Introduction to Group Theory (NMAG337)

Exercise sheet 10

6. 12. 2022

(18 days until Christmas)

Exercise 1. Show that a group of order 30 contains either a normal group of order 3 or a normal group of order 5. Using the Burnside theorem (a group of order $p^n q^m$ for p, q primes is solvable), counclude that there are no non-abelian simple groups with order $|G| < 60 = |A_5|$.

Exercise 2. Show that the groups $GL_2(\mathbb{Z}_2)$, $GL_2(\mathbb{Z}_3)$ are solvable. *Hint: Use the homomorphism* $det : GL_n(\mathbb{F}) \to \mathbb{F}$ *and the fact that if* $H \trianglelefteq G$, *then* (G *solvable*) $\iff$ (H *and* G/H *solvable*).

Definition. Let G be a group and $\pi : G \mapsto S_X, g \mapsto \pi_g$ be an action of G on a finite set X. We say that π is **transitive** if for every $x, y \in X$ there exists a $g \in G$ such that $\pi_g(x) = y$. We say that π is **doubly-transitive** if for every $x, x', y, y' \in X$ such that $x \neq x'$ and $y \neq y'$, there exists a $g \in G$ such that $\pi_g(x) = y$ and $\pi_g(x') = y'$.

Theorem 1. Suppose that $\pi : G \to S_X$ is doubly-transitive, $G = [G, G]$ and there exists an abelian $U \leq G$, which is a normal subgroup of some $St(x)$ and $\langle \bigcup Conj_G(U) \rangle = G$. Then $G/Ker(\pi)$ is a simple group.

Proof. The theorem can be proved with the tools you have, but we would spend most of the exercise session by proving it. If you are interested, you can find the proof here:

https://kconrad.math.uconn.edu/blurbs/grouptheory/PSLnsimple.pdf#theorem.2.10

Exercise 3. Use this theorem to (again) show that A_5 is simple. Observe that we cannot use it for A_n with $n > 5$, at least not with the usual action $A_n \to S_n$.

Exercise 4. Let $PSL_n(\mathbb{F}) := SL_n(\mathbb{F})/\{\omega I; \omega \in \mathbb{F}, \omega^n = 1\}$. Show that if $|\mathbb{F}| > 3$, $PSL_2(\mathbb{F})$ is simple:

- Show that the action of $SL_2(\mathbb{F})$ on the set of 1-dimensional subspaces of $\mathbb{F}^2$ is doubly-trasitive with the kernel $\{\pm I\}$.
- Show that $U = \left\{ \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right); \lambda \in \mathbb{F} \right\}$ is a normal abelian subgroup of $St(span\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right))$.
- Show that $U^T = \left\{ \left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right) \right\}$ is a conjugate of U and that $\langle U, U^T \rangle = SL_2(\mathbb{F})$.
  Hint: $\begin{pmatrix} 1 & 0 \\ (1-a)/a & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a-1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1/a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$
- Compute $\left[\left(\begin{smallmatrix} a & 0 \\ 0 & 1/a \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \right]$, show that for $|\mathbb{F}| > 3$ it holds that $U \leq [SL_2(\mathbb{F}), SL_2(\mathbb{F})]$ and conclude that $[SL_2(\mathbb{F}), SL_2(\mathbb{F})] = SL_2(\mathbb{F})$. Use the Theorem 1. to show that $PSL_2(\mathbb{F})$ is simple for $|\mathbb{F}| > 3$.

This can be analogously shown for $n > 2$, for that we don't need the assumption $|\mathbb{F}| > 3$.

Exercise 5. Only finitely many of $PSL_n(\mathbb{F}_{p^m})$ are isomorphic to some A_k. Show this for $n = 2$. *Hint: Find* $|PSL_2(\mathbb{F}_{p^m})|$ *and its the p-valuation, bound the p-valuation of* $k!$ *by* $\lfloor \frac{k}{p} \rfloor$, *show that* $|PSL_2(\mathbb{F}_{p^m})|$ *is not of the form* $k!$ *for* m *high enough.*