Introduction to Group Theory (NMAG337)

Exercise sheet 4

25. 10. 2022

Definition 1. Let G be a group, $H \leq G$. We say that H is:

- a characteristic subgroup of G if $\varphi(H) = H$ for any $\varphi \in Aut(G)$
- a fully characteristic subgroup if $\varphi(H) \subseteq H$ for any $\varphi \in End(G)$

Exercise 1. It is easy to see that (fully characteristic) $\Longrightarrow$ (characteristic) $\Longrightarrow$ (normal). Show that the opposite implications doesn't hold.

Exercise 2. Let $H \leq G$ be normal / characteristic / fully characteristic and $K \leq H$ be normal / characteristic / fully characteristic subgroup of H. What can you say about $K \leq G$?

Exercise 3. Let G be a group. Is $Z(G)$ characteristic/fully characteristic subgroup of G? What about $[G, G]$?

Exercise 4. Let $G \to S_X$ be a group action and $x \in X$. Show that $Fix(x)$ is a normal subgroup of G if and only if $Fix(x) = Fix(y)$ for any y in the orbit of x.

Exercise 5. Let G, H be finite groups, $gcd(|G|, |H|) = 1$. Show that $Aut(G \times H) = Aut(G) \times Aut(H)$.

Exercise 6. Show that $SU(2)/\{\pm I\} \cong SO(3)$:

- Define an isomorphism of vector spaces between $\mathbb{R}^3$ and the space S of purely imaginary quaternions given by

$$f : \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

- Check that

$$g : w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \mapsto \begin{pmatrix} w + \mathbf{i}x & y + \mathbf{i}z \\ -y + \mathbf{i}z & w - \mathbf{i}x \end{pmatrix}$$

  is an isomorphism between the group of quaternions of unit length (contating not just $\pm 1, \pm\mathbf{i}, \pm\mathbf{j}$ and $\pm\mathbf{k}$, but also elements such as $\sqrt{2} + \sqrt{2}\mathbf{i}$) and $SU(2)$.

- Let u be a quaternion of unit length and v be a purely imaginary quaternion. Show that $\varphi_u(v) := uvu^{-1}$ is a purely imaginary quaternion. Show that $f^{-1} \circ \varphi_u \circ f \in SO(3)$ and that this defines a homomorphism $\psi : SU(2) \to SO(3)$.
- Show $Ker(\psi) = \{\pm I\}$.
- Show that if $(x, y, z)^T$ is a unit vector in $\mathbb{R}^3$ and $\alpha \in \mathbb{R}$, then $\cos(\alpha) + \sin(\alpha)(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ correspond to the rotation by 2α around the axis $(x, y, z)^T$. Conclude that ψ is surjective. *Hint: This may be easier to see if you notice that the multiplication of purely imaginary quaternions behaves like a cross product of vectors in* $\mathbb{R}^3$.

Exercise 7. Let φ be the rotation by $180°$ in D_{4n}. Show that $D_{4n}/\{id, \varphi\} \cong D_{2n}$.