LECTURE 1
Relations on a set

PAVEL RŮŽIČKA

Abstract. We define the Cartesian products and the n^{th} Cartesian powers of sets. An n-ary relation on a set is a subset of its n^{th} Cartesian power. We study the most common properties of binary relations as reflexivity, transitivity and various kinds of symmetries and anti-symmetries. Via these properties we define equivalences, partial orders and pre-orders. Finally we describe the connection between equivalences and partitions of a given set.

1.1. **Cartesian product and relations.** A *Cartesian product* $M_1 \times \cdots \times M_n$ of sets $M_1, \dots, M_n$ is the set of all n-tuples $\langle m_1, \dots, m_n \rangle$ satisfying $m_i \in M_i$, for all $i = \{1, 2, \dots, n\}$. The Cartesian product of n-copies of a single set M is called an *n^{th}-Cartesian power*. We denote the *n^{th}-Cartesian power* of M by M^n. In particular, $M^1 = M$ and M^0 is the one-element set $\{\emptyset\}$.

An *n-ary relation* on a set M is a subset of M^n. Thus unary relations correspond to subsets of M, binary relations to subsets of $M^2 = M \times M$, etc.

1.2. **Binary relations.** As defined above, a *binary relation* on a set M is a subset of the Cartesian power $M^2 = M \times M$. Given such a relation, say $\mathrm{R} \subset M \times M$, we will usually use the notation $a \,\mathrm{R}\, b$ for $\langle a, b \rangle \in \mathrm{R}$, $a, b \in M$.

Let us list the some important properties of binary relations. By means of them we define the most common classes of binary relations, namely equivalences, partial orders and quasi-orders.

Definition 1.1. A binary relation R on a set M is said to be

- *reflexive* if $a \,\mathrm{R}\, a$, for all $a \in M$;
- *transitive* if $(a \,\mathrm{R}\, b$ and $b \,\mathrm{R}\, c) \implies a \,\mathrm{R}\, c$ for all $a, b, c \in M$;
- *symmetric* if $a \,\mathrm{R}\, b \implies b \,\mathrm{R}\, a$ for all $a, b \in M$;
- *anti-symmetric* if $(a \,\mathrm{R}\, b$ and $b \,\mathrm{R}\, a) \implies a = b$, for all $a, b \in M$;
- *asymmetric* if $a \,\mathrm{R}\, b \implies \neg(b \,\mathrm{R}\, a)$, for all $a, b \in M$.

Now we are ready to define the above mentioned important classes of binary relations.

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Definition 1.2. An *equivalence* on a set M is a binary relation on M that is reflexive, transitive and symmetric. A *partial order* on M is a reflexive, transitive, anti-symmetric relation on M while a *strict (partial) order* is a transitive and asymmetric relation on M.

Another important class of binary relations is the smallest class containing all equivalences and orders: By definition, a *quasi-order* is a reflexive and transitive binary relation.

1.3. **Equivalences and partitions.** Let E be an equivalence relation on a set M. The *block of* an element $a \in M$ is the set

$$[\,a\,] := \{b \in M \mid a \,\mathrm{E}\, b\}.$$

Before understanding the structure of blocks of an equivalence relation, we define a *partition* of a set M to be a collection P of pairwise disjoint subsets of M such that $\bigcup P = M$.

Lemma 1.3. *Let E be an equivalence on a set M. For ever $a, b \in M$,*

$$[\,a\,] = [\,b\,] \iff [\,a\,] \cap [\,b\,] \neq \emptyset.$$

Proof. It is clear that $[\,a\,] = [\,b\,] \implies [\,a\,] \cap [\,b\,] \neq \emptyset$. In order to prove the opposite implication, assume that $[\,a\,] \cap [\,b\,] \neq \emptyset$. Then we can pick $c \in [\,a\,] \cap [\,b\,]$. For every $d \in [\,a\,]$, we have $d \,\mathrm{E}\, a$, $a \,\mathrm{E}\, c$, and $c \,\mathrm{E}\, b$, due to symmetry. Applying transitivity of E, we conclude that $d \,\mathrm{E}\, b$, which says that $d \in [\,b\,]$. Thus $[\,a\,] \subseteq [\,b\,]$. The opposite inclusion is proved similarly. □

It readily follows from Lemma **??** that the blocks of an equivalence relation on a set M form a partition of M. Indeed, it follows that the blocks are pairwise disjoint and as $a \in [\,a\,]$ due to the reflexivity, their union is the entire M. Such a partition will be called *the partition induced by* the equivalence E. On the other hand, a partition P of a set M gives rise to a relation, say E, defined by $a \,\mathrm{E}\, b$ if and only if a and b belong to the same block of P. It is straightforward to verify that E is reflexive, transitive and symmetric. Moreover the partition P consists of the blocks of E. The outcome of this discussion shall be the observation that equivalence relations on a set M correspond to partitions of M.

1.4. **Orders and quasi-orders.** First observe that every partial order on a set M correspond to a unique strict order on M. Indeed, given an order R on a set M, the binary relation

$$\mathrm{S} := \{\langle a, b\rangle \mid a \,\mathrm{R}\, b \text{ and } \neg(b \,\mathrm{R}\, a)\}$$

is a strict order. Conversely, given a strict order S on the set M, the relation R defined by

$$a \,\mathrm{R}\, b \iff a \,\mathrm{S}\, b \text{ or } b \,\mathrm{S}\, a$$

is the corresponding partial order.

Let us show that a quasi-order on a set M *decomposes* into an equivalence relation on M and an order relation on the corresponding partition. Let Q be a quasi-order on M. We denote by E the binary relation defined by

$$a \,\mathrm{E}\, b \iff a \,\mathrm{Q}\, b \text{ and } b \,\mathrm{Q}\, a.$$

Lemma 1.4. *The relation* E *is an equivalence on* M.

Proof. Since Q is reflexive (by the definition), E is reflexive as well. Suppose that $a \,\mathrm{E}\, b$ and $b \,\mathrm{E}\, c$ for some $a, b, c \in M$. Then $a \,\mathrm{Q}\, b$ and $b \,\mathrm{Q}\, c$, whence $a \,\mathrm{Q}\, c$, due to the transitivity of Q. The symmetry of Q implies that $b \,\mathrm{Q}\, a$ and $c \,\mathrm{Q}\, b$, and so $c \,\mathrm{Q}\, a$. Since both $a \,\mathrm{Q}\, c$ and $c \,\mathrm{Q}\, a$, we conclude that $a \,\mathrm{E}\, c$. This proves that E is transitive. Symmetry of E is seen readily from its definition. These guarantee that E is an equivalence on M. □

Let P_{E} denote the partition of the set M induced by the equivalence relation E.

Lemma 1.5. *Let* $a \,\mathrm{E}\, a'$ *and* $b \,\mathrm{E}\, b'$ *for some* $a, a', b, b' \in M$. *Then* $a \,\mathrm{Q}\, b$ *if and only if* $a' \,\mathrm{Q}\, b'$.

Proof. Suppose that $a \,\mathrm{Q}\, b$. From $a \,\mathrm{E}\, a'$ we have that $a' \,\mathrm{Q}\, a$ and from $b \,\mathrm{E}\, b'$ we infer that $b \,\mathrm{Q}\, b'$. The transitivity of Q implies that $a' \,\mathrm{Q}\, b'$. The opposite implication is proven similarly. □

Lemma **??** allow us to define a relation R on P by $[\,a\,]\,\mathrm{R}[\,b\,]$ iff $a \,\mathrm{Q}\, b$, for all $a, b \in M$.

Lemma 1.6. *The relation* R *on* P *is reflexive, transitive and anti-symmetric, that is, it is a partial order on* P.

Proof. The reflexivity and the transitivity of R follows readily from the reflexivity and the transitivity of Q. In order to prove that R is anti-symmetric, suppose that, for some $a, b \in M$, $[\,a\,]\,\mathrm{R}[\,b\,]$ and $[\,b\,]\,\mathrm{R}[\,a\,]$. It follows from the definition of R that $a \,\mathrm{Q}\, b$ and $b \,\mathrm{Q}\, a$, which means that $a \,\mathrm{E}\, b$. Therefore $[\,a\,] = [\,b\,]$. This proves that the relation R is anti-symmetric. □

Exercises

We define the *diagonal relation*, the *transpose* of a binary relation, and the*composition* of binary relations as follows:

- The *diagonal relation* (on M) is the relation

$$\Delta := \{\langle a, a\rangle \mid a \in M\},$$

- The *transpose* of a relation R on M is defined as

$$R^T := \{\langle b, a\rangle \mid \langle a, b\rangle \in \mathrm{R}\},$$

- The *composition of relations* R and S on M is the relation

$$\mathrm{R} \circ \mathrm{S} := \{\langle a, c\rangle \mid (\exists b \in M)(a \,\mathrm{R}\, b \text{ and } b \,\mathrm{S}\, c\}.$$

Exercise 1.1. *Prove that given binary relations* R, S *and* T *on a set* M*, the following holds true:*

(i) $(\mathrm{R} \circ \mathrm{S}) \circ \mathrm{T} = \mathrm{R} \circ (\mathrm{S} \circ \mathrm{T})$;
(ii) $\mathrm{R} \circ \mathrm{S}^T = \mathrm{S}^T \circ \mathrm{R}^T$;
(iii) $\mathrm{R}^T \subseteq \mathrm{R} \iff \mathrm{R} \subseteq \mathrm{R}^T \iff \mathrm{R} = \mathrm{R}^T$.

Exercise 1.2. *Prove that a binary relation* R *on set* M *is*

(i) *reflexive if and only if* $\Delta \subseteq \mathrm{R}$,
(ii) *transitive if and only if* $\mathrm{R} \circ \mathrm{R} \subseteq \mathrm{R}$,
(iii) *symmetric if and only if* $\mathrm{R} = \mathrm{R}^T$,
(iv) *anti-symmetric if and only if* $\mathrm{R} \cap \mathrm{R}^T \subseteq \Delta$,
(v) *asymmetric if and only if* $\mathrm{R} \cap \mathrm{R}^T = \emptyset$.

Exercise 1.3. *Prove that a binary relation* R *on set* M *is*

(i) *a quasi-order if and only if* $\Delta \subseteq \mathrm{R} = \mathrm{R} \circ \mathrm{R}$;
(ii) *an equivalence if and only if* $\Delta \subseteq \mathrm{R}^T = \mathrm{R} \circ \mathrm{R}$;
(iii) *a partial order if and only if* $\mathrm{R} \circ \mathrm{R} \subseteq \mathrm{R}$ *and* $\mathrm{R} \cap \mathrm{R}^T = \Delta$;
(iv) *a strict order if and only if* $\mathrm{R} \circ \mathrm{R} \subseteq \mathrm{R}$ *and* $\mathrm{R} \cap \mathrm{R}^T = \emptyset$.

Exercise 1.4. *A* total order *on a set* M *is a partial order on* M *such that any two elements of* M *are comparable. Prove that a binary relation* R *on the set* M *is a total order if and only if* $\mathrm{R} \circ \mathrm{R} \subseteq \mathrm{R}$, $\mathrm{R} \cap \mathrm{R}^T = \Delta$, *and* $\mathrm{R} \cup \mathrm{R}^T = M \times M$.

Exercise 1.5. *Let* R *be a binary relation on a set* M*. For each natural number* n *put*

$$\mathrm{R}^{(n)} = \underbrace{\mathrm{R} \circ \cdots \circ \mathrm{R}}_{n\times}.$$

Prove that

$$\bigcup_{n \in \mathbb{N}} \mathrm{R}^{(n)}$$

is the least transitive relation containing R.

Exercise 1.6. *Prove that the composition* $\mathrm{E} \circ \mathrm{F}$ *of equivalence relations* E *and* F *on a set* M *is an equivalence on* M *if and only if* $\mathrm{E} \circ \mathrm{F} = \mathrm{F} \circ \mathrm{E}$.