Introduction to Group Theory (NMAG337)

Exercise sheet 9

29. 11. 2022

(25 days until Christmas)

Exercise 1. Let G be a group, such that every element of $G \setminus \{e\}$ has order 2. Show that G is abelian.

Exercise 2. Let $p \neq 2$ be a prime. Show that in the group of matrices of the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ over $\mathbb{Z}_p$, every non-identity element has order p, but the group is not abelian.

Exercise 3. Let G be a non-abelian group of order 8. Show that $G \cong D_8$ or $G \cong Q$. The result of Exercise 1, Sheet 3 might also help you ($G/Z(G)$ cannot be non-trivial cyclic).

Exercise 4. Use the previous exercise (together with exercises from previous sheets) to classify all groups of order $n \in \{1, 2, \ldots, 15\} \setminus \{12\}$.

Exercise 5. Let $n \in \mathbb{N}$ be such that $gcd(n, \varphi(n)) = 1$ (where φ is the Euler function) and G be a group of order n. Show that $G \cong \mathbb{Z}_n$:

- If G is abelian, use the classification of finitely generated abelian groups to show that $G \cong \mathbb{Z}_n$.
- Suppose G is not abelian. Observe that the condition $gcd(d, \varphi(d)) = 1$ also holds for every d dividing n. Conclude that you can use induction to assume that every non-trivial subgroup/quotient of G is cyclic.
- Show that $Z(G) = \{e\}$.
- Let $H \leq G$ be a maximal proper subgroup, $e \neq h \in H$. Show that the centralizer of h is H. Conclude that every two maximal proper subgroups have trivial intersection.
- Let H be a maximal proper subgroup. Suppose that the normalizer of H is not H, therefore it is the whole G. Show that conjugating H by some element $x \notin H$ is a non-trivial automorphism of H.
- Depending on the prime decomposition of $|H|$, what is $|Aut(H)|$? What orders of elements can appear in G, what in $|Aut(H)|$? Reach a contradiction, conclude that $N_G(H) = H$.
- Depending on $|H|$, what is $|Conj(H)|$? If $Conj(H) = \{H_1, \ldots, H_k\}$, what is $|\bigcup H_i|$? (be careful with the intersection).
- Show that there is another maximal proper subgroup H'. From $|\bigcup H_i \cup \bigcup H'_i|$ reach a contradiction.

Exercise 6. If $gcd(n, \varphi(n)) \neq 1$, find a non-cyclic group of order n. For which n can you find a non-abelian group?