Introduction to Group Theory (NMAG337)

Exercise sheet 12

3. 1. 2023

(-10 days until Christmas)

Exercise 1. Let G be a group with normal subgroups H_1, H_2, such that $G = H_1H_2$ and $H_1 \cap H_2 = \{e\}$. Show that $G \cong H_1 \times H_2$.

Exercise 2. Show that the groups D_8 (symmetries of a square) is not isomorphic to the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ (quaternions are non-commutative extension of complex numbers, their multiplication is defined by $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$).

Exercise 3. Show that a group of order $p_1^{n_1} \cdot \dots \cdot p_k^{n_k}$ (p_i primes) can be generated by $\sum_{i=1}^{k} n_k$ elements.

Exercise 4. Let G be group. Show that $G/Z(G)$ cannot be nontrivial cyclic.

Exercise 5. Show that a group G of order p^2 is isomorphic to $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$.

Exercise 6. Using the Burnside's theorem, count how many graphs on five vertices there are (up to isomorphism).

Exercise 7. Let G, H be finite groups, $gcd(|G|, |H|) = 1$. Show that $Aut(G \times H) = Aut(G) \times Aut(H)$.

Exercise 8. Let $H \leq G$ be normal / characteristic / fully characteristic and $K \leq H$ be normal / characteristic / fully characteristic subgroup of H. What can you say about $K \leq G$?

Exercise 9. Let G be a group. Is $Z(G)$ characteristic/fully characteristic subgroup of G? What about $[G, G] := \langle \{aba^{-1}b^{-1}; a, b \in G\} \rangle$?

Exercise 10. Let G be a group, $H \trianglelefteq G$, $K \leq G$, $G = HK$ and $H \cap K = \{e\}$. Show that $G \cong H \rtimes_\varphi K$ for some $\varphi : K \to Aut(H)$. Use this to show that $D_{2n} \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2$.

Exercise 11. Show that the quaternion group Q cannot be written as a semidirect product of smaller groups.

Exercise 12. Let $p < q$ be primes, $q \not\equiv 1 \pmod{p}$, let G be a group of order pq. Show that $G \cong \mathbb{Z}_{pq}$

Exercise 13. Let $p < q$ be primes, $q \equiv 1 \pmod{p}$. Show that there are only two groups (up to isomorphism) of order pq: $\mathbb{Z}_{pq}$ and a non-trivial semidirect product $\mathbb{Z}_q \rtimes \mathbb{Z}_p$.

Exercise 14. Show that a semidirect product $H \rtimes_\varphi K$ is abelian if and only if H and K are abelian and $\varphi_k = id_H$ for all $k \in K$ (i.e. the product is direct).

Exercise 15. Let $\varphi : \mathbb{Z}_2 \to Aut(S_3)$ be defined by $1 \mapsto (\pi \mapsto (1\ 2)\pi(1\ 2))$. Show that $S_3 \rtimes_\varphi \mathbb{Z}_2 \cong S_3 \times \mathbb{Z}_2$.

Exercise 16. Let G be a non-abelian group of order 8. Show that $G \cong D_8$ or $G \cong Q$.

Exercise 17. Classify all groups of order $n \in \{1, 2, \dots, 15\} \setminus \{12\}$.

Exercise 18. It can be shown that a subgroup of a finitely generated abelian subgroup is finitely generated. Show that this doesn't hold for groups in general: let G be the subgroup of $S_\mathbb{Z}$ generated by $a \mapsto a + 1$ and the transposition $(1\ 2)$, find a subgroup of G, which is not finitely generated.

Exercise 19. Show that $PSL(3, \mathbb{F}_4)$ and A_8 are simple groups of the same order. Show that $PSL(3, \mathbb{F}_4)$ does not contain an element of order 6, and conclude that they are not isomorphic.