VI. Products

1. Direct product

Giovanni Frattini
1852–1925

Definition: Let $G_1, \dots, G_n$ be groups. The set

$$G_1 \times \dots \times G_n = \{ \langle g_1, \dots, g_n \rangle \mid g_i \in G_i,\ i = 1, \dots, n \}$$

with multiplication given by

$$\langle g_1, \dots, g_n \rangle \langle h_1, \dots, h_n \rangle = \langle g_1 h_1, \dots, g_n h_n \rangle$$

forms a group. We will call the group $G_1 \times \dots \times G_n$ the direct product of $G_1, \dots, G_n$.

- The unit of $G_1 \times \dots \times G_n$ is the tuple $\langle 1, 1, \dots, 1 \rangle$.
- $\langle g_1, \dots, g_n \rangle^{-1} = \langle g_1^{-1}, \dots, g_n^{-1} \rangle$
- $\begin{aligned} G_i &\longrightarrow G_1 \times \dots \times G_n \\ g &\longmapsto \langle 1, \dots, \underset{\uparrow}{g}, 1, \dots, 1 \rangle \end{aligned}$ is an embedding. In particular $G_i \simeq \langle 1, 1, \dots, 1, G_i, 1, \dots, 1 \rangle$

  ith coordinate
- $\begin{aligned} G_1 \times \dots \times G_n &\longrightarrow G_i \\ \langle g_1, \dots, g_n \rangle &\longmapsto g_i \end{aligned}$ is a "canonical" projection. Its kernel is $\langle G_1, \dots, G_{i-1}, 1, G_{i+1}, \dots, G_n \rangle = G_1 \times \dots \times G_{i-1} \times 1 \times G_{i+1} \times \dots \times G_n$.

Lemma 6.1: Let G be a group, let $H_1, \dots, H_n$ be subgroups of G such that

① $G = \langle \bigcup_{i=1}^{n} H_i \rangle$ (i.e., G is generated by the union $\bigcup_{i=1}^{n} H_i$)

② $H_i \trianglelefteq G$ for all $i = 1, \dots, n$. (all H_i's are normal subgroups of G)

③ $H_i \cap \langle \bigcup_{j \neq i} H_j \rangle = 1$ (H_i has the trivial intersection with the subgroup generated by the rest of the groups.)

for all $i = 1, \dots, n$.

Then $G \simeq G_1 \times \dots \times G_n$.

Proof:

- If $a \in H_i$ and $b \in H_j$ for some $i \neq j$, then $ab = ba$.

  Proof: Since $H_i \trianglelefteq G$, $b a^{-1} b^{-1} \in H_i$. Since $H_j \trianglelefteq G$, $a b a^{-1} \in H_j$. It follows that $a b a^{-1} b^{-1} \in H_i \cap H_j = 1$ (by ③), hence $ab = ba$.

- Every element $g \in G$ is uniquely expressed as a product $g = h_1 \dots h_n$ with $h_i \in H_i$.

  Proof: Since $G = \langle \bigcup_{i=1}^{n} H_i \rangle$, g is a product of elements from $\bigcup_{i=1}^{n} H_i$. Since elements from distinct H_i's commute, we can write g as a product $h_1 \dots h_n$ with $h_i \in H_i$, for all $i = 1, \dots, n$.

Suppose that for some $h_i' \in H_i$, $i = 1, \dots, n$,

$$h_1 \dots h_n = h_1' \dots h_n'$$

Then for all $i = 1, \dots, n$:

$$h_i = h_{i-1}^{-1} \dots h_1^{-1} h_1' \dots h_{i-1}' h_i' h_{i+1}' \dots h_n' h_n^{-1} \dots h_{i+1}^{-1} .$$

Since h_i' commutes with all h_j', $j<i$ and all h_j, $j<i$,

$\cancel{h_i h_i'^{-1}}$

$$h_i'^{-1} h_i = h_{i-1}^{-1} \dots h_1^{-1} h_1' \dots h_{i-1}' h_{i+1}' \dots h_n' h_n^{-1} \dots h_{i+1} \in H_i \cap \left\langle \bigcup_{j \neq i} H_j \right\rangle = 1$$

Therefore $h_i'^{-1} h_i = 1$, and so $h_i = h_i'$. We conclude that the expression $g = h_1 \dots h_n$ is unique. Consequently, the map $\phi : G \longrightarrow H_1 \times H_2 \times \dots \times H_n$ is a bijection.

$$g \longmapsto \langle h_1, h_2, \dots, h_n \rangle$$

It remains to prove that $\overset{h_1 h_2 \dots h_n}{\overset{\|}{g}}$

- $\phi : G \times H_1 \times H_2 \times \dots \times H_n$ is a group homomorphism.

  $g \mapsto \langle h_1, \dots, h_n \rangle$

  Proof: Since elements from different H_i's commute, we get for $g = h_1 \dots h_n$ and $g' = h_1' \dots h_n'$ that $gg' = (h_1 h_1')(h_2 h_2') \dots (h_n h_n')$, therefore

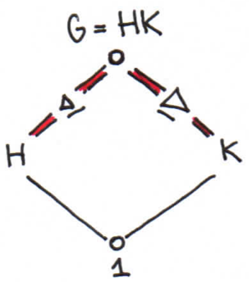
  $$\phi(gg') = (h_1 h_1')(h_2 h_2') \dots (h_n h_n') = h_1 h_2 \dots h_n h_1' h_2' \dots h_n' = \phi(g) . \phi(g') .$$ □

For a pair of normal subgroups we get that

Theorem 6.2: Let H and K be normal subgroups of a group G such that $H \cap K = 1$ and $HK = G$. Then $G \simeq H \times K$.

Proof. If $H \trianglelefteq G$ then $\langle H \cup K \rangle = HK$. We apply previous lemma. □

Remark: The situation can be depicted as follows:



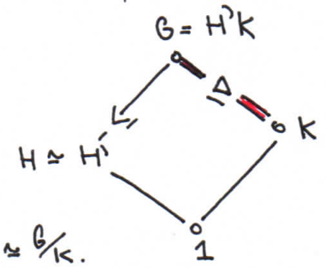
$\simeq H \times K$

2. SEMIDIRECT PRODUCT

Definition: A group G is a <u>semidirect product</u> of K by H if $K \trianglelefteq G$ and there is a subgroup H' of G isomorphic to H such that $H' \cap K = 1$ and $G = H'K$.

Notation: $G = K \rtimes H$ denotes that G is a semidirect product of K by H.

Remark: The semidirect product can be depicted as



(*)

- Observe that $H \simeq G/K$.

Lemma 6.3: Nechť $K \trianglelefteq G$. Potom je ekvivalentní

1) $G = K \rtimes G/K$,
2) there is a subgroup H of G such that every $g \in G$ is uniquely expressed as a product $g = xh$ with $x \in K$ and $h \in H$.

Proof. 1) ⇒ 2) Suppose that $G = K \rtimes G/K$. Then there exists a subgroup H of G such that $H \cap K = 1$ and $G = HK$. Let $g \in G$. Since $G = HK$, $g = xu$ for some $x \in K$, $u \in H$. Suppose that $xu = yv$ for another $y \in K$, $v \in H$. Then $y^{-1}x = vu^{-1} \in H \cap K = 1$. Therefore $x = y$ and $u = v$, and so the expression $g = xu$ is unique.

2) ⇒ 1) Suppose that there is a subgroup H of G such that each $g \in G$ is uniquely $g = xu$ with some $x \in K$, $u \in H$. From the existence of such an expression for every $g \in G$ we get that $G = HK$. From the uniqueness we get that if $g = xu \in H \cap K$, then $x = 1$ and $u = 1$, hence $H \cap K = 1$. □

Remark: The lemma says that the lattice (*) depicts exactly the semidirect product.

Lemma 6.4: Let $K \trianglelefteq G$. Denote by $\phi : G \to G/K$, $g \mapsto Kg$ the canonical ~~projection~~ homomorphism of G onto the quotient G/K.

The following are equivalent:

① $G = K \rtimes G/K$

② There is a homomorphism $\psi : G/K \to G$ s.t. $\phi\psi = 1_{G/K}$.

③ There is a homomorphism $\varepsilon : G \to G$ s.t. $\operatorname{Ker} \varepsilon = K$ and $\varepsilon^2 = \varepsilon$.

Proof: $(1 \Rightarrow 2)$ By the previous lemma there is a subgroup H of G s.t. every $g \in G$ is uniquely $g = xu$ with $x \in K$, $u \in H$.

Let $g_1 = x_1 u_1$, $g_2 = x_2 u_2$ with $x_1, x_2 \in K$, $u_1, u_2 \in H$ be two elements of G. Let $y \in K$, $v \in H$ be such that $g_1 g_2 = yv$. Then

$$g_1 g_2 = x_1 u_1 x_2 u_2 = x_1 (u_1 x_2 u_1^{-1}) u_1 u_2 .$$

Since K is a normal subgroup of G, $u_1 x_2 u_1^{-1} \in K$. From the uniqueness of the expression of each element of G we get that $y = x_1 (u_1 x_2 u_1^{-1}) \in K$ and $v = u_1 u_2 \in H$. It follows that the map

$$\psi : G/K \longrightarrow G$$
$$Kg \longmapsto u \quad \text{, where } g = xu;\ x \in K, u \in H$$

is a group homomorphism. Moreover $\phi\psi(Kg) = \phi(u) = Ku$ and since $g = xu$, $Kg = Kxu = Ku$. Therefore $\phi\psi = 1_{G/K}$.

Observe that $\operatorname{Im} \psi = H$.

$(2 \Rightarrow 3)$ We have the following situation:

$$\begin{tikzcd} G \arrow[r, shift left, "\phi"] & G/K \arrow[l, shift left, "\psi"] \arrow[loop right, "\phi\psi = 1_{G/K}"] \end{tikzcd}$$

Observe that $\phi\psi\phi = 1_{G/K}\phi = \phi$. Put $\varepsilon = \psi\phi : G \to G$.

Then

- $K = \operatorname{Ker} \phi = \operatorname{Ker} \phi\psi\phi = \operatorname{Ker} \phi\varepsilon \le \operatorname{Ker} \varepsilon = \operatorname{Ker} \psi\phi \le \operatorname{Ker} \phi = K$,

and so $\operatorname{Ker} \varepsilon = K$.

- $\varepsilon^2 = \psi\phi\psi\phi = \psi 1_{G/K} \phi = \psi\phi = \varepsilon$.

$(3 \Rightarrow 1)$ Let $\varepsilon : G \to G$ be a homomorphism such that $K = \operatorname{Ker} \varepsilon$ and $\varepsilon^2 = \varepsilon$.

Put $H = \operatorname{Im} \varepsilon$.

- Suppose that $x \in H \cap K$. Since $x \in H = \operatorname{Im} \varepsilon$, there is $g \in G$ with $x = \varepsilon(g)$. Since $x \in K$, $1 = \varepsilon(x) = \varepsilon\varepsilon(g) = \varepsilon(g) = x$. Therefore $H \cap K = 1$.
- Let $g \in G$. Then $\varepsilon(\varepsilon(g)^{-1} g) = \varepsilon(\varepsilon(g^{-1}) g) = \varepsilon\varepsilon(g^{-1})\varepsilon(g) = \varepsilon(g^{-1})\varepsilon(g) = 1$, hence $\varepsilon(g)^{-1} g \in K$. Clearly $g = \varepsilon(g)(\varepsilon(g)^{-1} g) \in HK$. Therefore $G = HK$. □

Example:

- S_n is a semidirect product of A_n by $\mathbb{Z}_2$.
- The dihedral group D_n is a semidirect product of $\mathbb{Z}_n$ by $\mathbb{Z}_2$

$$\begin{tikzcd} & S_n = A_n \cdot \langle (12) \rangle \arrow[dl, dash, "2"'] \arrow[dr, dash] & \\ A_n \arrow[dr, dash] & & \langle (12) \rangle \arrow[dl, dash] \\ & 1 & \end{tikzcd}$$

Lemma 6.5: Let $G = K \rtimes H$ be a semidirect product of K by H. Then there is a homomorphism

$$\begin{array}{rccc} \phi : & H & \longrightarrow & \mathrm{Aut}(K) \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & x & \longmapsto & \phi_x : K \to K \end{array} \qquad (\square)$$

such that

$$\phi_x(a) = x a x^{-1}$$

for all $x \in H$ and $a \in K$.

$$\begin{tikzcd} & G = K \rtimes H \arrow[dl, dash, "\trianglelefteq"'] \arrow[dr, dash] & \\ K \arrow[dr, dash] & & H \arrow[dl, dash] \\ & 1 & \end{tikzcd}$$

fig.

Proof: Since $K \trianglelefteq G$, $xKx^{-1} = K$ for all $x \in H$. Therefore the correspondence $x \mapsto \phi_x$ is a map $H \longrightarrow \mathrm{Aut}(K)$. It is easy to verify that $\phi_1 = 1_K$ and $\phi_y(\phi_x(a)) = \phi_{yx}(a)$ for all $a \in K$ and all $x, y \in H$. Therefore ϕ is a homomorphism from H to $\mathrm{Aut}(K)$. $\square$

Definition: Let K, H be groups. We say that a semidirect product <u>realizes</u> a homomorphism $\phi : H \longrightarrow \mathrm{Aut}(K)$, $x \mapsto \phi_x : K \to K$ if $\phi_x(a) = xax^{-1}$ for all $x \in H$ and all $a \in K$.

Definition: Given groups K and H and a homomorphism $\phi : H \longrightarrow \mathrm{Aut}(K)$, $x \mapsto \phi_x$, we define $G = K \rtimes_\phi H$ to be the set of all ordered pairs $\langle a, x \rangle \in K \times H$ with multiplication given by

$$\langle a, x \rangle \langle b, y \rangle = \langle a\, \phi_x(b), xy \rangle \qquad (*)$$

Theorem 6.6: $G = K \rtimes_\phi H$ is a semidirect product that realizes $\phi : H \longrightarrow \mathrm{Aut}(K)$.

Proof: Firs show that G with the operation given by $(*)$ is a group:

- associativity of the operation:

$$\big(\langle a,x\rangle \langle b,y\rangle\big)\langle c,z\rangle = \langle a\phi_x(b), xy\rangle\langle c,z\rangle = \langle a\phi_x(b)\phi_{xy}(c), xyz\rangle$$

$$\langle a,x\rangle\big(\langle b,y\rangle\langle c,z\rangle\big) = \langle a,x\rangle\langle b\phi_y(c), yz\rangle = \langle a\,\phi_x(b\phi_y(c)), xyz\rangle$$

and

$$\phi_x(b)\,\phi_{xy}(c) = \phi_x(b\,\phi_y(c)).$$

- the identity element:

$$\langle 1,1\rangle\langle a,x\rangle = \langle 1\,\phi_1(a), 1\cdot x\rangle = \langle a,x\rangle = \langle a\underbrace{\phi_x(1)}_{1}, x\cdot 1\rangle = \langle a,x\rangle\langle 1,1\rangle$$

- the inverse element:

$$\langle a,x\rangle\langle \phi_{x^{-1}}(a^{-1}), x^{-1}\rangle = \langle a\,\phi_x(\phi_{x^{-1}}(a^{-1})), xx^{-1}\rangle = \langle 1,1\rangle$$

and

$$\langle \phi_{x^{-1}}(a^{-1}), x^{-1}\rangle\langle a, x\rangle = \langle \phi_{x^{-1}}(a^{-1})\,\phi_{x^{-1}}(a), x^{-1}x\rangle = \langle \phi_{x^{-1}}(a^{-1}a), x^{-1}x\rangle = \langle 1,1\rangle .$$

Therefore G is a group.

- We prove that G is a semidirect product of K by H:

We identify K with the subgroup $\{\langle a,1\rangle \mid a\in K\}$ of G and define a map $\begin{array}{l}\sigma : G \longrightarrow H \\ \quad \langle a,x\rangle \longmapsto x\end{array}$. It is straightforward to see that σ is a homomorphism from G onto H with $\operatorname{Ker}\sigma = K$. Define a map $\begin{array}{l}\rho : H \longrightarrow G \\ \quad x \longmapsto \langle 1,x\rangle\end{array}$ and put $H' = \operatorname{Im}\rho = \{\langle 1,x\rangle \mid x\in H\}$. Then ρ is a homomorphism such that $\sigma\rho = 1_H$, $K\cap H' = 1$ and $G = KH'$ as $\langle a,x\rangle = \langle a,1\rangle\langle 1,x\rangle$ for all $a\in K$ and $x\in H$. We have proved that $G = K\rtimes H$.

Since $(1,x)(a,1)(1,x^{-1}) = \langle \phi_x(a), x\rangle\langle 1, x^{-1}\rangle = \langle \phi_x(a), 1\rangle$, the semidirect product realizes ϕ.

identified with x

identified with a

identified with x^{-1}

identified with $\phi_x(a)$

□

We can see that this construction characterizes semidirect product:

Theorem 6.7: If $G = K\rtimes H$, then there is a homomorphism $\phi : H\to \operatorname{Aut}(K)$ with $G\simeq K\rtimes_\phi H$.

Proof. Let $\phi : H\longrightarrow \operatorname{Aut}(K)$ be given by $\begin{array}{l} x\longmapsto \phi_x : K\longrightarrow K \\ \qquad\qquad a\longmapsto xax^{-1}\end{array}$ (cf. lemma □). Since

$$(ax)(by) = \cancel{(a\phi_x(b))y}\ \ axbx^{-1}xy = a\phi_x(b)\cdot xy$$

the map $\begin{array}{l} G\longrightarrow K\rtimes_\phi H \\ ax \longmapsto \langle ax\rangle\end{array}$ is an isomorphism: It is one-to-one since $ax = by \Rightarrow$ $b^{-1}a = yx^{-1}\in K\cap H = 1$, hence $a=b$ and $x=y$. □

3. WREATH. PRODUCT

Philip Hall 1904 – 1982

- Let G be a group acting on a set X on the left. If the kernel of the action is trivial, we say that X is a <u>G-set</u>.

- Let $\{D_\omega \mid \omega \in \Omega\}$ be a family of isomorphic copies of a group D. Put $K = \prod_{\omega \in \Omega} D_\omega$. We can view elements of K as maps $f : \Omega \longrightarrow D$ with coordinatewise multiplication.

A left action of a group Q on Ω can be transformed to a left action of Q on K via $q \cdot f(\omega) = f(q^{-1}\omega)$ for all $q \in Q$, $f \in K$ and $\omega \in \Omega$. Indeed $(q_2 q_1) \cdot f(\omega) = f((q_2 q_1)^{-1} \cdot \omega) = f(q_1^{-1} \cdot q_2^{-1} \omega) = q_1 f(q_2^{-1}\omega) =$ $= q_2 \cdot (q_1 f)(\omega)$. Therefore $(q_2 q_1) f = q_2 (q_1 f)$ for all $q_1, q_2 \in Q$, $f \in K$.

Observe that the map $\phi_Q : Q \longrightarrow \mathrm{Aut}(K)$ is a homomorphism into the group

$$q \longmapsto [f \mapsto qf]$$

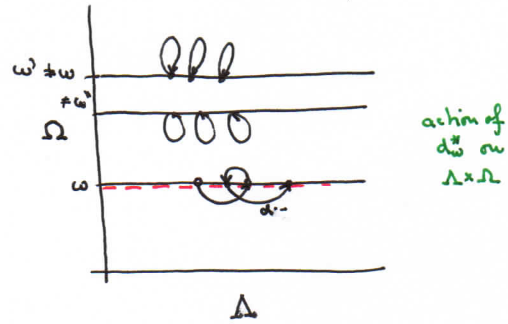
of automorphisms of the group K.

Definition. Let Ω be a Q-set, let $\{D_\omega \mid \omega \in \Omega\}$ be a collection of isomorphic copies of a group D indexed by the set Ω. Put $K = \prod_{\omega \in \Omega} D_\omega$ as above.
The <u>wreath product</u> of D by Q, denoted by $D \wr Q$, is the semidirect product of K by Q that realises the homomorphism ϕ_Q.
The normal subgroup K of $D \wr Q$ is called <u>the base</u> of the wreath product.

Permutation version of the wreath product

- Let Λ be a D-set. For each $d \in D$ and each $\omega \in \Omega$ (Ω is a set as above) define a permutation $d^*_\omega \in S_{\Lambda\times\Omega}$ as follows:

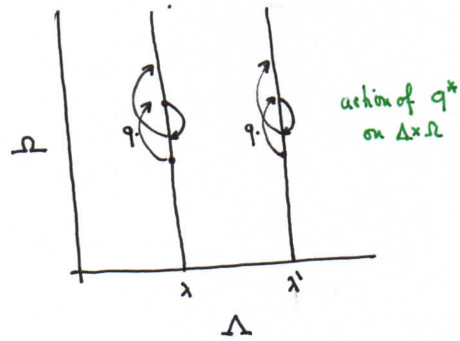
$$d^*_\omega(\lambda, \omega') = \begin{cases} \langle d\cdot\lambda, \omega\rangle & \text{if } \omega' = \omega \\ \langle \lambda, \omega'\rangle & \text{if } \omega' \neq \omega \end{cases}$$



$\triangleleft$ $D \longrightarrow D^*_\omega = \{d^*_\omega \mid d \in D\}$ is an isomorphism (since
$d \longmapsto d^*_\omega$
Λ is a D-set) on to a subgroup of $S_{\Lambda\times\Omega}$.

- Let Ω be a Q-set. For each $q \in Q$ define $q^* \in S_{\Lambda\times\Omega}$ by

$$q^*\cdot\langle\lambda,\omega\rangle = \langle\lambda, q\omega\rangle$$



$\triangleleft$ $Q \longrightarrow Q^* = \{q^* \mid q \in Q\}$ is an isomorphism
$q \longmapsto q^*$

Theorem 6.8: Let Λ be a D-set and Ω be a <u>finite</u> Q-set.
Then the wreath product $D \wr Q$ is isomorphic to the subgroup

$$W = \langle Q^*, D^*_\omega, \omega \in \Omega\rangle$$

of the group $S_{\Lambda\times\Omega}$.

Proof. • Put $K^* = \langle \bigcup_{\omega\in\Omega} D^*_\omega \rangle$; the subgroup of W generated by all the D^*_ω's.

$\triangleleft_1$ If $d, d' \in D$ and $\omega \neq \omega'$ in Ω, then the permutations d_ω and $d'_{\omega'}$ are independent.
Therefore $d_\omega \cdot d'_{\omega'} = d'_{\omega'} \cdot d_\omega$.

1. From $\triangleleft_1$ we infer that $D^*_\omega \trianglelefteq K^*$ for all $\omega \in \Omega$.

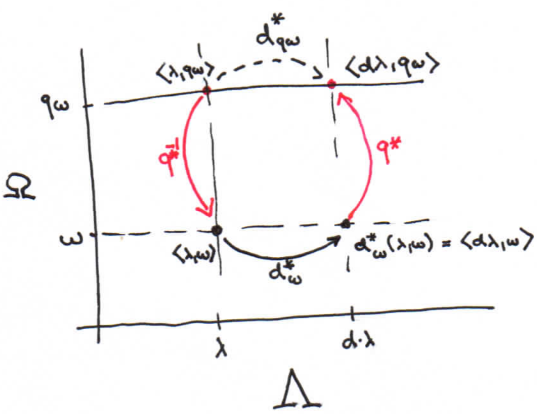
$\triangleleft_2$ Each element of $\langle \bigcup_{\omega'\neq\omega} D^*_{\omega'}\rangle$ fixes the set $\bigcap_{\omega'\neq\omega} \Lambda\times(\Omega\setminus\{\omega'\}) = \Lambda\times\{\omega\}$.

2. From $\triangleleft_2$ we infer that $D_\omega \cap \langle \bigcup_{\omega'\neq\omega} D^*_{\omega'}\rangle = 1$.

- From 1. and 2. we get that $K^* \simeq \prod_{\omega\in\Omega} D^*_\omega$.

$\triangleleft_3$ For all $d \in D$, $q \in Q$ and $\omega \in \Omega$:

$$q^* d^*_\omega q^{*-1} = d^*_{q\omega} \qquad (*)$$



3. From $\triangleleft_3$ we infer that $K^* \trianglelefteq W$.

4. By the definition $W = \langle Q^*, K^* \rangle$.

Since $K^* \trianglelefteq W$, $W = K^* Q^*$

$\sphericalangle_4$ Every element of K^* fixes the second coordinate of each $\langle \lambda, \omega \rangle$,

$\sphericalangle_5$ Every element of Q^* fixes the first coordinate of each $\langle \lambda, \omega \rangle$.

5. From $\sphericalangle_4$ and $\sphericalangle_5$, we infer that $K^* \cap Q^* = 1$.

Since $K^* \trianglelefteq W$, $W = K^* Q^*$ and $K^* \cap Q^* = 1$, we have that

$$W \simeq K^* \rtimes Q^*.$$

- We have supposed that the set ~~B~~ Ω is finite. Let $\Omega = \{\omega_1, \dots, \omega_k\}$.

given an element $f \in K = \prod_{i=1}^{k} D_{\omega_i}$, define $f^* = f(\omega_1)^*_{\omega_1} \cdot f(\omega_2)^*_{\omega_2} \cdots f(\omega_k)^*_{\omega_k}$, which is a product of mutually commuting permutations of $\Lambda \times \Omega$. Observe that

$\sphericalangle_6$
$$K \longrightarrow K^* \quad \text{is an isomorphism between } K \text{ and } K^*.$$
$$f \longmapsto f^*$$

- It follows from the definition that $\phi_{Q_q}(f)^* = f(q^{-1}\omega_1)^*_{\omega_1} \cdots f(q^{-1}\omega_k)^*_{\omega_k} =$

$$= f(\omega_1)^*_{q\omega_1} f(\omega_2)^*_{q\omega_2} \cdots f(\omega_k)^*_{q\omega_k} = (q^* f(\omega_1)^*_{\omega_1} q^{*-1})(q^* f(\omega_2)^*_{\omega_2} q^{*-1}) \cdots (q^* f(\omega_k)^*_{\omega_k} q^{*-1}) =$$

we permute the commuting elements and rewrite the indexes

due to $(*)$

$$= q^* f(\omega_1)^*_{\omega_1} \cdot f(\omega_2)^*_{\omega_2} \cdots f(\omega_k)^*_{\omega_k} q^{*-1} = q^* f^* q^{*-1}$$

We have verified that $\phi_{Q_q}(f)^* = q^* f^* q^{*-1}$.

- We conclude that the map

$$D \wr Q \longrightarrow W$$
$$\langle f, q \rangle \longmapsto f^* q^*$$

is an isomorphism. $\square$

Proposition 6.9: Let Λ be a D-set and Ω be a finite Q-set. Then

1. $D \wr Q$ acts on the set $\Lambda \times \Omega$ transitively (the action is given by identification of $D \wr Q$ with W)

2. For every $\langle \lambda, \omega \rangle \in \Lambda \times \Omega$:
$$St_W(\lambda, \omega) \cong St_D(\lambda) \times (D \wr St_Q(\omega))$$
and
$$[W : St_W(\lambda, \omega)] = [D : St_D(\lambda)][Q : St_Q(\omega)].$$

Proof:

1. Let $\langle \lambda_1, \omega_1 \rangle, \langle \lambda_2, \omega_2 \rangle \in \Lambda \times \Omega$.

Since D acts transitively on Λ, there is $d \in D$ such that $d\lambda_1 = \lambda_2$.

Since Q acts transitively on Ω, there is $q \in Q$ such that $q\omega_1 = \omega_2$.

Then
$$q^* d^*_{\omega_1} \langle \lambda_1, \omega_1 \rangle = q^* \langle d\cdot\lambda_1, \omega_1 \rangle = \langle d\cdot\lambda_1, q\cdot\omega_1 \rangle = \langle \lambda_2, \omega_2 \rangle.$$

2. Number elements of Ω, so that $\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$. Then each element of W is written as a product $d^*_{\omega_1} d^*_{\omega_2} \cdots d^*_{\omega_n} q^*$. For $\langle \lambda, \omega \rangle \in \Lambda \times \Omega$ we have

$$d^*_{\omega_1} d^*_{\omega_2} \cdots d^*_{\omega_n} q^* \cdot \langle \lambda, \omega \rangle = d^*_{\omega_1} d^*_{\omega_2} \cdots d^*_{\omega_n} \langle \lambda, q\omega \rangle = \langle d_{q\omega} \lambda, q\omega \rangle.$$

Therefore $d^*_{\omega_1} d^*_{\omega_2} \cdots d^*_{\omega_n} q^* \in St_W(\lambda, \omega)$ iff $q \in St_Q(\omega)$ and $d_\omega \in St_D(\lambda)$.

Recall that $D^*_\omega = \{ d^*_\omega \mid d \in D \}$. For $d, d' \in D$ and $\omega \neq \omega'$ in Ω we have that

- $d^*_\omega d'^*_{\omega'} = d'^*_{\omega'} d^*_\omega$, because the elements d^*_ω and $d'^*_{\omega'}$ represent independent permutations of Ω. For $d \in D$, $\omega \in \Omega$ and $q \in St_Q(\omega)$ we have that

- $q^* d^*_\omega q^{*-1} = d^*_{q\omega} = d^*_\omega$, hence $q^* d^*_\omega = d^*_\omega q^*$.

Therefore $St_D(\lambda)^*_\omega = \{ d^*_\omega \mid d_\omega \in St_D(\lambda) \}$ is disjoint with $\langle D^*_{\omega'}, \omega' \neq \omega ; St_Q(\omega)^* \rangle$ and centralizes it (i.e, commutes with every element of the subgroup). It follows that

$$St_W\langle \lambda, \omega \rangle = \left\langle St_D(\lambda)^*_\omega, \prod_{\omega' \neq \omega} D^*_{\omega'}, St_Q(\omega)^* \right\rangle \simeq$$
$$\simeq St_D(\lambda)^*_\omega \times \left\langle \prod_{\omega' \neq \omega} D^*_{\omega'}, St_Q(\omega)^* \right\rangle$$
$$\simeq St_D(\lambda)^*_\omega \times \left(D \wr St_Q(\omega) \right)$$

Here $St_Q(\omega)$ acts on the set $\Omega \setminus \{\omega\}$.

- We can count the elements:

$$|St_W\langle \lambda, \omega \rangle| = |St_D(\lambda)^*_\omega| \cdot |D \wr St_Q(\omega)| = |St_D(\lambda)| \cdot |D|^{|\Omega|-1} \cdot |St_Q(\omega)|$$

Then for the indexes of subgroups we have that

$$\cancel{[W : St_W\langle \lambda, w \rangle] = |D|^{|\Omega|} |Q| / |St_D(\lambda)||St_Q(\omega)||D|^{|\Omega|-1} = \frac{|Q|}{|St_Q(\omega)|} \cdot \frac{|D|}{|St_D(\lambda)|}} = [Q : St_D(\lambda)][D : St_D(\lambda)].$$

$$[W : St_W\langle \lambda, w \rangle] = \frac{|D|^{|\Omega|} \cdot |Q|}{|St_D(\lambda)||D|^{|\Omega|-1} \cdot |St_Q(\omega)|} = [D : St_D(\lambda)] \cdot [Q : St_Q(\omega)].$$

$\square$

Theorem 6.10: Let T, D, Q be groups. Let Ω be a finite Q-set, Λ a finite D-set and let Δ be a T-set. Then

$$T \wr (D \wr Q) \cong (T \wr D) \wr Q$$

Proof. We compare permutation versions of both wreath products. They operate on the cartesian product $\Delta \times \Lambda \times \Omega$.

- $T \wr (D \wr Q)$ is a subgroup of $S_{\Delta \times \Lambda \times \Omega}$ generated by permutations:
  - $q^* : \quad \langle \delta, \lambda, \omega \rangle \mapsto \langle \delta, \lambda, q\omega \rangle$
  - $d^*_\omega : \quad \langle \delta, \lambda, \omega' \rangle \mapsto \begin{cases} \langle \delta, d \cdot \lambda, \omega \rangle & \text{if } \omega' = \omega \\ \langle \delta, \lambda, \omega \rangle & \text{otherwise.} \end{cases}$
  - $t^*_{\langle \lambda, \omega \rangle} : \quad \langle \delta, \lambda', \omega' \rangle \mapsto \begin{cases} \langle t\delta, \lambda, \omega \rangle & \text{if } \langle \lambda', \omega' \rangle = \langle \lambda, \omega \rangle \\ \langle t, \lambda', \omega' \rangle & \text{otherwise.} \end{cases}$

- $(T \wr D) \wr Q$ is a subgroup of $S_{\Delta \times \Lambda \times \Omega}$ generated by permutations:
  - $q^* : \quad \langle \delta, \lambda, \omega \rangle \mapsto \langle \delta, \lambda, q\omega \rangle$
  - $d^*_\omega : \quad \langle \delta, \lambda, \omega' \rangle \mapsto \begin{cases} \langle \delta, d\lambda, \omega \rangle & \text{if } \omega' = \omega \\ \langle \delta, \lambda, \omega' \rangle & \text{otherwise} \end{cases}$
  - $t^*_{\lambda\omega} : \quad \langle \delta, \lambda', \omega' \rangle \mapsto \begin{cases} \langle t\delta, \lambda, \omega \rangle & \text{if } \langle \lambda', \omega' \rangle = \langle \lambda, \omega \rangle \\ \langle t', \lambda', \omega' \rangle & \text{if } \langle \lambda', \omega' \rangle \neq \langle \lambda, \omega \rangle. \end{cases}$
  - $T \wr D$ is a subgroup of $S_{\Delta \times \Lambda}$ generated by
    - $d^* : \langle \delta, \lambda \rangle \mapsto \langle \delta, d\lambda \rangle$
    - $t^*_\lambda : \langle \delta, \lambda' \rangle \mapsto \begin{cases} \langle t\delta, \lambda' \rangle & \text{if } \lambda' = \lambda \\ \langle \delta, \lambda' \rangle & \text{otherwis} \end{cases}$

We see that the permutation versions of both the groups coincide. Therefore they are isomorphic. $\square$