Introduction to Group Theory (NMAG337)

Exercise sheet 5

1. 11. 2022

Definition. Let G be a group, $a, b \in G$. The commutator of a and b (denoted $[a, b]$) is $aba^{-1}b^{-1}$. The commutator subgroup $[G, G]$ (sometimes also G' or $G^{(1)}$) is defined as $\langle [a, b]; a, b \in G \rangle$.

Exercise 1. Let G be a group, $H \trianglelefteq G$. Show that G/H is abelian if and only if $[G, G] \leq H$.

Exercise 2. Compute $[G, G]$ for $G \in \{Q, A_4, S_4, D_{2n}\}$. *Hint: The previous exercise might be helpful.*

Exercise 3. Let G be a group of order pq, $p \neq q$ primes. Use the Sylow theorems to show that G is not simple (it has a non-trivial normal subgroup).

Exercise 4. Let $p < q$ be primes, $gcd(p, q-1) = 0$, let G be a group of order pq. Show that $G \cong \mathbb{Z}_{pq}$

Exercise 5. Let G be a finite group and for every prime p let G_p be some p-Sylow subgroup of G. Show that $G = \langle \bigcup G_p \rangle$.

Exercise 6. Find a p-Sylow subgroup of S_n:

- Show that we have an injective homomorphism $S_a \times S_b \to S_{a+b}$.
- Let $n = \sum a_i p^i$. Show that we have an injective homomorphism $\prod (S_{p^i})^{a_i} \to S_n$.
- Show that if $G_{p,i}$ is a $p-$Sylow subgroup of S_{p^i}, then $\prod (G_{p,p^i})^{a_i}$ (when considered as a subgroup of S_n) is a $p-$Sylow subgroup. Conclude that we only need to find $G_{p,i}$.
- Observe that $(G_{p,i})^p$ is almost a p-Sylow subgroup of $S_{p^{i+1}}$. To be more specific, show that $[S_{p^{i+1}} : (G_{p,i})^p] = pq$, $gcd(p, q) = 1$.
- Find a subgroup $G_{p,i+1}$ satisfying $(G_{p,i})^p \leq G_{p,i+1} \leq S_{p^{i+1}}$ and $[G_{p,i+1} : (G_{p,i})^p] = p$.

Exercise 7. Let G be a (possibly infinite group), P its maximal p-subgroup, such that $Conj(P) = \{gPg^{-1}; g \in G\}$ is finite. Show that $|Conj(P)| \equiv 1 \pmod{p}$.

Exercise 8. Let G be a group of order p^2q, $p \neq q$ primes. Show that G is not simple. *Hint: Use the Sylow theorems to find bounds between p and q. You may want to do the case $|G| = 12$ separately.*