriction (b^{p_2} \cap f_*(X_1)))$. It is straightforward that $[\beta]_{a^F} \in \operatorname{dom} \Phi(F)^{a,b}$ and $\Phi(F)^b([\beta]_{a^F}, \alpha) = \beta$. $\square$

Corollary 3.9. *Let $F = (f, f') : p_1 \to p_2$ be an efficient morphism in $\boldsymbol{D_f}$. Then* $\Lambda([\Phi(F)]) = \mathrm{Bo}(f)$.

Lemma 3.10. *Let*

$$F: \begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ L_1 & \xrightarrow[f']{} & L_2 \end{array} \qquad \text{and} \qquad G: \begin{array}{ccc} X_2 & \xrightarrow{g} & X_3 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_3} \\ L_2 & \xrightarrow[g']{} & L_3 \end{array}$$

be morphisms in $\boldsymbol{D_f}$. Then $[\Phi(G \circ F)] = [\Phi(G)] \circ [\Phi(F)]$.

Proof. According to the definition of the composition of premorphisms in the category $\mathbf{c}$, for every $a \in J(L_1)$ and $c \in J(L_3)$,

$$\mathrm{dom}(\Phi(G) \circ \Phi(F))^{a,c} = \bigcup_{b \in J(L_2)} \mathrm{dom}\,\Phi(G)^{b,c} \times \mathrm{dom}\,\Phi(F)^{a,b},$$

and $(\Phi(G) \circ \Phi(F))^c$ is a map defined by the correspondence

$$(([\gamma']_{b^G}, [\beta']_{a^F}), \alpha) \mapsto \Phi(G)^c \left([\gamma']_{b^G}, \Phi(F)^b([\beta']_{a^F}, \alpha)\right).$$

In order to prove that $[\Phi(G) \circ \Phi(F)] = [\Phi(G \circ F)]$, we define maps $g^{a,c}$ from $\mathrm{dom}(\Phi(G) \circ \Phi(F))^{a,c}$ to $\mathrm{dom}\,\Phi(G \circ F)^{a,c}$ by

$$g^{a,c}\left([\gamma']_{b^G}, [\beta']_{a^F}\right) = [\gamma'']_{a^{G \circ F}},$$

where $\gamma'' \in TO(c^{p_3})$ satisfies $\gamma'' \sim_{b^G} \gamma'$ and $\gamma'' \restriction b^G \sim_{a^{G\circ F}} g'\beta'$. Again it is easily seen that these properties determine γ'' uniquely.

Let $a \in J(L_1)$, $b \in J(L_2)$, and $c \in J(L_3)$ satisfy $f'(a) \geq b$ and $g'(b) \geq c$. Let $[\gamma']_{b^G} \in \mathrm{dom}\,\Phi(G)^{b,c}$, $[\beta']_{a^F} \in \mathrm{dom}\,\Phi(F)^{a,b}$, and $\alpha \in \Phi(p_1)^a$. We verify that

$$(\Phi(G) \circ \Phi(F))^c\left(([\gamma']_{b^G}, [\beta']_{a^F}), \alpha\right) = \Phi(G \circ F)^{a,c}\left(g^{a,c}\left([\gamma']_{b^G}, [\beta']_{a^F}\right), \alpha\right). \tag{3.2}$$

First we evaluate the left hand side of (3.2):

$$\begin{aligned}(\Phi(G) \circ \Phi(F))^c\left(([\gamma']_{b^G}, [\beta']_{a^F}), \alpha\right) &= \Phi(G)^c\left([\gamma']_{b^G}, \Phi(F)^b([\beta']_{a^F}, \alpha)\right) \\ &= \Phi(G)^c([\gamma']_{b^G}, \beta),\end{aligned}$$

where $\beta \in TO(b^{p_2})$ satisfies $\beta \sim_{a^F} \beta'$ and $\beta \restriction a^F = f\alpha$, and

$$\Phi(G)^c([\gamma']_{b^G}, \beta) = \gamma,$$

where $\gamma \in TO(c^{p_3})$ satisfies $\gamma \sim_{b^G} \gamma'$, and $\gamma \restriction b^G = g\beta$.

Now we evaluate the right hand side of (3.2):

$$\Phi(G \circ F)^{a,c}\left(g^{a,c}\left([\gamma']_{b^G}, [\beta']_{a^F}\right), \alpha\right) = \Phi(G \circ F)^{a,c}\left([\gamma'']_{a^{G\circ F}}, \alpha\right),$$

where $\gamma'' \in TO(c^{p_3})$ satisfies $\gamma'' \sim_{b^G} \gamma'$ and $\gamma'' \restriction b^G \sim_{a^{G\circ F}} g\beta'$, and

$$\Phi(G \circ F)^{a,c}\left([\gamma'']_{a^{G\circ F}}, \alpha\right) = \delta,$$

where $\delta \in TO(c^{p_3})$ satisfies $\delta \sim_{a^{G\circ F}} \gamma''$ and $(g \circ f)\alpha = \delta \restriction a^{G\circ F}$.

It remains to compare γ and δ. Since $f'(a) \geq b$, $a^{G\circ F} \subseteq b^G$. The equality $\beta \restriction a^F = f\alpha$ implies $g\beta \restriction a^{G\circ F} = (g \circ f)\alpha$, and since $\gamma \restriction b^G = g\beta$, $\gamma \restriction a^{G\circ F} = (\gamma \restriction b^G) \restriction a^{G\circ F} = (g \circ f)\alpha$. Now $\gamma \restriction b^G = g\beta$ and $\beta \sim_{a^F} \beta'$, thus $\gamma \restriction b^G \sim_{a^{G\circ F}} g\beta'$, and since $\gamma \sim_{b^G} \gamma'$, we conclude that $\gamma \sim_{a^{G\circ F}} \gamma''$. This together with the equality $(g \circ f)\alpha = \gamma \restriction a^{G\circ F}$ proves that $\delta = \gamma$. □

Lemma 3.11. *For every element $p \in \boldsymbol{D_f}$, $[\Phi(\mathbf{1}_p)] = \mathbf{1}_{\Phi(p)}$.*

Proof. Let

$$\mathbf{1}_p : \begin{tikzcd} X \arrow[r, "\mathbf{1}_X"] \arrow[d, "p"'] & X \arrow[d, "p"] \\ L \arrow[r, "\mathbf{1}_L"'] & L \end{tikzcd}$$

be the identity morphism at p in $\mathbf{D_f}$. Let $a,\, b \in J(L)$. If $a \ngeq b$, then by the definition $\operatorname{dom}\Phi(\mathbf{1}_p)^{a,b} = \emptyset$. If $a \geq b$, then $\operatorname{dom}\Phi(\mathbf{1}_p)^{a,b}$ is a set of all $[\beta']_{a^{\mathbf{1}_p}}$ satisfying $a^p = a^{\mathbf{1}_p} \in H(\beta')$ and if $b' \in J(L)$ satisfies $b < b' \leq a$, then $b'^p \notin H(\beta')$. It follows that $a = b$, and in this case $\beta' \sim_{a^{\mathbf{1}_p}} \beta''$ for all $\beta',\, \beta'' \in TO(b^p)$, whence $\operatorname{dom}\Phi(\mathbf{1}_p)^{a,b}$ is a one-element set. This proves that $[\Phi(\mathbf{1}_p)] = \mathbf{1}_{\Phi(p)}$. $\square$

For an object p, resp. a morphism F of $\mathbf{D_f}$ set $[\Phi](p) = \Phi(p)$, resp. $[\Phi](F) = [\Phi(F)]$.

Corollary 3.12. $[\Phi]$ *is a functor from the category $\boldsymbol{D_f}$ to the category $\boldsymbol{c}$.*

The situation we have got at the moment is illustrated on Figure 5. The arrow $\mathrm{Bo} : \mathbf{S_{fd}} \to \mathbf{S_{fb}}$ is dotted since Bo is not a functor; it only preserves the composition of morphisms. The trapezium on the left is not commutative but it commutes if we restrict ourselves to efficient morphisms.

$$\begin{tikzcd} \mathbf{D_f} \arrow[r, "{[\Phi]}"] \arrow[d, "\Theta_f"'] & \mathbf{c} \arrow[rr, "A"] \arrow[dr, "\Lambda"'] \arrow[rr, phantom, shift right=6, "\overset{\simeq}{\Leftarrow}\,\eta"] & & \mathbf{m}_k \arrow[dl, "\mathrm{Id}_c"] \\ \mathbf{S_{fd}} \arrow[rr, dotted, "\mathrm{Bo}"'] & & \mathbf{S_{fb}} & \end{tikzcd}$$

FIGURE 5

4. LIFTING OF THE FUNCTOR Θ_∞ WITH RESPECT TO Id_c

Denote by $\mathbf{D_\infty}$ the full subcategory of the category $\mathbf{D}$ whose objects are projections $P : X \to L$ such that $P^*(\{a\})$ is infinite for every $a \in L$, and let Θ_∞ denote the restriction $\Theta \restriction \mathbf{D_\infty}$. Given an object $P : X \to L \in \mathbf{D}$, denote by $\operatorname{Fin}(P)$ the set $\{p \in \mathbf{D_f} \mid p \subseteq P\}$ of all finite sub objects of P, and for every $p \subseteq q$ in $\operatorname{Fin}(P)$ denote by $I_{p,q}$, resp. $I_{p,P}$ the inclusion morphism from p to q, resp. from p to P. Define an order "$\ll$" on the set $\operatorname{Fin}(P)$: $p \ll q$ if the inclusion morphism $I_{p,q}$ is efficient.

Given $p \in \mathbf{D_f}$, denote by u_p, v_p the morphisms between $\Theta(p)$ defined by correspondences (2.1). Now we are going to define a functor $\Psi : \mathbf{D} \to \mathbf{M}_k$. By Corollary 3.12, $[\Phi]$ is a functor, and we define the restriction $\Psi \restriction \mathbf{D_f} : \mathbf{D_f} \to \mathbf{m}_k$ as the composition of the functors A and $[\Phi]$.

Let $P : X \to L$ be an object in $\mathbf{D_\infty}$. Then the set $\operatorname{Fin}(P)$ partially ordered by "$\ll$" is upwards directed, and $\langle P, I_{p,P}\rangle_{p\in\operatorname{Fin}(P)}$ is a direct limit of the directed system $\langle p, I_{p,q}\rangle_{p\ll q \text{ in } \operatorname{Fin}(P)}$ in $\mathbf{D}$. We define $\Psi(P) = \varinjlim\langle\Psi(p), \Psi(I_{p,q})\rangle_{p\ll q \text{ in } \operatorname{Fin}(P)}$, and we let $\Psi(I_{p,P})$ be the corresponding limiting morphisms.

Let $F : P_1 \to P_2$ be morphism $\mathbf{D}$. For every $p \in \operatorname{Fin}(P_1)$ select $p^\star \in \operatorname{Fin}(P_2)$ so that the image of $F \restriction p$ is contained in $p^\star$, and denote by F_p the morphism in

$\mathbf{D_f}(p,p^\star)$ which coincides with the restriction $F \restriction p$. It is straightforward that if $p \ll q$ in $\mathrm{Fin}(P_1)$ and $p^\star, q^\star \ll r$ in $\mathrm{Fin}(P_2)$, then

$$I_{p^\star,r} \circ F_p = I_{q^\star,r} \circ F_q \circ I_{p,q}. \tag{4.1}$$

Thus $\Psi(I_{p^\star,r}) \circ \Psi(F_p) = \Psi(I_{q^\star,r}) \circ \Psi(F_q) \circ \Psi(I_{p,q})$, and, by Lemma 2.2, there exists a unique k-linear ring homomorphism $h : \Psi(P_1) \to \Psi(P_2)$ such that

$$h \circ \Psi(I_{p,P_1}) = \Psi(I_{p^\star,P_2}) \circ \Psi(F_p), \tag{4.2}$$

for every $p \in \mathrm{Fin}(P)$.

Lemma 4.1. *The map h does not depend on the choice of the elements $p^\star$.*

Proof. For every $p \in \mathrm{Fin}(P_1)$ select another $p^{\star\star} \in \mathrm{Fin}(P_2)$ so that the image of $F \restriction p$ is contained in $p^{\star\star}$, and denote by $F_p^\star$ the morphism in $\mathbf{D_f}(p,p^{\star\star})$ which coincides with the restriction $F \restriction p$. Then, as above, there exists a unique k-linear ring homomorphism $h^\star$ such that

$$h^\star \circ \Psi(I_{p,P_1}) = \Psi(I_{p^{\star\star},P_2}) \circ \Psi(F_p^\star),$$

for every $p \in \mathrm{Fin}(P)$. Now, for each $p \in \mathrm{Fin}(P_1)$ select $p^\dagger \in \mathrm{Fin}(P_2)$ with $p^\star$, $p^{\star\star} \ll p^\dagger$, and denote by $F_p^\dagger$ the morphism in $\mathbf{D_f}(p,p^\dagger)$ corresponding to the restriction $F \restriction p$. Since

$$I_{p^\star,p^\dagger} \circ F_p = F_p^\dagger,$$

we have that

$$\Psi(I_{p^\star,P_2}) \circ \Psi(F_p) = \Psi(I_{p^\dagger,P_2}) \circ \Psi(I_{p^\star,p^\dagger}) \circ \Psi(F_p) = \Psi(I_{p^\dagger,P_2}) \circ \Psi(F_p^\dagger),$$

whence the map h satisfies the equality

$$h \circ \Psi(I_{p,P_1}) = \Psi(I_{p^\dagger,P_2}) \circ \Psi(F_p^\dagger),$$

for every $p \in \mathrm{Fin}(P_1)$. Similarly we get that $h^\star$ satisfies

$$h^\star \circ \Psi(I_{p,P_1}) = \Psi(I_{p^\dagger,P_2}) \circ \Psi(F_p^\dagger),$$

for every $p \in \mathrm{Fin}(P_1)$, and from the unicity of such a map we deduce that $h = h^\star$. □

Define $\Psi(F) = h$. It is straightforward that $\Psi : \mathbf{D} \to \mathbf{M}_k$ is a functor which respects direct limits.

Proposition 4.2. *The functor Ψ lifts Θ_∞ with respect to Id_c.*

Proof. We have defined $\mathrm{Id}_c\Psi(p) = \mathrm{Id}_c A([\Phi](p))$, for every $p \in \mathbf{D_f}$, and so $\eta_{[\Phi](p)}$ is an isomorphism from $\mathrm{Id}_c\Psi(p)$ to $\mathrm{Bo}(\Theta(p))$. We abbreviate the notation setting $\eta_p = \eta_{[\Phi](p)}$. Let $F : p_1 \to p_2$ be a morphism in $\mathbf{D_f}$. By Corollary 3.9, $\mathrm{Bo}(\Theta(F)) = \lambda([\Theta(F)])$, and since $\eta : \mathbf{1}_c \circ [\Phi] \to \Lambda$ is a natural equivalence,

$$\mathrm{Bo}(\Theta(F)) = \eta_{p_2} \circ \mathrm{Id}_c\Psi(F) \circ \eta_{p_1}^{-1}. \tag{4.3}$$

Let $P : X \to L$ be an object in $\mathbf{D_\infty}$. By the definition

$$\langle \Psi(P), \psi(I_{p,P}) \rangle_{p \in \mathrm{Fin}(P)} = \varinjlim \langle \Psi(p), \Psi(I_{p,q}) \rangle_{p \ll q \text{ in } \mathrm{Fin}(P)},$$

and, since the functor Id_c respects direct limits, it follows that

$$\langle \mathrm{Id}_c\Psi(P), \mathrm{Id}_c\Psi(I_{p,P}) \rangle_{p \in \mathrm{Fin}(P)} = \varinjlim \langle \mathrm{Id}_c\Psi(p), \mathrm{Id}_c\Psi(I_{p,q}) \rangle_{p \ll q \text{ in } \mathrm{Fin}(P)}.$$

By (4.3), the directed system $\langle \mathrm{Id}_c\Psi(p), \mathrm{Id}_c\Psi(I_{p,q})\rangle_{p\subsetneq q \text{ in } \mathrm{Fin}(P)}$ is isomorphic, via the family of isomorphisms $\{\eta_p \mid p \in \mathrm{Fin}(P)\}$, to the directed system

$$\langle \mathrm{Bo}(\Theta(p)), \mathrm{Bo}(\Theta(I_{p,q}))\rangle_{p\ll q \text{ in } \mathrm{Fin}(P)}.$$

Since $P \in \mathbf{D}_{\infty}$, the partially ordered set $\mathrm{Fin}(P)$ has no maximal elements, hence, by Lemma 2.1,

$$\langle \Theta(P), \Theta(I_{p,P}) \circ u_p\rangle_{p\in \mathrm{Fin}(P)} = \varinjlim \langle \mathrm{Bo}(\Theta(p)), \mathrm{Bo}(\Theta(I_{p,q}))\rangle_{p\ll q \text{ in } \mathrm{Fin}(P)}.$$

The isomorphisms $\{\eta_p \mid p \in \mathrm{Fin}(P)\}$ induce a unique isomorphism $\eta_P : \mathrm{Id}_c\Psi(P) \to \Theta(P)$ such that for every $p \in \mathrm{Fin}(P)$:

$$\eta_P \circ \mathrm{Id}_c\Psi(I_{p,P}) = \Theta(I_{p,P}) \circ u_p \circ \eta_p. \tag{4.4}$$

Let $F : P_1 \to P_2$ be a morphism in $\mathbf{D}_{\infty}$. Select for each $p \in \mathrm{Fin}(P_1)$ an object $p^\star \in \mathrm{Fin}(P_2)$ and define the morphism F_p, as above. Then, it follows from (4.1) that for every $p \subseteq q$ in $\mathrm{Fin}(P_1)$ and every $r \in \mathrm{Fin}(P_2)$ with $p^\star, q^\star \ll r$,

$$\mathrm{Bo}\Theta(I_{p^\star,r}) \circ \mathrm{Bo}\Theta(F_p) = \mathrm{Bo}\Theta(I_{q^\star,r}) \circ \mathrm{Bo}\Theta(F_q) \circ \mathrm{Bo}\Theta(I_{p,q}),$$

and so, by Lemma 2.2, there is a unique homomorphism $h : \Theta(P_1) \to \Theta(P_2)$ such that, for every $p \in \mathrm{Fin}(P)$,

$$h \circ \Theta(I_{p,P_1}) \circ u_p = \Theta(I_{p^\star,P_2}) \circ u_{p^\star} \circ \mathrm{Bo}\Theta(F_p). \tag{4.5}$$

Now, it follows from (4.2) that

$$\mathrm{Id}_c\Psi(F) \circ \mathrm{Id}_c\Psi(I_{p,P}) = \mathrm{Id}_c\Psi(I_{p^\star,P_2}) \circ \mathrm{Id}_c\Psi(F_p). \tag{4.6}$$

Applying (4.4), we derive from (4.6) that

$$\mathrm{Id}_c\Psi(F) \circ \eta_{P_1}^{-1} \circ \Theta(I_{p,P_1}) \circ u_p \circ \eta_p = \eta_{P_2}^{-1} \circ \Theta(I_{p^\star,P_2}) \circ u_{p^\star} \circ \mathrm{Bo}\Theta(F_p) \circ \eta_p \tag{4.7}$$

Composing (4.7) with η_{P_2}, η_p^{-1} from the left, right, respectively, gives

$$\eta_{P_2} \circ \mathrm{Id}_c\Psi(F) \circ \eta_{P_1}^{-1} \circ \Theta(I_{p,P_1}) \circ u_p = \Theta(I_{p^\star,P_2}) \circ u_{p^\star} \circ \mathrm{Bo}\Theta(F_p),$$

which, by Lemma 2.2, implies that $\eta_{P_2} \circ \mathrm{Id}_c\Psi(F) \circ \eta_{P_1}^{-1} = h$. Finally, since, by the definition, $F \circ I_{p,P_1} = I_{p^\star,P_2} \circ F_p$, for every $p \in \mathrm{Fin}(P)$,

$$\Theta(F) \circ \Theta(I_{p,P_1}) \circ u_p = \Theta(I_{p^\star,P_2}) \circ u_{p^\star} \circ \mathrm{Bo}\Theta(F_p),$$

as well, and so $\eta_{P_2} \circ \mathrm{Id}_c\Psi(F) \circ \eta_{P_1}^{-1} = h = \Theta(F)$. This concludes the proof. □

Corollary 4.3. *Let Υ be a functor from a category $\boldsymbol{C}$ to a category $\boldsymbol{S_d}$ whose image is in $\boldsymbol{L_d}$. Then the functor Υ can be lifted with respect to Θ_∞ if and only if it can be lifted with respect to* Id_c.

Proof. ($\Rightarrow$) Let Φ be a functor from $\mathbf{C}$ to $\mathbf{D}_{\infty}$ that lifts Υ with respect to Θ_∞. Then, by Proposition 4.2, $\mathrm{Id}_c\Psi\Phi = \Theta_\infty\Phi = \Upsilon$, whence the functor $\Psi\Phi$ lifts Υ with respect to Id_c.

($\Leftarrow$) Suppose that a functor Γ lifts Υ with respect to Id_c. For an element a of a locally matricial k-algebra A denote by $\langle a\rangle$ the two-sided ideal of A generated by

a. Define a functor $\Delta : \mathbf{C} \to \mathbf{D_\infty}$ as follows: $\Delta(C) : \Gamma(C) \times \omega \to \mathrm{Id}_c\Gamma(C)$ is given by the correspondence $(a, n) \to \langle a\rangle$, for every object $C \in \mathbf{C}$, and

$$\Delta(f): \quad \begin{array}{ccc} \Gamma(C_1) \times \omega & \xrightarrow{\Gamma(f)\times\omega} & \Gamma(C_2) \times \omega \\ {\scriptstyle\Delta(C_1)}\downarrow & & \downarrow{\scriptstyle\Delta(C_2)} \\ \mathrm{Id}_c\Gamma(C_1) & \xrightarrow[\mathrm{Id}_c\Gamma(f)]{} & \mathrm{Id}_c\Gamma(C_2) \end{array} ,$$

for every morphism $f : C_1 \to C_2$ in $\mathbf{C}$. The functor Δ lifts Υ with respect to Θ_∞. □

5. Existence and non-existence of liftings

Denote by $\mathbf{L_{d1}}$ the category whose objects are distributive $\langle 0, 1\rangle$-lattices and whose morphisms are one-to-one $\langle\vee, \wedge, 0, 1\rangle$-homomorphisms. We apply Corollary 4.3 to prove that the category $\mathbf{L_{d1}}$ as well as every diagram in $\mathbf{L_d}$ has a lifting with respect to the Id_c functor. Lets start with the $\mathbf{L_{d1}}$ case.

Theorem 5.1. *The category $\mathbf{L_{d1}}$ has a lifting with respect to* Id_c.

Proof. By Corollary 4.3, it suffices to find a lifting Π of the category $\mathbf{L_{d1}}$ with respect to the functor Θ_∞. It is easy, we only have to guarantee that its image is in $\mathbf{D_\infty}$. Let M be an infinite set and given a distributive $\langle 0, 1\rangle$-lattice L, define $\Pi(L)$ to be the map $L \times M \to L$ sending (a, m) to a, and given a $\langle\vee, \wedge, 0, 1\rangle$-embedding $f : L_1 \to L_2$, let $\Pi(f)$ be the morphism

$$\Pi(f): \quad \begin{array}{ccc} L_1 \times M & \xrightarrow{f\times\mathbf{1}_M} & L_2 \times M \\ {\scriptstyle\Pi(L_1)}\downarrow & & \downarrow{\scriptstyle\Pi(L_2)} \\ L_1 & \xrightarrow[f]{} & L_2 \end{array}$$

in $\mathbf{D_\infty}$. □

As opposed to the Theorem 5.1, even a simple finite subcategory of the whole category $\mathbf{L_d}$ cannot be lifted with respect to Id_c, which is demonstrated in Example 5.3. First we need the following definition.

Definition 5.2. We say that a lattice homomorphism $f : L_1 \to L_2$ *separates zero* if $f(a) > 0$ for every nonzero $a \in L_1$. Observe if $\mathrm{Id}_c(\varphi)$ separates zero for a homomorphism $\varphi : A_1 \to A_2$ in $\mathbf{M}_k$ then the homomorphism φ is one-to-one.

For an ordinal number λ denote by C_λ a well-ordered chain of the ordinal type λ and for ordinal numbers λ and δ let $f_{\lambda,\delta} : C_\lambda \to C_\delta$ be a homomorphism satisfying

$$f_{\lambda,\delta}(a) = \begin{cases} 1 & : a > 0; \\ 0 & : a = 0. \end{cases}$$

Example 5.3. There is not lifting of the category $\mathbf{C_f}$ displayed in Figure 6 with respect to Id_c.

Proof. Assume the contrary. Then, since $f_{3,2}$ separates zero, $\Psi(f_{3,2})$ is one-to-one. It follows that

$$\Psi(f_{3,2} \circ f_{3,3}) = \Psi(f_{3,2}) \circ \Psi(f_{3,3}) \neq \Psi(f_{3,2}) \circ \Psi(\mathbf{1}_{C_3}) = \Psi(f_{3,2} \circ \mathbf{1}_{C_3}) = \Psi(f_{3,2}),$$

while $f_{3,2} \circ f_{3,3} = f_{3,2}$, which is a contradiction. □

$$\begin{tikzcd} C_3 \arrow[r, bend left, "f_{3,3}"] \arrow[r, bend right, "\mathbf{1}_{C_3}"'] & C_3 \arrow[r, "f_{3,2}"] & C_2 \end{tikzcd}$$

Figure 6

Theorem 5.4. *Every diagram of $\boldsymbol{L_d}$ has a lifting with respect to* Id_c.

Proof. Let I be a partially ordered set and $\mathcal{D} : I \to \mathbf{L_d}$ a diagram of $\mathbf{L_d}$. Again, by Corollary 4.3, it suffice to find a lifting $\mathcal{E}$ of $\mathcal{D}$ with respect to Θ. Let $\{M_i \mid i \in I\}$ be a collection of infinite pairwise disjoint sets. For every $i \in I$ set

$$X_i = \bigcup_{j \leq i \text{ in } I} \mathcal{D}(j) \times M_j,$$

and let $\mathcal{E}(i)$ be the map sending $(a, m) \in \mathcal{D}(j) \times M_j$ to $\mathcal{D}(j \to i)(a)$ (observe that the map is a projection since it includes the projection $\mathcal{D}(i) \times M_i \to \mathcal{D}(i)$). Finally, to a homomorphism morphism $\mathcal{D}(i \to j)$ assign a morphism

$$\mathcal{E}(i \to j) : \quad \begin{tikzcd} X_i \arrow[r, "\subseteq"] \arrow[d, "\mathcal{E}(i)"'] & X_j \arrow[d, "\mathcal{E}(j)"] \\ \mathcal{D}(i) \arrow[r, "\mathcal{D}(i \to j)"'] & \mathcal{D}(j) \end{tikzcd}$$

in $\mathbf{D_\infty}$. □

The last example represents a subcategory $\mathbf{C}_\vartriangle$ of $\mathbf{L_d}$ corresponding to a partially ordered class (see Figure 7) which cannot be lifted with respect to Id_c.

$$\begin{tikzcd} & & C_2 & & \\ C_2 \arrow[urr] & C_3 \arrow[ur] & \cdots & C_\lambda \arrow[ul] & \cdots \end{tikzcd}$$

Figure 7

Example 5.5. Let On denote the class of all ordinal numbers and denote by $\mathbf{C}_\vartriangle$ a subcategory of $\mathbf{L_d}$ whose objects are lattices $\{C_\lambda \mid 2 \leq \lambda \in \mathrm{On}\}$ and whose morphisms are $\{f_{\lambda,2} \mid \lambda \in \mathrm{On}\}$ and identities. The category $\mathbf{C}_\vartriangle$ has no lifting with respect to Id_c.

Proof. Assume the contrary. Let λ be an ordinal number whose cardinality is bigger than $|\Psi(C_2)|$. Since $f_{\lambda,2}$ separates zero, $\Psi(f) : \Psi(C_\lambda) \to \Psi(C_2)$ is an embedding. But $|\Psi(C_\lambda)| \geq |\lambda| > |\Psi(C_2)|$, which is a contradiction. □

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