ALGEBRA I (LECTURE NOTES 2017/2018) LECTURE 4 - THE ALTERNATING GROUP OF PERMUTATIONS

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4.1. **The signum of a permutation.** Let $\pi \in S_n$ be a permutation. The *signum* of π is defined as $\operatorname{sgn}\pi = (-1)^{n-m}$, where m is the number of blocks of π (both singleton and non-singleton). A permutation π is *even* if $\operatorname{sgn}\pi = 1$ and *odd* if $\operatorname{sgn}\pi = -1$.

Observe that a unit permutation $\upsilon_n \in \boldsymbol{S_n}$ has exactly n blocks. Therefore $\operatorname{sgn}\upsilon_n = (-1)^{n-n} = 1$. Next note that if $\gamma = (a_1, \dots, a_n)$ is a cycle, then $\gamma^{-1} = (a_n, \dots, a_1)$. Finally, if a permutation $\pi = \gamma_1 \cdots \gamma_m$ is a product of independent cycles (including the trivial ones), then $\pi^{-1} = \gamma_m^{-1} \dots \gamma_1^{-1}$. Therefore the permutations π and π^{-1} have the same blocks. We conclude that

Lemma 4.1. *The unit permutation is even, and* $\operatorname{sgn}\pi = \operatorname{sgn}\pi^{-1}$*, for every* $\pi \in \boldsymbol{S_n}$*. In particular, the inverse of an even permutation is even.*

Exercise 4.1. *Let* $\pi \in \boldsymbol{S_n}$*. Prove that* $\operatorname{sgn}\pi = (-1)^k$ *where* k *is the number of blocks of* π *of even size.*

It is customary to call 2-cycles *transpositions*.

Lemma 4.2. *Let* $\pi, \tau \in \boldsymbol{S_n}$*. If* τ *is a transposition, then*

$$\operatorname{sgn}\tau \cdot \pi = -\operatorname{sgn}\pi. \tag{4.1}$$

Proof. Let $B_1, \dots, B_m$ be blocks of π and $\pi = \gamma_1 \cdots \gamma_m$ the decomposition of the permutation π into the product of independent cycles (including the trivial ones that correspond to singleton blocks). By the definition, $\operatorname{sgn}\pi = (-1)^{n-m}$. The proof splits into two cases:

Case 1: $\operatorname{supp}\tau$ *is contained in a block of* π*.* We can without loss of generality assume that $\operatorname{supp}\tau \subseteq B_1$ (since the cycles are independent, and so permutable), $\tau = (a_1, a_i)$ and $\gamma_1 = (a_1, \dots, a_k)$. Then we compute that

$$\tau \cdot \gamma_1 = (a_1, a_i) \cdot (a_1, \dots, a_{i-1}, a_i, \dots, a_k) = (a_1, \dots, a_{i-1}) \cdot (a_i, \dots, a_k),$$

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hence

$$\tau \cdot \pi = \tau \cdot \gamma_1 \cdot \gamma_2 \cdots \gamma_m = (a_1, \dots, a_{i-1})(a_i, \dots, a_k) \cdot \gamma_2 \cdots \gamma_m.$$

It follows that the permutation $\tau \cdot \pi$ is a product of $m+1$ independent cycles (including the trivial ones), and so it has $m+1$ blocks. We conclude that $\operatorname{sgn} \tau \cdot \pi = (-1)^{n-(m+1)} = -\operatorname{sgn} \pi$.

Case 2: $\operatorname{supp} \tau$ *meets two different blocks of* π. By suitably permuting the cycles $\gamma_1 \cdots \gamma_m$, we can assume that $\operatorname{supp} \tau \subseteq B_1 \cup B_2$, $\tau = (a_1, b_1)$, $\gamma_1 = (a_1, \dots, a_k)$, and $\gamma_2 = (b_1, \dots, b_l)$. We compute that

$$\tau \cdot \gamma_1 \cdot \gamma_2 = (a_1, b_1) \cdot (a_1, \dots, a_k) \cdot (b_1, \dots, b_l) = (a_1, \dots, a_k, b_1, \dots, b_l),$$

hence

$$\tau \cdot \pi = \tau \cdot \gamma_1 \cdot \gamma_2 \cdot \gamma_3 \cdots \gamma_m = (a_1, \dots, a_k, b_1, \dots, b_l) \cdot \gamma_3 \cdots \gamma_m.$$

It follows that the permutation $\tau \cdot \pi$ is a product of $m-1$ independent cycles (including the trivial ones), and so it has $m-1$ blocks. We conclude that $\operatorname{sgn} \tau \cdot \pi = (-1)^{n-(m-1)} = -\operatorname{sgn} \pi$. □

Lemma 4.3. *Every permutation on at least two-element set is a product of transpositions.*

Proof. Let n be a positive integer. The identity equals $(1,2) \cdot (1,2)$. Since every permutation is a product of cyclic permutations due to Theorem 3.3, it suffices to prove that every cyclic permutation is a product of transpositions. Its straightforward, for we have that

$$(a_1, \dots, a_k) = (a_1, a_k) \cdot (a_1, a_{k-1}) \cdots (a_1, a_2),$$

for every cycle $(a_1, \dots, a_k)$. □

Exercise 4.2. *Prove that a cycle of the length k is not a product of less than $k-1$ transpositions.*

Lemma 4.4. *If a permutation $\pi = \tau_1 \cdots \tau_k$ is a product of transpositions $\tau_1, \dots, \tau_k$, then $\operatorname{sgn} \pi = (-1)^k$.*

Proof. By induction on k applying Lemma 4.2. □

There are many ways how to write a permutation as a product o transpositions but the signum can be computed from any of them.

Lemma 4.5. *Let π, ρ be permutations on at least two element set. Then*

$$\operatorname{sgn}(\pi \cdot \rho) = \operatorname{sgn} \pi \cdot \operatorname{sgn} \rho.$$

Proof. Permutations π and ρ can be decomposed as products of transpositions, say $\pi = \tau_1 \cdots \tau_m$ and $\rho = \sigma_1 \cdots \sigma_k$, due to Lemma 4.3. Then $\pi \cdot \rho = \tau_1 \cdots \tau_m \cdot \sigma_1 \cdots \sigma_k$, and, by Lemma 4.4,

$$\operatorname{sgn}(\pi \cdot \rho) = (-1)^{m+k} = (-1)^m(-1)^k = \operatorname{sgn}\pi \cdot \operatorname{sgn}\rho.$$

□

Corollary 4.6. *The product of*

- *even permutations is an even permutation,*
- *two odd permutations is an even permutation,*
- *even and odd permutation is an odd permutation.*

Let A_n denote the set of all even permutations on the set $\{1, 2, \ldots, n\}$. It follows from Corollary 4.6 that A_n is a sub-universe of the symmetric group $\boldsymbol{S_n} = (S_n, \cdot)$. From Lemma 4.1 we see that A_n is an underlying set of a subgroup of $\boldsymbol{S_n}$. The subgroup is called the *alternating group of permutations* and is denoted by $\boldsymbol{A_n}$.

4.2. **The 15 puzzle.** The *15 puzzle* is a game invented by invented by Noyes Palmer Chapman, a postmaster in Canastota, NY, around the year 1875. In 1880, the game spread from USA to Canada and Europe and later to Asia and it gained a world-wide popularity.

The puzzle is often credited to Sam Loyd, who falsely claimed its authorship. He is known for offering a \$ 1000 prize to a solver of the *advertising position* (see Figure:1). We will see that the advertising position is unsolvable.

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

FIGURE 1. The advertising position

The game consists of a box containing fifteen numbered squared tiles and one empty square. You can slide neighboring tiles to the empty square, and so change the position. The aim of the game is to transform a given starting position to the final position in which the tiles are numbered gradually from the upper left corner to the lower right corner, where the empty square is (see Figure 2).

A starting position

5	2	7	8
3	13	1	4
15	11	9	6
14	10	12	

$\xrightarrow{?}$

The final position

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

FIGURE 2. The 15 puzzle

We can assign the number 16 to the empty space, and so we can write down each position as a permutation of the set $\{1, 2, \ldots, 16\}$, where the final position corresponds to the unit permutation. For example, the starting position on Figure 2 is written as

$$(1, 7, 3, 5) \cdot (4, 8) \cdot (6, 12, 15, 9, 11, 10, 14, 13).$$

We call a position *standard* if the empty square is in the lower right corner (i.e., position 16). We clam that

Proposition 4.7. *Standard positions corresponding to odd permutations are unsolvable.*

Proof. We will use the *chessboard trick*. Lets call one slide of a tile to the empty square *a move*. Color the box as in Figure 3 making it a small chessboard, and observe that one-move changes the color of the empty square, from black to white and conversely.

Let the starting position corresponds to an odd permutation, say σ. A sequence of moves compose to a pemutation, say π, and the resulting position corresponds to the product $\pi \cdot \sigma$. Since the starting position is standard, and so the empty square is black, the color of the empty space in the resulting position is black if and only if π is even. Since the product of even and odd permutation is odd (due to Corollary 4.6), the empty space in the resulting position is black if and only if the corresponding permutation is odd. However, the final position corresponds to the unit permutation, which is even. It follows that the starting position is unsolvable.

□

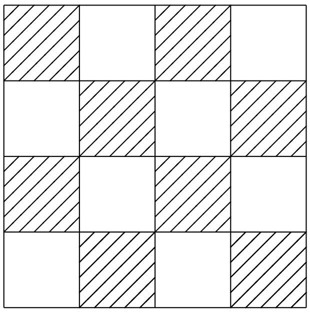

FIGURE 3. The small chessboard

Exercise 4.3. *Decide whether the area on Figure 4 can be covered by domino tiles.* [Hint: Use the chessboard trick.]

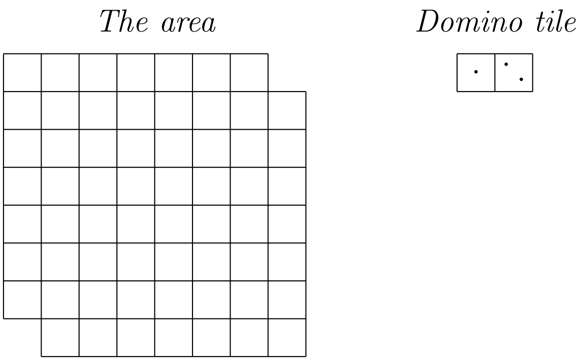

FIGURE 4. Covering by domino tiles