ALGEBRA I (LECTURE NOTES 2017/2018)
LECTURE 1 - RELATIONS ON A SET

PAVEL RŮŽIČKA

1.1. **Cartesian product and relations.** A *cartesian product* of sets $M_1, \dots, M_n$ is a set of all n-tuples $\langle m_1, \dots, m_n \rangle$ such that $m_i \in M_i$, for all $i = \{1, 2, \dots, n\}$. The cartesian product of n-copies of a single set M is called a *cartesian power* of M and is be denoted by M^n. In particular, M^1 is just the set M and $M^0 = \{\emptyset\}$ is an one-element set.

An n-ary *relation* on a set M is a subset of M^n. For instance, unary relations correspond to subsets of M, binary relations are exactly subsets of $M^2 = M \times M$, etc.

1.2. **Binary relations.** As defined above, a *binary relation* on a set M is a subset of the cartesian power $M^2 = M \times M$. Given such a relation $\mathrm{R} \subset M \times M$, we will usually use the notation $a \,\mathrm{R}\, b$ for $\langle a, b \rangle \in \mathrm{R}$, $a, b \in M$. We define the *diagonal relation*, the *transpose* of a binary relation, and the*composition* of binary relations as follows:

- The *diagonal relation* (on M) is the relation
$$\Delta := \{\langle a, a \rangle \mid a \in M\},$$
- The *transpose* of a relation R on M is defined as
$$R^T := \{\langle b, a \rangle \mid \langle a, b \rangle \in \mathrm{R}\},$$
- The *composition of relations* R and S on M is the relation
$$\mathrm{R} \circ \mathrm{S} := \{\langle a, c \rangle \mid (\exists b \in M)(a \,\mathrm{R}\, b \text{ and } b \,\mathrm{S}\, c\}.$$

Exercise 1.1. *Prove that given binary relations* R, S *and* T *on a set* M*, the following holds true:*

(i) $(\mathrm{R} \circ \mathrm{S}) \circ \mathrm{T} = \mathrm{R} \circ (\mathrm{S} \circ \mathrm{T})$*;*
(ii) $\mathrm{R} \circ \mathrm{S}^T = \mathrm{S}^T \circ \mathrm{R}^T$*;*
(iii) $\mathrm{R}^T \subseteq \mathrm{R} \iff \mathrm{R} \subseteq \mathrm{R}^T \iff \mathrm{R} = \mathrm{R}^T$.

There are some important properties of binary relations, which enable us to define practically useful classes of binary relations as partial orders and equivalences. Let us define some of them: A binary relation R on a set M is said to be

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- *reflexive* if $a \,\mathrm{R}\, a$, for all $a \in M$;
- *transitive* if $(a \,\mathrm{R}\, b$ and $b \,\mathrm{R}\, c) \implies a \,\mathrm{R}\, c$ for all $a, b, c \in M$;
- *symmetric* if $a \,\mathrm{R}\, b \implies b \,\mathrm{R}\, a$ for all $a, b \in M$;
- *anti-symmetric* if $(a \,\mathrm{R}\, b$ and $b \,\mathrm{R}\, a) \implies a = b$, for all $a, b \in M$;
- *asymmetric* if $a \,\mathrm{R}\, b \implies \neg(b \,\mathrm{R}\, a)$, for all $a, b \in M$.

Exercise 1.2. *Prove that a binary relation* R *on set* M *is*

(i) *reflexive if and only if* $\Delta \subseteq \mathrm{R}$*,*
(ii) *transitive if and only if* $\mathrm{R} \circ \mathrm{R} \subseteq \mathrm{R}$*,*
(iii) *symmetric if and only if* $\mathrm{R} = \mathrm{R}^T$*,*
(iv) *anti-symmetric if and only if* $\mathrm{R} \cap \mathrm{R}^T \subseteq \Delta$*,*
(v) *asymmetric if and only if* $\mathrm{R} \cap \mathrm{R}^T = \emptyset$*.*

Now we are ready to define the most important classes of binary relations. An *equivalence* on a set M is a binary relation on M that is reflexive, transitive and symmetric. A *partial order* on M is a reflexive, transitive, anti-symmetric relation on M while a *strict (partial) order* is a transitive and asymmetric relation on M.

Another important class of binary relations is the smallest class containing all equivalences and orders: By definition, a *quasi-order* is a reflexive and transitive binary relation.

1.3. **Equivalences and partitions.** Let E be an equivalence relation on a set M. Given an element $a \in M$, the *block of* a is the set

$$[a] := \{b \in M \mid a \,\mathrm{E}\, b\}.$$

Before understanding the structure of blocks of an equivalence relation, we define a *partition* of a set M as a collection $\mathcal{P}$ of pairwise disjoint subsets of M such that $\bigcup \mathcal{P} = M$.

Lemma 1.1. *Let* E *be an equivalence on a set* M*. For ever* $a, b \in M$*,*

$$[a] = [b] \iff [a] \cap [b] \neq \emptyset.$$

Proof. It is clear that $[a] = [b] \implies [a] \cap [b] \neq \emptyset$. In order to prove the opposite implication, assume that $[a] \cap [b] \neq \emptyset$. Then we can pick $c \in [a] \cap [b]$. For every $d \in [a]$, we have $d \,\mathrm{E}\, a$, $a \,\mathrm{E}\, c$, and $c \,\mathrm{E}\, b$, due to symmetry. Applying transitivity of E, we conclude that $d \,\mathrm{E}\, b$, which says that $d \in [b]$. Thus $[a] \subseteq [b]$. The opposite inclusion is proved similarly. □

Observe that Lemma 1.1 says that the blocks of an equivalence relation on a set M form a partition of M, indeed, it follows that they are pairwise disjoint and as $a \in [a]$, their union is the entire M. On the other hand, a partition $\mathcal{P}$ of a set M gives rise to a relation, say E, defined by $a \,\mathrm{E}\, b$ if and only if a and b belong to the same block of $\mathcal{P}$. It is straightforward to verify that E is reflexive, transitive and symmetric. The outcome of this discussion shall be the observation that equivalence relations on a set M correspond to partitions of M.

Exercise 1.3. *Prove that the composition* $\mathrm{E} \circ \mathrm{F}$ *of equivalence relations* E *and* F *on a set* M *is an equivalence on* M *if and only if* $\mathrm{E} \circ \mathrm{F} = \mathrm{F} \circ \mathrm{E}$.

1.4. **Orders and quasi-orders.** First observe that every partial order on a set M correspond to a unique strict order on M. In particular, given an order R on a set M, the corresponding strict order is $\mathrm{R} \setminus \Delta$ while a strict order S correspond to a partial order $\mathrm{S} \cup \Delta$.

Let us see that a quasi-order on a set M *decomposes* into an equivalence relation on M and an order relation on the corresponding partition. Let Q be a quasi-order on M. We denote by E the binary relation on M defined by

$$a \,\mathrm{E}\, b \iff a \,\mathrm{Q}\, b \text{ and } b \,\mathrm{Q}\, a,$$

that is, $\mathrm{E} := \mathrm{Q} \cap \mathrm{Q}^T$.

Lemma 1.2. *The relation* E *is an equivalence on* M.

Proof. Since Q is reflexive (by the definition), E is reflexive as well. Suppose that $a \,\mathrm{E}\, b$ and $b \,\mathrm{E}\, c$ for some $a, b, c \in M$. Then $a \,\mathrm{Q}\, b$ and $b \,\mathrm{Q}\, c$, whence $a \,\mathrm{Q}\, c$, due to the transitivity of Q. The symmetry of E implies that $b \,\mathrm{Q}\, a$ and $c \,\mathrm{Q}\, b$, and so $c \,\mathrm{Q}\, a$, due to transitivity of Q. Since both $a \,\mathrm{Q}\, c$ and $c \,\mathrm{Q}\, a$, we conclude that $a \,\mathrm{E}\, c$. This proves that E is transitive. Symmetry of E is seen readily from its definition. These guarantee that E is an equivalence on M. $\square$

Let $\mathcal{P}$ denote the partition oof the set M corresponding to the equivalence relation E.

Lemma 1.3. *Let* $a \,\mathrm{E}\, a'$ *and* $b \,\mathrm{E}\, b'$ *for some* $a, a', b, b' \in M$. *Then* $a \,\mathrm{Q}\, b$ *if and only if* $a' \,\mathrm{Q}\, b'$.

Proof. Suppose that $a \,\mathrm{Q}\, b$. From $a \,\mathrm{E}\, a'$ we have that $a' \,\mathrm{Q}\, a$ and from $b \,\mathrm{E}\, b'$ we get that $b \,\mathrm{Q}\, b'$. The transitivity of Q implies that $a' \,\mathrm{Q}\, b'$. The opposite implication is proven similarly. $\square$

Lemma 1.3 allow us to define a relation R on $\mathcal{P}$ by $[a] \,\mathrm{R}[b]$ if $a \,\mathrm{Q}\, b$, for all $a, b \in M$.

Lemma 1.4. *The relation* R *on* $\mathcal{P}$ *is reflexive, transitive and antisymmetric, that is, it is a partial order on* $\mathcal{P}$.

Proof. The reflexivity and the transitivity of R follows readily from the reflexivity and the transitivity of Q. In order to prove that R is antisymmetric, suppose that, for some $a, b \in M$, $[a] \,\mathrm{R}[b]$ and $[b] \,\mathrm{R}[a]$. The definition of R gives that $a \mathrel{\mathrm{Q}} b$ and $b \mathrel{\mathrm{Q}} a$, which means that $a \mathrel{\mathrm{E}} b$, that is, $[a] = [b]$. This shows that R is anti-symmetric. $\square$