4.10 Use for example min-var model

a) $\min \frac{1}{2} w^T \Sigma w$

s.t. $w^T\mu \geq \bar{\mu}$

$w^T \mathbb{1} = 1$

$w \geq 0$

b) $L(w, \alpha_0, \alpha_1, \beta) = \frac{1}{2} w^T \Sigma w + \alpha_0 (\bar{\mu} - w^T\mu) + \alpha_1 (1 - w^T \mathbb{1})$

$- \beta^T w$

KKT:

- stationarity

$\frac{\partial L}{\partial w} = 0 \Rightarrow \Sigma w - \alpha_0 \mu - \alpha_1 \mathbb{1} - \beta = 0$

- primal feasibility

$\frac{\partial L}{\partial \alpha_0} \leq 0 \Rightarrow \bar{\mu} - w^T\mu \leq 0$

$\frac{\partial L}{\partial \alpha_1} = 0 \Rightarrow 1 - w^T \mathbb{1} = 0$

$\frac{\partial L}{\partial \beta} \leq 0 \Rightarrow -w \leq 0$

- dual feasibility

$\alpha_0 \geq 0$

$\beta \geq 0$

- complementary-slackness

$\alpha_0 \cdot (\bar{\mu} - w^T\mu) = 0$

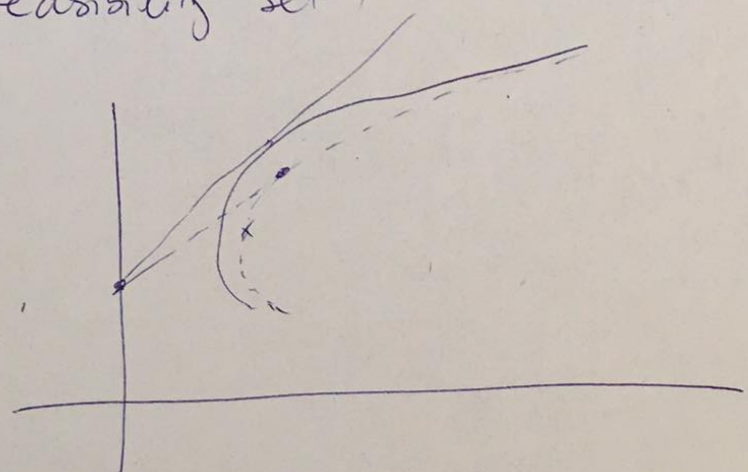
$\beta^T w = 0$

c)

1) use $w^T \mathbb{1} = 1$, plug in for one of the weights and get rid of this inequality constraint, use Lagrange multipliers to calculate the optimal value on a compact set

2) its convex problem (even quadratic programming problem), hence methods used for this class of problems can be employed.
(feasibility set is a set of linear cuts..)

d) mean-variance frontier
→ would be "smaller", subset of the original feasibility set, and not "smooth"



CML → only from risk free asset to the new tangency portfolio, and then prolonged by the eff. frontier

4.11 Basically the same as 4.10

a)

$$\min \frac{1}{2} w^T \Sigma w$$
$$\text{s.t. } w^T \mu \geq \bar{\mu}$$
$$w^T \mathbb{1} = 1$$
$$w \geq 0$$

$$\Rightarrow \min \frac{1}{2} (w_1\ w_2\ w_3) \begin{pmatrix} 0.06 & 0.0009 & 0.0018 \\ 0.0009 & 0.03 & 0.0018 \\ 0.0018 & 0.0018 & 0.15 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

$$\text{s.t. } (w_1\ w_2\ w_3) \begin{pmatrix} 0.14 \\ 0.08 \\ 0.2 \end{pmatrix} \geq 0.15$$
$$w_1 + w_2 + w_3 = 1$$
$$w_1, w_2, w_3 \geq 0$$

b) $L(w_1, w_2, w_3, \alpha_0, \alpha_1, \beta_1, \beta_2, \beta_3) =$

$$= \frac{1}{2} (w_1\ w_2\ w_3) \left(\Sigma \right) \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + \alpha_0 \left(0.15 - (w_1\ w_2\ w_3) \begin{pmatrix} 0.14 \\ 0.08 \\ 0.2 \end{pmatrix} \right)$$
$$+ \alpha_1 (1 - w_1 - w_2 - w_3) - \beta_1 w_1 - \beta_2 w_2 - \beta_3 w_3$$

- stationarity

$$\left(\Sigma \right) \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} - \alpha_0 \begin{pmatrix} 0.14 \\ 0.08 \\ 0.2 \end{pmatrix} - \alpha_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = 0$$

- primal feas.

$$0.15 - (w_1\ w_2\ w_3) \begin{pmatrix} 0.14 \\ 0.08 \\ 0.2 \end{pmatrix} \leq 0$$
$$1 - w_1 - w_2 - w_3 = 0$$
$$-w_1 \leq 0, \quad -w_2 \leq 0, \quad -w_3 \leq 0$$

- dual feas.

$$\alpha_0 \geq 0, \quad \beta_3 \geq 0, \quad \beta_1 \geq 0, \quad \beta_2 \geq 0$$

- complementary slackness

$$\alpha_0 \cdot (\bar{\mu} - w^T \mu) = 0$$
$$\beta_1 w_1 + \beta_2 w_2 + \beta_3 w_3 = 0$$

c) solution via any software (quadr. programming)

$$w^* = \begin{pmatrix} 0.479 \\ 0.177 \\ 0.344 \end{pmatrix}$$

4.12 Risk = Variance

a) first, we need to calculate μ and Σ

$$\mu = \tfrac{1}{4}s_1 + \tfrac{1}{2}s_2 + \tfrac{1}{4}s_3 = \frac{1}{4}\begin{pmatrix}24\\28\\4\end{pmatrix} + \frac{1}{2}\begin{pmatrix}12\\12\\8\end{pmatrix} + \frac{1}{4}\begin{pmatrix}0\\4\\28\end{pmatrix} = \begin{pmatrix}12\\14\\12\end{pmatrix}$$

$$\Sigma = \sum_{i=1}^{3} p_i (s_i - \mu)(s_i - \mu)^T$$

$$= \frac{1}{4}\begin{pmatrix}12\\14\\-8\end{pmatrix}(12\ 14\ -8) + \frac{1}{2}\begin{pmatrix}0\\-2\\-4\end{pmatrix}(0\ -2\ -4) + \frac{1}{4}\begin{pmatrix}-12\\-18\\16\end{pmatrix}(-12\ -18\ 16)$$

$$= \begin{pmatrix}72 & 72 & -72\\72 & 76 & -64\\-72 & -64 & 88\end{pmatrix} = 4\cdot\begin{pmatrix}18 & 18 & -18\\18 & 19 & -16\\-18 & -16 & 22\end{pmatrix}.$$

model: $\min \frac{1}{2}\omega^T\Sigma\omega$ $\qquad \bar\mu = 14$

s.t. $\omega^T\mu \geq \bar\mu$ $\qquad \mu = (12\ 14\ 12)$

$\omega^T\mathbb{1} = 1$ $\qquad \Sigma = 4\begin{pmatrix}18 & 18 & -18\\18 & 19 & -16\\-18 & -16 & 22\end{pmatrix}.$

b) optimal solution via formula does not exist, as Σ is singular and hence not invertible.

c) -> find x such that $x^T\Sigma x = 0$

$$\left(\begin{array}{ccc|ccc}18 & 18 & -18 & 1 & 0 & 0\\18 & 19 & -16 & 0 & 1 & 0\\-18 & -16 & 22 & 0 & 0 & 1\end{array}\right) \sim \left(\begin{array}{ccc|ccc}1 & 1 & -1 & 1/18 & 0 & 0\\0 & 1 & 2 & -1 & 1 & 0\\0 & 2 & 4 & 1 & 0 & 1\end{array}\right) \sim \left(\begin{array}{ccc|ccc}1 & 1 & -1 & 1/18 & 0 & 0\\0 & 1 & 2 & -1 & 1 & 0\\0 & 0 & 0 & -1 & -2 & 1\end{array}\right)$$

=> $x = (3\ \ -2\ \ 1)^T$

-> can we invest so we have no risk?

$$\left.\begin{array}{l}\omega_1 + \omega_2 + \omega_3 = 1\\ \omega_3 = t,\ \omega_2 = -2t,\ \omega_1 = 3t\end{array}\right\} \quad 3t + t - 2t = 1 \ \rightarrow t = 1/2$$

$\rightarrow$ $\omega = \begin{pmatrix}3/2\\-1\\1/2\end{pmatrix}$ has zero risk.

-> optimal return is only 10, we want 14.

→ let us plug in

$$\omega_3 = 1 - \omega_1 - \omega_2$$

then we have

$$\min_{\omega_1, \omega_2} \quad \frac{1}{2} \begin{pmatrix} \omega_1 & \omega_2 & 1-\omega_1-\omega_2 \end{pmatrix} \begin{pmatrix} 18 & 18 & \boxed{-18} \\ +18 & 19 & \boxed{-16} \\ \boxed{-18} & \boxed{-16} & 22 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ 1-\omega_1-\omega_2 \end{pmatrix}$$

+2

$$\text{s.t.} \quad \begin{pmatrix} \omega_1 & \omega_2 & 1-\omega_1-\omega_2 \end{pmatrix} \begin{pmatrix} 12 \\ 14 \\ 12 \end{pmatrix} \geq 14$$

→ that is minimization of a quadratic function with a condition

$$12\omega_1 + 14\omega_2 + (1-\omega_1-\omega_2) \cdot 12 \geq 14$$

$$\underline{\omega_2 \geq 1}$$

→ its minimum is 0, and that is at point $[3/2, -1]$, which does not meet the criteria $\omega_2 \geq 1$.

→ hence the extreme must lie on the boundary of the space

→ note that because it is a quadratic function with positive coefficients for ω_1^2 and ω_2^2, we know that the extreme must exist.

⇒ that implies that $\omega_2 = 1$, hence we can plug this into the derivative and minimize for ω_1.

$$f(\omega_1) = \frac{1}{2} \begin{pmatrix} \omega_1 & 1 & -\omega_1 \end{pmatrix} \begin{pmatrix} 18 & 18 & -18 \\ 18 & 19 & -16 \\ -18 & -16 & 22 \end{pmatrix} \begin{pmatrix} \omega_1 \\ 1 \\ -\omega_1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 36\omega_1 + 18 & 34\omega_1 + 19 & -40\omega_1 - 16 \end{pmatrix} \begin{pmatrix} \omega_1 \\ 1 \\ -\omega_1 \end{pmatrix}$$

$$= \frac{1}{2} \left(36\omega_1^2 + 18\omega_1 + 34\omega_1 + 19 + 40\omega_1^2 + 16\omega_1 \right)$$

$$= \frac{1}{2} \left(76\omega_1^2 + 68\omega_1 + 19 \right)$$

$$\frac{\partial f(\omega_1)}{\partial \omega_1} = 2 \cdot 76\omega_1 + 68 \qquad \frac{\partial f(\omega_1)}{\partial \omega_1} = 0 \iff \omega_1 = -\frac{68}{2 \cdot 76} = -\frac{17}{38}$$

$$\Rightarrow w^* = \left(-\frac{17}{38}, 1, \frac{17}{38} \right)$$