# ALGEBRA I (LECTURE NOTES 2017/2018) LECTURE 8 - THE ISOMORPHISM THEOREMS, CONGRUENCES, AND KERNELS

PAVEL RŮŽIČKA

## 8.1. The isomorphisms theorems.

**Theorem 8.1** (The 1st isomorphism theorem)**.** *Let $\varphi\colon \boldsymbol{G} \to \boldsymbol{H}$ be a group homomorphism. The image $\varphi(\boldsymbol{G})$ of $\varphi$ is a subgroup of $\boldsymbol{H}$ and*

$$\varphi(\boldsymbol{G}) \simeq \boldsymbol{G}/\ker\varphi.$$

$$\begin{tikzcd}
\ker\varphi \arrow[d, equal] \arrow[dr, hook, "\subseteq"] & & & \\
\ker\varphi \arrow[r, hook, "\subseteq"'] & \boldsymbol{G} \arrow[r, "\pi_{\boldsymbol{G}/\ker\varphi}"] \arrow[drr, "\varphi"'] & \boldsymbol{G}/\ker\varphi \arrow[d, "\simeq"] & \\
& & \varphi(\boldsymbol{G}) \arrow[r, hook, dotted, "\subseteq"'] & \boldsymbol{H}
\end{tikzcd}$$

Figure 1. The 1st isomorphism theorem

*Proof.* Since $\varphi$ is a group homomorphism, we have that $\varphi(g)\cdot\varphi(h)^{-1} = \varphi(g\cdot h^{-1}) \in \varphi(G)$, for all $g, h \in G$. Therefore $\varphi(\boldsymbol{G})$ is a subgroup of $\boldsymbol{H}$. It follow from Theorem 7.10 that there is an embedding $\psi\colon \boldsymbol{G}/\ker\varphi \to \varphi(\boldsymbol{G})$ such that $\varphi = \psi \circ \pi_{\boldsymbol{G}/\ker\varphi}$. Thus $\psi$ induces an isomorphism between $\boldsymbol{G}/\ker\varphi$ and $\varphi(\boldsymbol{G})$. □

**Lemma 8.2.** *Let $\boldsymbol{N}$, $\boldsymbol{H}$ be subgroups of a group $\boldsymbol{G}$.*

(i) *If $\boldsymbol{N} \trianglelefteq \boldsymbol{G}$ or $\boldsymbol{H} \trianglelefteq \boldsymbol{G}$, then $\boldsymbol{N}\cdot\boldsymbol{H}$ is a subgroup of $\boldsymbol{G}$.*
(ii) *If both $\boldsymbol{N} \trianglelefteq \boldsymbol{G}$ and $\boldsymbol{H} \trianglelefteq \boldsymbol{G}$, then $\boldsymbol{N}\cdot\boldsymbol{H} \trianglelefteq \boldsymbol{G}$.*

*Proof.* (i) Since $\boldsymbol{N} \trianglelefteq \boldsymbol{G}$ or $\boldsymbol{H} \trianglelefteq \boldsymbol{G}$, then $N\cdot H = H\cdot N$. It follows that

$$(N\cdot H)\cdot(N\cdot H) = N\cdot N\cdot H\cdot H = N\cdot H,$$

*Date*: November 20, 2017.

hence $N \cdot H$ is a sub-universe of $\boldsymbol{G}$. For all $n \in N$ and $h \in H$, we have that $(n \cdot h)^{-1} = h^{-1} \cdot n^{-1} \in H \cdot N = N \cdot H$. We conclude that $\boldsymbol{N} \cdot \boldsymbol{H}$ is a subgroup of $\boldsymbol{G}$.

(ii) If both $\boldsymbol{N}$ and $\boldsymbol{H}$ are normal subgroup of $\boldsymbol{G}$, then $g \cdot N \cdot H = N \cdot g \cdot H = N \cdot H \cdot g$, for all $g \in G$. It follows the subgroup $\boldsymbol{N} \cdot \boldsymbol{H}$ is normal due to Lemma 6.6. $\square$

**Remark 8.3.** Observe that $\boldsymbol{N} \cdot \boldsymbol{H}$ is the least subgroup (resp. normal subgroup) of $\boldsymbol{G}$ containing both the groups $\boldsymbol{N}$ and $\boldsymbol{H}$ and if at least one of the subgroups $\boldsymbol{N}$ and $\boldsymbol{H}$ is normal (resp. both the subgroups $\boldsymbol{N}$ and $\boldsymbol{H}$ are normal), then $\boldsymbol{N} \cap \boldsymbol{H}$ is the greatest common subgroup (resp. normal subgroup) of $\boldsymbol{N}$ and $\boldsymbol{H}$.

**Theorem 8.4** (The 2nd isomorphism theorem)**.** *Let $\boldsymbol{G}$ be a group, $\boldsymbol{H}$ a subgroup of $\boldsymbol{G}$, and $\boldsymbol{N}$ a normal subgroups of $\boldsymbol{G}$. Then $\boldsymbol{N} \cdot \boldsymbol{H}$ is a subgroup $\boldsymbol{G}$,*

$$\boldsymbol{N} \cap \boldsymbol{H} \trianglelefteq \boldsymbol{H}, \quad \textit{and} \quad \boldsymbol{H}/(\boldsymbol{N} \cap \boldsymbol{H}) \simeq (\boldsymbol{N} \cdot \boldsymbol{H})/\boldsymbol{N}.$$

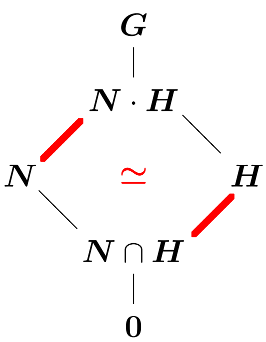


FIGURE 2. The 2nd isomorphism theorem

*Proof.* The product $\boldsymbol{N} \cdot \boldsymbol{H}$ is a subgroup of $\boldsymbol{G}$ due to Lemma 8.2 (i). Since $\boldsymbol{N} \trianglelefteq \boldsymbol{H}$, we have according to Lemma 6.6 that $h \cdot (N \cap H) \cdot h^{-1} \subseteq (h \cdot N \cdot h^{-1}) \cap (h \cdot H \cdot h^{-1}) \subseteq N \cap H$, for all $h \in H$. It follows that $\boldsymbol{N} \cap \boldsymbol{H} \trianglelefteq \boldsymbol{H}$. Note that

$$g \cdot h^{-1} \in N \cap H \text{ if and only if } g \cdot h^{-1} \in N,$$

for all $g, h \in H$. Applying Corollary 5.3, we infer that

$$g \cdot (N \cap H) = h \cdot (N \cap H) \text{ if and only if } g \cdot N = h \cdot N,$$

for all $g, h \in H$. It follows that we can define a group embedding $\boldsymbol{H}/(\boldsymbol{N} \cap \boldsymbol{H}) \to (\boldsymbol{N} \cdot \boldsymbol{H})/\boldsymbol{N}$ by $h \cdot (N \cap H) \mapsto h \cdot N$. It is clear that $(N \cdot H)/N = \{h \cdot N \mid h \in H\}$, hence the map is an isomorphism. $\square$

**Theorem 8.5** (The 3rd isomorphism theorem)**.** *Let $\boldsymbol{G}$ be a group and $\boldsymbol{N}$, $\boldsymbol{K}$ normal subgroups of $\boldsymbol{G}$. If $K \subseteq N$, then*

$$\boldsymbol{N}/\boldsymbol{K} \trianglelefteq \boldsymbol{G}/\boldsymbol{K} \quad \textit{and} \quad \boldsymbol{G}/\boldsymbol{N} \simeq (\boldsymbol{G}/\boldsymbol{K})/(\boldsymbol{N}/\boldsymbol{K}).$$

*Proof.* Since $\boldsymbol{K} \trianglelefteq \boldsymbol{G}$, we have that

$$(g \cdot K) \cdot (n \cdot K) \cdot (g \cdot K)^{-1} = g \cdot K \cdot n \cdot K \cdot g^{-1} \cdot K = g \cdot n \cdot g^{-1} \cdot K$$

for all $n \in N$ and $g \in G$. Since $\boldsymbol{N} \trianglelefteq \boldsymbol{G}$, there is $n' \in N$ such that $g \cdot n \cdot g^{-1} = n'$, hence

$$(g \cdot K) \cdot (n \cdot K) \cdot (g \cdot K)^{-1} = n' \cdot K,$$

for some $n' \in N$. It follows that $\boldsymbol{N}/\boldsymbol{K} \trianglelefteq \boldsymbol{G}/\boldsymbol{K}$.

Let $g, h \in G$. If $g \cdot h^{-1} \cdot K \in N/K$, then $g \cdot h^{-1} \cdot K = n \cdot K$, for some $n \in N$. It follows that $g \cdot h^{-1} \in N$. From this we infer that

$$g \cdot h^{-1} \in N \text{ if and only if } g \cdot h^{-1} \cdot K \in N/K, \text{ for all } g, h \in G,$$

and so

$$g \cdot N = h \cdot N \text{ if and only if } (g \cdot K) \cdot N/K = (h \cdot K) \cdot N/K,$$

for all $g, h \in G$. Therefore we can define a map

$$\boldsymbol{G}/\boldsymbol{N} \to (\boldsymbol{G}/\boldsymbol{K})/(\boldsymbol{N}/\boldsymbol{K})$$

by $g \cdot N \mapsto (g \cdot K) \cdot (N/K)$. It is straightforward to verify that the map is a group isomorphism. $\square$

8.2. **Congruences of algebras.** Let $\boldsymbol{A}$ be an algebra of a given signature $\mathcal{I} = \langle I_0, I_1, \ldots \rangle$ (cf. Subsection 2.3).

**Definition 8.6.** A *congruence* of the algebra $\boldsymbol{A}$ is an equivalence relation $\theta$ on the set $A$ satisfying

$$\forall\, i = 1, \ldots, k\colon a_i \equiv_\theta b_i \implies f_i^k(a_1, \ldots, a_k) \equiv_\theta f_i^k(b_1, \ldots, b_k), \tag{8.1}$$

for all $k \in \mathbb{N}_0$, $i \in I_k$, and $a_1, \ldots, a_k, b_1, \ldots, b_k \in A$.

We denote by $[a]_\theta$ the $\theta$-block of $a \in A$, that is,

$$[a]_\theta := \{b \in A \mid a \equiv_\theta b\},$$

and we set

$$A/\theta := \{[a]_\theta \mid a \in A\}.$$

It follows from (8.1) that we can define

$$f_i^k([a_1]_\theta, \ldots, [a_k]_\theta) := [f_i^k(a_1, \ldots, a_k)]_\theta,$$

for all $k \in \mathbb{N}_0$, $i \in I_k$, and $a_1, \ldots, a_k, b_1, \ldots, b_k \in A$ and so make the set $A/\theta$ an algebra $\boldsymbol{A}/\theta$ of the signature $\mathcal{I}$. The algebra $\boldsymbol{A}/\theta$ is called a *factor algebra* of $\boldsymbol{A}$ by the congruence $\theta$.

**Definition 8.7.** Let $\mathcal{I} = \langle I_0, I_1, \dots \rangle$ be a signature, and $\boldsymbol{A}$, $\boldsymbol{B}$ algebras of the signature $\mathcal{I}$. A map $\varphi\colon A \to B$ is a *homomorphism* from the algebra $\boldsymbol{A}$ to the algebra $\boldsymbol{B}$ provided that

$$\varphi(f_i^k(a_1, \dots, a_k)) = f_i^k(\varphi(a_1), \dots, \varphi(a_k)), \tag{8.2}$$

for all $k \in \mathbb{N}_0$, $i \in I_k$, and $a_1, \dots, a_k \in A$. That is *a homomorphism is a map preserving all operations.*

**Lemma 8.8.** *Let $\boldsymbol{A}$ be an algebra of a signature $\mathcal{I} = \langle I_0, I_1, \dots \rangle$, let $\theta$ be a congruence of $\boldsymbol{A}$. The map $\pi_{\boldsymbol{A}/\theta}\colon A \to A/\theta$ given by the correspondence $a \mapsto [a]_\theta$ induces a homomorphism $\pi_{\boldsymbol{A}/\theta}\colon \boldsymbol{A} \to \boldsymbol{A}/\theta$. It is called the canonical homomorphism onto the factor algebra $\boldsymbol{A}/\theta$.*

*Proof.* Let $k \in \mathbb{N}_0$, $i \in I_k$, and $a_1, \dots, a_k, b_1, \dots, b_k \in A$. It follows from (8.1) that

$$\begin{aligned} \pi_{\boldsymbol{A}/\theta}(f_i^k(a_1, \dots, a_k)) &= [f_i^k(a_1, \dots, a_k)]_\theta = f_i^k([a_1]_\theta, \dots, [a_k]_\theta) \\ &= f_i^k(\pi_{\boldsymbol{A}/\theta}(a_1), \dots, \pi_{\boldsymbol{A}/\theta}(a_k)). \end{aligned}$$

Therefore $\pi_{\boldsymbol{A}/\theta}\colon \boldsymbol{A} \to \boldsymbol{A}/\theta$ is a homomorphism. $\square$

**Definition 8.9.** Let $\boldsymbol{A}$, $\boldsymbol{B}$ be algebras of the same signature and $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$ their homomorphism. The *kernel* of the homomorphism $\varphi$ is an equivalence on $A$ defined by

$$\ker \varphi := \{(a, b) \in A \times A \mid \varphi(a) = \varphi(b)\}.$$

It is straightforward from the definitions that the kernel of a homomorphism $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$ is a congruence of $\boldsymbol{A}$. On the other hand a congruence $\theta$ of an algebra $\boldsymbol{A}$ is the kernel of the canonical homomorphism $\pi_{\boldsymbol{A}/\theta}\colon \boldsymbol{A} \to \boldsymbol{A}/\theta$. That is, *congruences correspond to kernels of homomorphisms.*

The homomorphism theorem and the three isomorphism theorems can be reformulated in the setting of general algebras and their homomorphisms. We will limit ourselves to the first two of them, Theorems 7.10 and 8.1. Their proves closely follow these for groups and we will omit them.

**Theorem 8.10** (The homomorphism theorem)**.** *Let $\boldsymbol{A}$, $\boldsymbol{B}$ be algebras of the same signature and $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$ their homomorphism. Let $\theta$ be congruence of the algebra $A$.There is a homomorphism $\psi\colon \boldsymbol{A}/\theta \to \boldsymbol{B}$ such that $\varphi = \psi \circ \pi_{\boldsymbol{A}/\theta}$ is and only if $\theta \subseteq \ker \varphi$.*

*Moreover $\psi$ is an embedding if and only if $\theta = \ker \varphi$.*

**Theorem 8.11** (The 1st isomorphism theorem)**.** *Let $\boldsymbol{A}$, $\boldsymbol{B}$ be algebras of the same signature and $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$ their homomorphism. The image*

*$\varphi(A)$ is a sub-universe of $\boldsymbol{B}$ and*

$$\varphi(\boldsymbol{A}) \simeq \boldsymbol{A}/\ker\varphi.$$

8.3. **Congruences modulo normal subgroups.** Let $\boldsymbol{N}$ be a subgroup of a group $\boldsymbol{G}$. We proved in Lemma 5.4 that the relation $\equiv_N$ defined by $f \equiv_N g$ if $g^{-1} \cdot f \in N$, for all $f, g \in G$, is an equivalence on $G$ and that the blocks of $\equiv_N$ correspond to left cosets of $\boldsymbol{N}$.

**Lemma 8.12.** *If $\boldsymbol{N}$ is a normal subgroup of a group $\boldsymbol{G}$, then $\equiv_N$ is a congruence of $\boldsymbol{G}$.*

*Proof.* Suppose that $f_1 \equiv_N g_1$ and $f_2 \equiv_N g_2$ for some $f_1, f_2, g_1, g_2 \in G$. By the definition we have that $g_i^{-1} \cdot f_i \in N$, for both $i = 1, 2$. Since $\boldsymbol{N}$ is a normal subgroup of $\boldsymbol{G}$, we infer from Lemma 6.6 that $f_2 \cdot N = N \cdot f_2$. It follows that $(g_1 \cdot g_2)^{-1} \cdot (f_1 \cdot f_2) = g_2^{-1} \cdot g_1^{-1} \cdot f_1 \cdot f_2 \in g_2^{-1} \cdot N \cdot f_2 = g_2^{-1} \cdot f_2 \cdot N \subseteq N$, hence $f_1 \cdot f_2 \equiv_N g_1 \cdot g_2$.

Let $f, g \in G$. Since $\boldsymbol{N}$ is a normal subgroup of $\boldsymbol{G}$, we have according to Corollary 6.9 that if $g^{-1} \cdot f \in N$, then $f \cdot g^{-1} = g^{-1} \cdot f \in N$, hence $(g^{-1})^{-1} \cdot f^{-1} = g \cdot f^{-1} = (f \cdot g^{-1})^{-1} \in N$. Therefore $f \equiv_N g$ implies $f^{-1} \equiv_N g^{-1}$. □

We call $\equiv_N$ the *congruence modulo* $N$ and if $f \equiv_N g$, we say that $f$ *is congruent to* $g$ *modulo* $\boldsymbol{N}$.

**Lemma 8.13.** *Let $\theta$ be a congruence of a group $\boldsymbol{G}$. Then the block of unit $N := [u_{\boldsymbol{G}}]_\theta$ is a normal subgroup of $\boldsymbol{G}$ and $\theta$ equals $\equiv_N$.*

*Proof.* Since $N = [u_{\boldsymbol{G}}]_\theta = \{g \in G \mid \pi_{\boldsymbol{G}/\theta}(g) = u_{\boldsymbol{G}/\theta}\}$, $N$ forms a normal subgroup of $\boldsymbol{G}$ due to Lemma 7.7. For all $f, g \in G$ we have that

$$g^{-1} \cdot f \in N \iff \pi_{\boldsymbol{G}/\theta}(g^{-1} \cdot f) = u_{\boldsymbol{G}/N} \iff \pi_{\boldsymbol{G}/\theta}(f) = \pi_{\boldsymbol{G}/\theta}(g),$$

hence $g \equiv_N f \iff g \equiv_\theta f$. □

It follows that the kernels of group homomorphism defined by Definition 8.9 are determined by their blocks of the unit and the blocks are normal subgroups. Moreover a normal subgroup is a block of unit of a unique group congruence. This justifies Definition 7.6.