# Introduction to Group Theory (NMAG337)

Exercise sheet 8

22. 11. 2022

This exercise session, we will prove the following theorem:

**Theorem 1. (Classification of finitely generated abelian groups)** Let $G$ be a finitely generated abelian group. Then $G \cong \mathbb{Z}^a \times \prod_{p \text{ prime}, i \in \mathbb{N}} (\mathbb{Z}_{p^i})^{a_{p,i}}$, where the coefficients $a, a_{p,i} \in \mathbb{N} \cup \{0\}$ are uniquely determined by $G$.

**Exercise 1.** Let $G$ be a finite abelian group. Show that $G$ is a direct product of its Sylow subgroups.

**Exercise 2.** Let $G$ be a finite abelian $p$-group. Show by induction on $|G|$ that $G \cong \prod_{i \in \mathbb{N}} (\mathbb{Z}_{p^i})^{a_i}$ for some $a_i \in \mathbb{N} \cup \{0\}$:

- Let $g \in G$ be an element with the highest order, $ord(g) = p^k$. By the induction hypothesis $G/\langle g \rangle \cong \prod_{i \in \mathbb{N}} (\mathbb{Z}_{p^i})^{b_i}$. Fix an isomorphism of this form, let $b = \sum b_i$ and for $j \in \{1, \dots, b\}$, let $G_j$ be the subgroup of $G/\langle g \rangle$ containing the elements with zero in all components except the $j$-th (when considering $G/\langle g \rangle$ as $\mathbb{Z}_{p^{i_1}} \times \dots \mathbb{Z}_{p^{i_b}}$).
- For each $G_j$ fix a generator $h_j$ and show that there exists an element $g_j \in G$, such that $h_j = g_j + \langle g \rangle$ and $ord(g_j) = ord(h_j)$.
- Show that $\langle g_1, \dots, g_b \rangle \cong G/\langle g \rangle$.
- Show that $G \cong \langle g_1, \dots, g_b \rangle \times \langle g \rangle$.

*If you know what an injective module is, you may also consider $G$ as a $\mathbb{Z}_{p^k}$-module and use the Baer's criterion of injectivity to show that $\langle g \rangle$ is an injective $\mathbb{Z}_{p^k}$-module and therefore the short exact sequence $0 \to \langle g \rangle \to G \to G/\langle g \rangle \to 0$ splits.*

**Definition.** Let $G$ be an abelian group. A torsion of $G$ is the subgroup $T(G) := \{g \in G; ord(g) < \infty\}$. We say that $G$ is torsion-free if $T(G) = 0$.

**Exercise 3.** Let $G$ be a finitely generated torsion-free abelian group. Show that $G \cong \mathbb{Z}^b$:

- Suppose $G = \langle g_1, \dots, g_n \rangle$. Show that if $g_i$ are linearly independent (for $a_i \in \mathbb{Z}$ holds $\sum a_i g_i = 0 \iff (\forall i \; a_i = 0)$), then $G \cong \mathbb{Z}^n$.
- Suppose $g_i$ are not linearly independent, let $S_{(g_1,\dots,g_n)} = \{(a_i) \in \mathbb{Z}^n; \sum a_i g_i = 0\}$. For $a = (a_i) \in S_{(g_1,\dots,g_n)}$ define $||a|| = \sum |a_i|$. Take an $n$-element generating set $(h_1, \dots, h_n)$ and $a \in S_{(h_1,\dots,h_n)}$, such that $||a||$ is the smallest possible.
- For $i \neq j \in \{1, \dots, n\}$ and $\lambda \in \mathbb{Z}$, observe that $h' = (h_1, \dots, h_{i-1}, h_i + \lambda h_j, h_{i+1}, \dots, h_n)$ is a generating set with $a' = (a_1, \dots a_{j-1}, a_j - \lambda a_i, a_{j+1}, \dots, a_n) \in S_{h'}$. Show that $a$ must be zero in all components except one.
- Show that $||a|| = 1$ and conclude that there exists an $(n-1)$-element generating set of $G$. Use the induction to show that $G \cong \mathbb{Z}^b$.

**Exercise 4.** Let $G$ be a finitely generated abelian group. Show that $G/T(G)$ is torsion-free and finitely generated, $G \cong T(G) \times (G/T(G))$ (similarly as in the Exercise 2.), $T(G)$ is finitely generated and therefore finite. Conclude the existence part of the theorem.

**Exercise 5.** Show that if $G \cong \prod_{i \in \mathbb{N}} (\mathbb{Z}_{p^i})^{a_i}$, then you can determine $a_i$ from the orders of elements. Show that if $G \cong \mathbb{Z}^a$, then $|Hom(G, \mathbb{Z}_2)| = 2^a$. Use this to prove the uniqueness part of the theorem.