Introduction to Group Theory (NMAG337)

Exercise sheet 7

15. 11. 2022

Exercise 1. Let $H \rtimes_\varphi K$ be a semidirect product, $K' \leq K$, $\varphi' = \varphi|_{K'}$. Show that $H \rtimes_{\varphi'} K' \leq H \rtimes_\varphi K$. Show that this subgroup is normal if and only if $K' \trianglelefteq K$.

Exercise 2. Let $H \rtimes_\varphi K$ be a semidirect product, $H' \leq H$ be a subgroup, such that $\varphi_k(H') = H'$ for all $k \in K$. Show that $H' \rtimes_{\varphi'} K \leq H \rtimes_\varphi K$ (where $\varphi'_k = \varphi_k|_{H'}$). Show that this subgroup is normal if and only if $H' \trianglelefteq H$.

Exercise 3. Let $H \rtimes_\varphi K$ be a semidirect product. Suppose $H' \leq H$, $K' \leq K$ are subgroups, such that the subset $H' \times K' \subseteq H \rtimes K$ is a subgroup. Show that this subgroup is a semidirect product of H' and K'.

Exercise 4. Show that a semidirect product $H \rtimes_\varphi K$ is abelian if and only if H and K are abelian and $\varphi_k = id_H$ for all $k \in K$ (i.e. the product is direct).

Exercise 5. Let $H \rtimes_\varphi K$ be a semidirect product. Show that (as subsets of $H \rtimes K$)

$$(Z(H) \cap Fix(\varphi)) \times (Z(K) \cap Ker(\varphi)) \subseteq Z(H \rtimes_\varphi K) \subseteq H \times Z(K)$$

If you want, you may also use Exercise 1 and 2 to show that sets on the left and right are in fact subgroups of $H \rtimes_\varphi K$. *Hint: Recall that $Z(G)$ is a characteristic subgroup of G.*

Exercise 6. Show that the set $(Z(H) \cap Fix(\varphi))$ doesn't have to be the whole $Z(H \rtimes_\varphi K)$: Let H be a non-abelian group, $h \in H \setminus Z(H)$, $K = \mathbb{Z}_n$, $\varphi_k : g \mapsto h^k g h^{-k}$. Show that $(h, -1) \in Z(H \rtimes_\varphi K)$. (If you don't want to do it in the general case, try $H = S_3$, $h = (1\ 2\ 3)$.)

Exercise 7. Let $H \rtimes_\varphi K$ be a semidirect product of finite groups, p be a prime. Show that there exist a Sylow p-subgroup of the form $H' \rtimes K'$ (in the sense of Exercise 3), where H' is Sylow p-subgroup of H and K' is a Sylow p-subgroup of K:

- Let H_0 be a Sylow p-subgroup of H. Let G_0 be a Sylow p-subgroup of $H \rtimes K$ containing H_0. From the Sylow theorems and the normality of H conclude that for every Sylow p-subgroup G_1, it holds that $G_1 \cap H$ is a Sylow p-subgroup of H.
- Choose a Sylow p-subgroup K' of K, find a Sylow p-subgroup G' of $H \rtimes K$ containing it. Show that G' is of the required form.