Introduction to Group Theory (NMAG337)

Exercise sheet 1

4. 10. 2022

Exercise 1. Let $(G, \cdot, ^{-1}, e)$ be a group, suppose $H \neq \emptyset$ is a finite subset of G closed under multiplication. Show that H is a subgroup of G (it also contains the neutral element e and it is closed under inverses).

Exercise 2. Show that a group of order 4 is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ (or more generally, that a group of order p^2 is isomorphic to $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$).

Exercise 3. Show that for a field F, $SL(n, F) \trianglelefteq GL(n, F)$ (the group $GL(n, F)$ consists of invertible $n \times n$ matrices over F, $SL(n, F)$ is its subgroup consisting of matrices with unit determinant).

Exercise 4. Let G be a finite group, H its subgroup satisfying $[G : H] = 2$. Show that $H \trianglelefteq G$.

Exercise 5. How many homomorphisms $\mathbb{Z}_n \to \mathbb{Z}_m$ are there? If $m = n$, which of them are isomorphisms?

Exercise 6. Find the subgroups and normal subgroups of D_{12} (the 12-element group of symmetries of a hexagon).

Exercise 7. In the group S_4, consider its subgroup $K = \{id, (12)(34), (13)(24), (14)(23)\}$ and $L = \{id, (12)(34)\}$. Check that $L \trianglelefteq K$ and $K \trianglelefteq S_4$, but $L \not\trianglelefteq S_4$.

Exercise 8. Let G be a cyclic group and $H \leq G$. Show that H is also cyclic.

Exercise 9. If the groups in the previous exercise are finite, show that (for a fixed G) the subgroup H is uniquely determined by its order.

Exercise 10. Let G be a group, $A \trianglelefteq G$ and G/A is abelian. Show that for any $A \leq B \leq G$ also holds $B \trianglelefteq G$.

Exercise 11. Show that the groups D_8 (symmetries of a square) is not isomorphic to the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ (quaternions are non-commutative extension of complex numbers, their multiplication is defined by $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$).

Exercise 12. Let $G \leq GL(n, F)$ be the subgroup consisting of (invertible) upper-triangular matrices, let H be its subgroup consisting of upper-triangular matrices with diagonal $(1, 1, \ldots, 1)$. Show that $G \trianglelefteq H$.

Exercise 13. Let H be the same as in the previous exercise, $F = \mathbb{Z}_2$, $n = 3$. Is H is isomorphic to Q? To D_8?

Exercise 14. Suppose that $A, B \leq G$ and $A \cup B \leq G$. Show that $A \leq B$ or $B \leq A$.

Exercise 15. Let $A, B \leq G$. Show that AB is a subgroup of G if and only if $AB = BA$. Specifically show that if $A \trianglelefteq G$, $AB \leq G$.

Exercise 16. Let p be a prime. Show that a group of order p^n is cyclic if and only if it is abelian and contains only one subgroup of order p.

Exercise 17. All subgroups of an abelian group are normal. Does the converse implication hold - if all subgroups of G are normal, is G abelian?