Investment Analysis (NMFP533)

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Abstract

This document serves as a learning material for students taking course Investment Analysis (NMFP533) at Charles University, Faculty of Mathematics and Physics. It covers basics from interest rates, bond evaluation, duration and convexity of an investment and portfolio selection methods based on risk/yield criteria.

Contents

1 Interest Rate Basics

In this chapter, we will discuss the relationship between values of different amounts of money at different time horizons. This will bring us to a concept of interest rates and the definition of the interest rate curve. We will also mention different approaches of compounding of interest rate and couple of day count conventions.

1.1 Time Value of Money

Problem definition

In general, people think economically, and most of them, if you give them a choice, whether to have one million CZK now or one million CZK next year, they would prefer to have the money now. One can also ask slightly modified question in a similar fashion. Say, what would be the amount of money you want to have now so you would be indifferent between having this money now and having one million next year?

This brings us to the general problem, that people value differently the same amount of money at different times. This phenomena is called time value of money and it is explained by the concept of interest rates.

Present and Future Value

We introduce a concept of *present value* PV and *future value* FV_t to describe relationship between present value (at *time* now, $t = 0$) and future value (at *future time*, $t = t > 0$) of some claim. In general, we write

$$\mathrm{PV} = \mathrm{FV}_t \cdot d(i_t, t), \quad \mathrm{FV}_t = \mathrm{PV} \cdot \frac{1}{d(i_t, t)},$$

where t denotes time, i_t is the interest rate applicable for time period t, $d(i_t, t)$ is the discount function, which gives the factor how much people value money at time t compared to now. Moreover, we should add, that i_t is usually quoted as *annualized*, i.e. standardized to describe how much interest rate you pay in one year. The relationship $t \to i_t$ depicts the *interest rate curve*, or *yield curve*, where for any time-to-maturity we have some interest rate specifying how people value money at this time horizon.

Compounding

There are different types of discount function depending on the approach used to perform *compounding* of interest. This in general means how we interest the interest already paid for the claim.

Assume a loan with *nominal* N_0 of one CZK ($PV = 1$) is given to a person now, and the person is expected to repay it back at some future time t. Then, we have few possibilities:

- *Simple compounding:* repay all the money at time t together with interest for the given period of time for the nominal:

$$\mathrm{FV}_t = N_0 \cdot (1 + i_t t).$$

- *Periodic compounding:* It has some associated frequency m, which describes how many times a year the interest is added to the nominal. In other words, at the end of such a period, we add the interest realized to the nominal for this period to form the nominal for the next period. There, it is again subject to interest.

$$\mathrm{FV}_t = N_0 \cdot \left(1 + \frac{i_t}{m}\right)^{t \cdot m}.$$

- *Continuous compounding:* is a limit of the periodic compounding, when m goes to infinity (interest is immediately subject to interest).

$$\mathrm{FV}_t = \lim_{m\to\infty} N_0 \cdot \left(1 + \frac{i_t}{m}\right)^{t\cdot m} = N_0 \cdot e^{i_t t}.$$

Day Count Conventions

Another important aspect to correctly calculate the interest rate associated with some notional is to apply correct day count convention, i.e. a formula how to calculate the time quantity t. This is usually given in a form of a fraction

$$\frac{\text{Number of days subject to interest}}{\text{Number of days in a year}}.$$

There are many day count conventions, some overview can be found here: `https://en.wikipedia.org/wiki/Day_count_convention`, with the most frequent conventions being:

- *30/360*: year has 360 days, while every full month between start date and end date of the period of interest has 30 days.
- *ACT/365*: one calculates exactly the number of days between start date and end date of the period of interest and divides by 365.
- *ACT/360*: one calculates exactly the number of days between start date and end date of the period of interest and divides by 360.

For all the calculations, when counting the number of days in the starting and ending month, one does not include the start date and includes the end date.

1.2 Cash-Flows and Investment

Usually, an investment can be formulated/defined by some sort of cash-flows, which occur at some given time. These can be both certain and uncertain, but for now, assume that they are all certain. Let us denote the times, at which these cash-flows occur as $0 = t_0 < t_1 < \ldots < t_K$ and the cash-flows itself as $\mathrm{CF}_k, k = 0, \ldots, K$. Note that these can be negative as well.

For $k = 0 \ldots, K$, we define present value of one cash-flow as

$$\mathrm{PV}_k = \mathrm{CF}_k \cdot d(i_{t_k}, t_k)$$

and the present value of the investment as

$$\mathrm{PV} = \sum_{k=0}^{K} \mathrm{PV}_k.$$

Note that for each time t_k we use different interest rate i_{t_k}, which we get from the yield curve. Also the choice of the discount function depends on the choice of compounding of interest rate. Often, for given investment, we know its properties (structure of cash-flow, discount function which shall be used) and also its market price (= present value). In such a setting, interest rate i, which meets the condition

$$\mathrm{PV} = \sum_{k=0}^{K} \mathrm{CF}_k \cdot d(i, t_k)$$

is called the *internal rate of return* of the investment.

1.3 Examples and Exercises

Example 1.1. Zuzka came 28^{th} November 2019 to a bank to borrow 50 000 CZK and promised to repay back the entire loan with interest on 12^{th} March 2020. Banker Anna has offered her a loan with 12% interest, simple compounding and day count convention ACT/365. However, two other bankers, Borek (30/360, simple compounding) and Cyril (ACT/365, periodic compounding with period one year) offered Zuzka different deals. Anna, Borek and Cyril are, however, from the same bank and hence all the loans had to have equal value for the bank. What were the offered rates by Borek and Cyril?

Proof. We will express the future values ($\mathrm{FV}_A, \mathrm{FV}_B, \mathrm{FV}_C$) of all the loans ($A, B, C$). These have to be equal for all the loans.

$$\begin{aligned} \mathrm{FV}_A &= N_0 \cdot (1 + i_A \cdot t_A) \\ \mathrm{FV}_B &= N_0 \cdot (1 + i_B \cdot t_B) \\ \mathrm{FV}_C &= N_0 \cdot (1 + i_C)^{t_C}, \end{aligned}$$

where $N_0 = 50000$ and $i_A = 0.12$. Using that $\mathrm{FV}_A = \mathrm{FV}_B = \mathrm{FV}_C$ we can solve for i_B and i_C.

$$\begin{aligned} i_B &= i_A \cdot \frac{t_A}{t_B} \\ i_C &= (1 + i_A \cdot t_A)^{1/t_C} - 1. \end{aligned}$$

Moreover, we have

$$\begin{aligned} t_A &= \frac{2 + 31 + 31 + 29 + 12}{365} = \frac{105}{365}, \\ t_B &= \frac{2 + 30 + 30 + 30 + 12}{360} = \frac{104}{360}, \\ t_C &= t_A, \end{aligned}$$

which gives

$$\begin{aligned} i_B &= 0.12 \cdot \frac{105 \cdot 360}{104 \cdot 365} \doteq 0.1195 \\ i_C &= \left(1 + 0.12 \cdot \frac{105}{365}\right)^{365/105} - 1 \doteq 0.1252. \end{aligned}$$

□

Exercise 1.2. Consider a loan with fixed nominal and interest rate and four types of compounding:

(a) simple compounding,

(b) periodic compounding with one-year period,

(c) periodic compounding with one-month period,

(d) continuous compounding.

Which one of these four would you choose if you wanted to borrow for

1. one week,
2. three months,
3. two years.

Can you conclude some general relationships between the four types of compounding?

Exercise 1.3. Calculate the time periods under 30/360, ACT/365 and ACT/360 day count conventions.

(a) 24^{th} June 2020 to 6^{th} September 2020,

(b) 1^{st} January 2020 to 1^{st} January 2021

(c) 17^{th} December 2020 to 12^{th} January 2025.

Example 1.4. A share, which ensures that its owner receives 4.65% from a nominal of 1000 CZK each year, is traded at the market for 619 CZK. What is the internal rate of return of this asset? (Assume periodic compounding with one year period).

Proof. First, we formulate an equation for the present value (market price) of the share.

$$\mathrm{PV} = \sum_{t=1}^{\infty} N_0 \cdot \frac{d}{(1+i)^t} = N_0 \cdot d \cdot \frac{1}{1+i} \sum_{t=0}^{\infty} \frac{1}{(1+i)^t} = N_0 \cdot d \cdot \frac{1}{1+i} \cdot \frac{1+i}{i},$$

where $\mathrm{PV} = 619, N_0 = 1000, d = 0.0465$ is the dividend and i is the internal rate of return. In the calculations, we used a formula for a sum of a geometric sequence. From there, we can express

$$i = \frac{N_0 \cdot d}{\mathrm{PV}} \doteq 0.0751.$$

□

Exercise 1.5. A bank has offered a nine month long loan with a nominal of 50 000 CZK and interest rate 12.6%. However, one of the conditions was that on the bank account atleast 20% of the borrowed amount will be placed all the time. What are the true properties (nominal and interest rate) of such a loan?

Example 1.6. A father has stated in his will that an amount of 3 mil. CZK will be transferred to a special account, from which each of his three children will receive the same amount of money when they turn 18. The special account is interested by a rate 8% with periodic compounding with period six months. How much money will the children receive, if we assume that they celebrated 11, 13 and 16 birthdays at the time of father's death? Assume 30/360 day count convention and that the children's birthdays are in April.

Proof. Let us denote P the payout the children will receive when they turn 18. That is the future value at the time of their birthday. We need to discount this amount to present values $(PV_{16}, PV_{13}, PV_{11})$, which need to sum up to the initial amount their father deposits on the account.

$$\begin{aligned}
\mathrm{PV}_{16} &= P \cdot \frac{1}{(1+i/2)^4}, \\
\mathrm{PV}_{13} &= P \cdot \frac{1}{(1+i/2)^{10}}, \\
\mathrm{PV}_{11} &= P \cdot \frac{1}{(1+i/2)^{14}},
\end{aligned}$$

where $i = 0.08$. Using that $\mathrm{PV} = \mathrm{PV}_{11} + \mathrm{PV}_{13} + \mathrm{PV}_{16}$, we obtain

$$P = \mathrm{PV} \Big/ \left(\frac{1}{(1+i/2)^{14}} + \frac{1}{(1+i/2)^{10}} + \frac{1}{(1+i/2)^4} \right).$$

Plugging in for $\mathrm{PV} = 3\,000\,000$ and $i = 0.08$, we obtain $P = 1\,423\,256$. □

Exercise 1.7. In building savings, we deposit 20 000 CZK at the beginning of each year for 6 consecutive years. On 1^{st} April, 10% from last year contributions, but 2000 CZK maximum, is contributed by the Czech state. How much money will be deposited on the account at the beginning of April in the seventh year, if the bank interests the account by periodic compounding with annual rate 1% and period one month? Assume that there is a tax of 15% on credited interest. How much of these money are our contributions, interest rate payments and state contributions? And what is the internal rate of return of this investment? (You can use excel for these calculations).

Exercise 1.8. You have a plan to buy a flat in two years. Currently, you have 1 mil. CZK which you look to deposit somewhere. You have two options.

(a) an account with periodic compounding with monthly period and interest rate 3.04%,

(b) two-year long term deposit (simple compounding) with interest rate 3.45%.

Which option would you choose? And what would be the real return (= nominal return[%] - inflation), if the inflation is 2.4% a year and interests are taxed by 15%?

Exercise 1.9. You own an oil pipeline, which in next year is expected to earn you 2 mil. USD. The amount of oil transferred is, however, decreasing and you expect your profit to lower each year by 4%. Consider the long term interest rate to be 3.5%.

(a) What is the present value of the pipeline, if you assume it is going to work forever?

(b) What would be the present value, if you would expect the pipeline to be decommissioned in 20 years and moreover, after that, you would need to pay 1 mil. USD for its removal and re-cultivation of the country?

Exercise 1.10. Harold is 30 years old and earns 20 000 USD per year. He believes that his salary will increase by 5% each year up until his retirement. He is due to retire when turning 60.

(a) What is the present value of all Harold's salaries, if we consider fixed interest rate 4% and that Harold gets his salary at the end of the year?

(b) How much money will Harold have by the time of his retirement, if he saves 5% of his yearly salary each year? (Saving account is interested by a rate 4%.)

(c) Harold expects himself to die at 80, so how much money he would be able to spend in retirement each year, if he wants to spend his savings equally (the same amount each year) and withdraws money from the saving account at the beginning of each year?

Example 1.11. There are six risk-free bonds available at the market as given in table below. There, half-

	nominal	time-to-maturity	half-year coupon	market price
Bond A	1000	0.25	0	989
Bond B	1000	0.5	0	977
Bond C	1000	1.0	100	1140
Bond D	1000	1.5	40	1020
Bond E	1000	2.0	60	1090
Bond F	1000	2.5	50	1050

year coupon means that every half-year till the maturity of the bond a coupon of the specified amount is paid to the holder of the bond. The task is to construct the yield curve from the market quotations of the bond if continuous compounding is used.

Proof. We need to calculate interest rate i_t for time-to-maturity t for all available times. We proceed from the shortest available quotation.

Bond A means, that in three months the holder will be paid the nominal N of the bond. From the present value formula we obtain

$$989 = \mathrm{PV} = N \cdot e^{-i_t t} = 1000 \cdot e^{-i_{0.25} \cdot 0.25},$$

which yields $i_{0.25} = \log(N/\mathrm{PV})/0.25 = 0.0442$.

Next, we continue to bond B, there, we also have only the nominal payment and hence, in a similar fashion as in the previous case, we can conclude that

$$i_{0.5} = \log(N/\mathrm{PV})/0.5 = \log(1000/977)/(0.5) = 0.0465.$$

Further, we go to bond C, there, after half a year, the holder receives the coupon c and at the end of the maturity, he also receives the nominal and the coupon. We again construct the present value equation:

$$1140 = \mathrm{PV} = c \cdot e^{-i_{0.5} \cdot 0.5} + (c + N) \cdot e^{-i_1 \cdot 1}.$$

Note, that the only unknown variable here is i_1, because we have already calculated the value of $i_{0.5}$. This gives us

$$i_1 = \log\left(\frac{c+N}{\mathrm{PV} - c \cdot e^{-i_{0.5} \cdot 0.5}}\right) = 0.05388.$$

Continuing in a similar way for bonds D, E and F (*exercise!*), we obtain $i_{1.5} = 0.0653, i_2 = 0.0708$ and $i_{2.5} = 0.0773$. Finally, if we plot these values, we obtain the figure of a yield curve.

Figure 1: Interest rate curve.

□

Such a method, when we calculate the interest rate curve from some interest rate derivatives (usually bonds or interest rate swaps), is called *bootstrapping*. Do not confuse this term with statistical bootstrap/-bootstrapping as a method of random sampling with replacement.

2 Financial Arbitrage

Arbitrage is an important concept in finance/financial mathematics, which basically forms the foundations on which the equilibrium theories and pricing formulas are derived. Often, one can see assumptions such as *an arbitrage-free market* or *under the no arbitrage condition*. We present the following in order to understand what authors mean by such remarks.

2.1 Definition

From wiki: *An arbitrage is a transaction that involves no negative cash flow at any probabilistic or temporal state and a positive cash flow in at least one state.*

In other words, if you are able to complete a transaction/create a portfolio, such as

- There is absolutely no chance you loose money.
- There is a positive chance, that you earn money,

then, you have an arbitrage. In theory, markets are assumed to be *efficient*, meaning that there are no arbitrage opportunities. However, in reality, these opportunities arise and people (mostly speculators) aim to exploit them. Often, arbitrage opportunities are created due to some human or coding errors. There have been cases of success stories, where people found some arbitrage opportunities in the markets, but also big failures, as in the real uncertain environment, even a safe bet can end up as a big disaster.

2.2 Examples and Exercises

Example 2.1. An investor has 2 mil. USD and is looking into *forex* (foreign exchange) market for possible investments. He spots offers for currency exchange of three banks as following:

- Bank A: offers 0.894 EUR for 1 USD or 1.095 USD for 1 EUR.
- Bank B: offers 0.678 GBP for 1 USD or 1.432 USD for 1 GBP.
- Bank C: offers 0.784 GBP for 1 EUR or 1.252 EUR for 1 GBP.

Can he create an arbitrage transaction from these offers?

Proof. Yes, he can. First, he goes to bank A to exchange USD for EUR and receives 1.788 mil EUR. Then, he goes to bank C to exchange EUR for GBP, so he obtains 1.402 mil GBP and finally he visits bank B to get back USD. After this transaction, he has 2 007 366 USD and hence he gained 7 366 USD with no risk. □

Exercise 2.2. There are three risk-free bonds available at the market with nominal and yearly coupon (given in percent of the nominal) as specified in the table below. Assume that you can both buy and sell these bonds in any fraction for the quoted market price. Decide, whether there is an arbitrage opportunity.

	nominal	time-to-maturity	coupon	market price
Bond A	1000	2	12%	1030
Bond B	1000	2	8%	970
Bond C	1000	1	10%	1000

Hint: try to replicate cash-flows first in the second year (so you buy/sell bond A and sell/buy bond B such that in year 2 your incomes would be the same as expenses) and then buy/sell the Bond C to match the cash-flows in the first year as well. Conclude according to your payments you would have to make to purchase such a portfolio.

Can you propose a new price for Bond A so there would be no arbitrage?

3 Duration and Convexity

3.1 Definitions

Assume that a portfolio is formed by a set of cash-flows $\mathrm{CF}_k, k = 0, \ldots, K$ at times $0 = t_0 < t_1 < \ldots < t_K$. We define duration D and convexity CX of this portfolio as

$$D = -\frac{1+i}{\mathrm{PV}} \cdot \frac{\partial \mathrm{PV}(i)}{\partial i}, \tag{1}$$

$$CX = \frac{1}{\mathrm{PV}} \cdot \frac{\partial^2 \mathrm{PV}(i)}{\partial i^2}, \tag{2}$$

where i is the internal rate of return of the portfolio. Duration measures the sensitivity of market price of the portfolio to the interest rate, as it is the market price's derivative (up to a multiplicative constant). Convexity is basically the second term in the Taylor expansion of a market price of the portfolio by the interest rate.

3.2 Duration Properties

Example 3.1. Duration has the following properties:

(a) Duration is an average time-to-maturity of the portfolio,

(b) duration is a time, at which the portfolio value "does not change" when interest rate changes.

Proof. (a): Let as calculate the derivative of the present value with respect to i.

$$\begin{aligned}
D &= -\frac{1+i}{\mathrm{PV}} \cdot \frac{\partial \mathrm{PV}(i)}{\partial i} \\
&= -\frac{1+i}{\mathrm{PV}} \cdot \sum_{k=0}^{K} \frac{\partial}{\partial i} \frac{\mathrm{CF}_k}{(1+i)^{t_k}} \\
&= -\frac{1+i}{\mathrm{PV}} \cdot \sum_{k=0}^{K} \frac{\mathrm{CF}_k}{(1+i)^{t_k+1}}(-t_k) \\
&= \frac{1}{\mathrm{PV}} \cdot \sum_{k=0}^{K} \frac{t_k \mathrm{CF}_k}{(1+i)^{t_k}} \\
&= \sum_{k=0}^{K} t_k \frac{\mathrm{PV}_k}{\mathrm{PV}}.
\end{aligned}$$

There, the ratio $\mathrm{PV}_k/\mathrm{PV}$ depicts the significance of the k-th cash-flow in the portfolio and in the sum it weights the time when it is scheduled. Hence we can interpret duration as the average time-to-maturity.

(b) We want to show that the change in time D value of a portfolio is insensitive to interest rate changes. This can be shown via the following:

$$\frac{\partial \left[\mathrm{PV}(i)(1+i)^D\right]}{\partial i} = \frac{\partial \mathrm{PV}(i)}{\partial i}(1+i)^D + \mathrm{PV}(i)\frac{\partial (1+i)^D}{\partial i} = \frac{-D \cdot \mathrm{PV}(i)}{1+i}(1+i)^D + \mathrm{PV}(i) D (1+i)^{D-1} = 0.$$

In the process, we used the definition of a duration (1) and we assumed that duration is not a function of interest rate. This is obviously not true, but in general, duration does not change at all when interest rate changes slightly so this effect can be neglected. □

Example 3.2. Let $t(\Delta i)$ be a solution of an equation

$$PV(i)(1+i)^{t(\Delta i)} = PV(i+\Delta i) \cdot (1+i+\Delta i)^{t(\Delta i)},$$

show that $D = \lim_{\Delta i \to 0} t(\Delta i)$.

Proof. First, let us express $t(\Delta i)$ by taking logs of both sides in the above equation.

$$t(\Delta i) = \log\left(\frac{\mathrm{PV}(i+\Delta i)}{\mathrm{PV}(i)}\right) / \log\left(\frac{1+i}{1+i+\Delta i}\right),$$

next, we take the limit and use the l'Hospital rule:

$$\begin{aligned}
\lim_{\Delta i \to 0} t(\Delta i) &= \lim_{\Delta i \to 0} \log\left(\frac{\mathrm{PV}(i+\Delta i)}{\mathrm{PV}(i)}\right) / \log\left(\frac{1+i}{1+i+\Delta i}\right) \\
&= \lim_{\Delta i \to 0} \frac{1}{PV(i+\Delta i)} \frac{\partial PV(i+\Delta i)}{\partial \Delta i} \Big/ \frac{-1}{1+i+\Delta i} \\
&= \lim_{\Delta i \to 0} -\frac{1+i+\Delta i}{PV(i+\Delta i)} \frac{\partial PV(i)}{\partial i} \\
&= D.
\end{aligned}$$

□

Exercise 3.3. Show that

$$\frac{\Delta \mathrm{PV}}{\mathrm{PV}} \sim -D\frac{\Delta i}{1+i} + \frac{(\Delta i)^2}{2} CX.$$

Hint: use Taylor expansion of the second order of the present value formula and plug in for the derivatives the definitions of duration (1) *and convexity* (2)*.*

The statement above gives us an approximation of the relative price change (present value change) of a portfolio given some movement of interest rate.

3.3 Duration of Multiple Assets

Next, assume that we have $l = 1, \ldots, L$ assets, each with present value PV_l and duration D_l. Moreover, assume that these assets have the same internal rate of return. Then, we can express the duration of the portfolio of all the assets as

$$D = \frac{\sum_{l=1}^{L} D_l \mathrm{PV}_l}{\sum_{l=1}^{L} \mathrm{PV}_l}. \tag{3}$$

Proof.

$$D = -\frac{1+i}{\mathrm{PV}} \cdot \frac{\partial \mathrm{PV}(i)}{\partial i} = -\frac{1+i}{\mathrm{PV}} \cdot \frac{\partial \sum_{l=1}^{L} \mathrm{PV}_l(i)}{\partial i} = \frac{1}{\mathrm{PV}} \cdot \sum_{l=1}^{L} -(1+i)\frac{\partial \mathrm{PV}_l(i)}{\partial i} = \frac{1}{\mathrm{PV}} \sum_{l=1}^{L} D_l \mathrm{PV}_l.$$

□

This formula is frequently used even when the condition on the same internal rate of return is not satisfied. In this case, one needs to bear in mind that the result is only an approximation.

Exercise 3.4. Let us consider three bonds with yearly coupons as in the table below.

	nominal	time-to-maturity	coupon
Bond A	1000	1	4%
Bond B	1000	2	6%
Bond C	1100	2	0%

(a) Determine the market price and duration of the bonds if the interest rate is such one-year interest rate $y_1 = 4\%$ and two-year interest rate $y_2 = 6\%$.

(b) An investor wants to buy 3 bonds A and 1 bond B. What would be duration of such a portfolio?

(c) Afterwards, he spots bond C available at the market and buys bond C instead of bond B. He calculates the duration of the portfolio via (3). What value did he obtain and how big error did he make by using that formula?

Exercise 3.5. There are two bonds with yearly coupons available as in the table below. Market interest rate is set at 10%. An investor wants a portfolio from these bonds with duration 3 years. How much of each asset should he buy?

	nominal	time-to-maturity	coupon
Bond A	1000	2	10%
Bond B	1000	4	10%

4 Shares and Other Uncertain Financial Instruments

Next, we move to a much wider class of financial instruments and that is instruments with some uncertain behaviour. Between these, the most simple and the most frequently traded are shares. Shares give to its holder a decision power on the company management as well as the right to earn dividends the company pays to its shareholders.

Dividends are paid out with some frequency (usually yearly/quarterly) and distribute the earnings of the company between its owners. This dividend is, however, not guaranteed, the amount of money paid each time varies and it can happen that the dividend is not paid at all. What also needs to be taken into account is that the share itself has some value/price and that changes as well.

For each company, we should understand what are its assets and its liabilities. Roughly speaking, assets are things from which the company can expect to earn money, things which in future will bring economic benefit. On the other hand, liabilities are things for which the company will need to pay money. Assets add value to the company, liabilities decrease it.

Example 4.1. For a bank, a mortgage given to a client is an asset, as client will pay money to the bank for the mortgage. A savings account of another client is a liability, as the bank will need to pay money to the client.

Exercise 4.2. You are a manager of a food chain, split the following objects between assets and liabilities:

- cash
- 3000l of milk you cannot sell
- a car used by you for work purposes
- leasing loan for the car
- trolleys used by your customers
- a mortgage for one of the supermarket buildings
- that building you have the mortgage for
- employees of the company
- savings stored in a bank
- know-how of how to store meat so it lasts longer
- bank loan
- your customers
- brand of the company
- tax obligations

One can see, there are various assets and liabilities the company can have, and many more could be listed too. The overall value of the company is quite difficult to judge given the nature of many of its assets and liabilities, which are difficult to evaluate (asses their real economic value). Pricing is done by investors, who either buy or sell the shares of the company and hence the final price is determined by the market equilibrium. It is quite frequent that the value of some company is misspecified, which leads to unexpected jumps at the market. These can be caused due to some new information which completely changes a value of some of the assets/liabilities as well.

Exercise 4.3. Give three examples in latest years when a value of a company changed rapidly due to some external factor/new information.

Asset-liability management is a complex discipline, where also mathematical and optimization methods find their utilization. We have seen numerous application of statistical models used to predict future assets/liabilites and various stochastic optimization models which aim to find best decisions to improve the balance sheet and hence the value of the company.

Another important name is equity. Equity represents the value that would be returned to the company's shareholders if all of the assets were liquidated (sold) and all of the company's debts were paid off (total assets - total liabilities). In another words, once you quantify economic values of all assets and liabilities the company has, then the leftover is the equity. As this value is owned by shareholders and the company in the end should pay this to them, then equity is also a liability

4.1 Financial Indicators

For any joint-stock company (a company with shares traded at the market), we can follow various financial indicators. Between them we should list at least the following

- ROE - *return on equity* = profit/equity
- ROA - *return on assets* = profit/assets,
- DR - *debt ratio* = (liabilities-equity)/assets,
- LR - *leverage ratio* = (liabilities-equity)/equity,
- DPS - *dividend per share* = total dividends/no of shares,
- EPS - *earnings per share* = total earnings/no of shares,
- PR - *payout ratio* = DPS/EPS,
- P/E - *price/earnings* = share price / earnings per share.

One way, how we can calculate the price of a share is by discounting the estimated future cash-flows. In other words, we calculate

$$\mathrm{PV} = \sum_{t=1}^{\infty} \frac{D_t}{(1+i)^t},$$

where D_t are the estimated future cash-flows. In reality, dividends are not certain and hence this formula needs to be viewed as approximate. One could propose to use the expected dividend instead, however, it is clear that the uncertainty/risk has also its influence on the final price and so such a formula would also be just a simplification.

Usually, instead of prices, we work with returns. These correspond to relative change in price and they can be quoted as daily, monthly or annual. When working with data of returns, one needs to take into account, when dividends are paid, or to use prices adjusted for dividends. There is usually a price jump on a dividend ex-date (the decisive date which determines to whom the dividend is paid). Hence the share price decreases, meaning the return is negative, but because the holder receives the dividend, no loss is incurred. In the following, we assume that the returns are adjusted for these jumps and correspond to real returns the investor makes when buying the share at one point and selling it at another point of time.

Exercise 4.4. Chief Medical Inc. is a company which develops lung ventilators. Given current environment, there is a great uncertainty in their dividend payments. Albert believes, that they will grow by 5% each year up until infinity. Bukayo, on the other hand, thinks that they will grow by 20% in the next four years and then by 4% till infinity. The last dividends paid out by the company were 3$ per share, while the interest rate used for discounting dividends of companies with similar profile is currently 12%.

(a) How much should Albert value one share of the company?

(b) What should be a price of one share according to Bukayo?

4.2 Portfolio Selection

One of the common questions you hear at the market is which shares to buy or how to form one's portfolio? And given that in recent world, everyone is allowed to have an opinion, people continuously come up with ideas and answers. These are based on various arguments, some on fundamental ones, some on more psychological ones and some on completely wrong ones. In the following, we will present one of the most basic fundamental arguments, which develops strategies based on expected return/risk criteria.

Let us have K shares with returns $\mathbf{r} = (r_1, \ldots, r_K)^T$, volatilities $\sigma_1, \ldots, \sigma_K$, correlations $\rho_{i,j}, i, j = 1, \ldots, K$ and variance-covariance matrix Σ. Return of one share r_k We consider returns to be random, with some expected value (expected return μ_k) and variance σ_k^2. Next, we define a vector of weights $\mathbf{w} = (w_1, \ldots, w_K)^T, \sum_k w_k = 1$, which governs the proportions of the shares in the portfolio.

We denote $\boldsymbol{\mu} = \mathbb{E}\mathbf{r}$ the expected returns of assets and $r_p = \mathbf{w}^T\mathbf{r}$, $\mu_p = \mathbf{w}^T\boldsymbol{\mu}$ and $\sigma_p^2 = \mathbf{w}^T\Sigma\mathbf{w}$ the portfolio return, expected portfolio return and its variance respectively. The mean-variance approach stems from finding optimal portfolio composition such we maximize expected return and minimize its variance. Given that we have two criteria, which we are interested in, there are different formulations of the optimization model. Compare:

$$\begin{aligned} \max_{\mathbf{w}} \quad & \theta\mathbf{w}^T\boldsymbol{\mu} - \mathbf{w}^T\Sigma\mathbf{w} \\ \text{s.t.} \quad & \mathbf{w}^T\mathbf{1} = 1, \end{aligned} \tag{4}$$

where θ is some predetermined parameter, to

$$\begin{aligned} \min_{\mathbf{w}} \quad & \frac{1}{2}\mathbf{w}^T\Sigma\mathbf{w} \\ \text{s.t.} \quad & \mathbf{w}^T\boldsymbol{\mu} \geq \bar{\mu} \\ & \mathbf{w}^T\mathbf{1} = 1, \end{aligned} \tag{5}$$

with $\bar{\mu}$ some benchmark mean return, and to

$$\begin{aligned} \max_{\mathbf{w}} \quad & \mathbf{w}^T\boldsymbol{\mu} \\ \text{s.t.} \quad & \frac{1}{2}\mathbf{w}^T\Sigma\mathbf{w} \leq \bar{\sigma}^2 \\ & \mathbf{w}^T\mathbf{1} = 1, \end{aligned} \tag{6}$$

where $\bar{\sigma}^2$ is the upper limit of portfolio return variance. These are three different formulations of a relatively similar problem, and surprisingly, they have one thing in common. If you consider all possible parameters θ for (4), $\bar{\mu}$ for (5) and $\bar{\sigma}^2$ for (6), the obtained solutions are exactly the same. These optimal portfolios are called *efficient portfolios* and form the so called *mean-variance frontier*.

We can slightly modify the above problem by introducing the risk-free asset, an asset with guaranteed return and hence zero variance. Let us denote the return of this asset $r_0 = \mu_0$ and assign him weight w_0. Then reformulation of (5) gives us

$$\begin{aligned} \min_{\mathbf{w}} \quad & \frac{1}{2}\mathbf{w}^T\Sigma\mathbf{w} \\ \text{s.t.} \quad & \mathbf{w}^T\boldsymbol{\mu} + w_0\mu_0 \geq \bar{\mu} \\ & \mathbf{w}^T\mathbf{1} + w_0 = 1, \end{aligned}$$

which can be further simplified to

$$\begin{aligned}\min_{\mathbf{w}} \quad & \frac{1}{2}\mathbf{w}^T\Sigma\mathbf{w} \\ \text{s.t.} \quad & \mathbf{w}^T(\boldsymbol{\mu}-\mu_0\mathbf{1}) \geq \bar{\mu}-\mu_0.\end{aligned} \tag{7}$$

Exercise 4.5. Find solution of the problem (7).

Hint: formulate Lagrange function for the minimization problem, denote $A = \mathbf{1}^T\Sigma^{-1}\mathbf{1}, B = \mathbf{1}^T\Sigma^{-1}\boldsymbol{\mu}, C = \boldsymbol{\mu}^T\Sigma^{-1}\boldsymbol{\mu}$ *and perform algebraic manipulations to obtain solution for* $w_0, \mathbf{w}$.

The solution is

$$\begin{aligned}\mathbf{w}^* &= \frac{(\bar{\mu}-\mu_0)\Sigma^{-1}(\boldsymbol{\mu}-\mu_0\mathbf{1})}{C-2B\mu_0+A\mu_0^2}, \\ w_0^* &= 1-\mathbf{1}^T\mathbf{w}^*.\end{aligned}$$

Moreover, if we denote

$$\begin{aligned}\delta &= 1-\frac{(\bar{\mu}-\mu_0)(B-\mu_0A)}{C-2B\mu_0+A\mu_0^2}, \\ 1-\delta &= \frac{(\bar{\mu}-\mu_0)(B-\mu_0A)}{C-2B\mu_0+A\mu_0^2}, \\ \mathbf{w}_T &= \frac{\Sigma^{-1}(\boldsymbol{\mu}-\mu_0\mathbf{1})}{(B-\mu_0A)},\end{aligned} \tag{8}$$

one can see that

$$\begin{aligned}\mathbf{w}^* &= (1-\delta)\mathbf{w}_T, \\ w_0^* &= \delta.\end{aligned}$$

This has an extremely nice interpretation, as we see that we invest proportion of δ of our income to the risk-free asset and the rest $(1-\delta)$ to some *tangency portfolio* or *market portfolio* $\mathbf{w}_T$. This *tangency portfolio* is the only portfolio lying on the *efficient variance frontier*, which is also efficient in the new settings with risk-free asset.

Exercise 4.6. Calculate the expected return and variance of a portfolio with weights $(w_0^*, \mathbf{w}^*)$.

Hint: just plug-in formulas from above to the expected return/variance formula and perform algebraic manipulations.

The solution gives

$$\begin{aligned}\mu_p &= \delta\mu_0 + (1-\delta)\mathbf{w}_T^T\boldsymbol{\mu} = \bar{\mu}, \\ \sigma_p &= (1-\delta)\sqrt{\mathbf{w}_T^T\Sigma\mathbf{w}_T} = \frac{\bar{\mu}-\mu_0}{\sqrt{\mu_0^2A-2\mu_0B+C}}.\end{aligned} \tag{9}$$

4.3 Capital Market Line

To conclude, we see that after the introduction of the risk-free asset, the optimal portfolios are just convex combinations of the tangency portfolio and the risk-free asset. Moreover, the relationship between the expected return and the standard deviation of such portfolios is linear. This is then summarized by the *Capital Market Line* (CML).

$$\mu_p = \mu_0 + \frac{\sigma_p}{\sigma_T}(\mu_T-\mu_0), \tag{10}$$

where μ_T and σ_T are the mean return and standard deviation of the tangency portfolio $\mathbf{w}_T$. Note that this relationship is only theoretical and it holds only for efficient portfolio. The capital market line together with the mean-variance frontier, tangency portfolio and the risk-free asset can be summarized in a chart as shown in Figure 2.

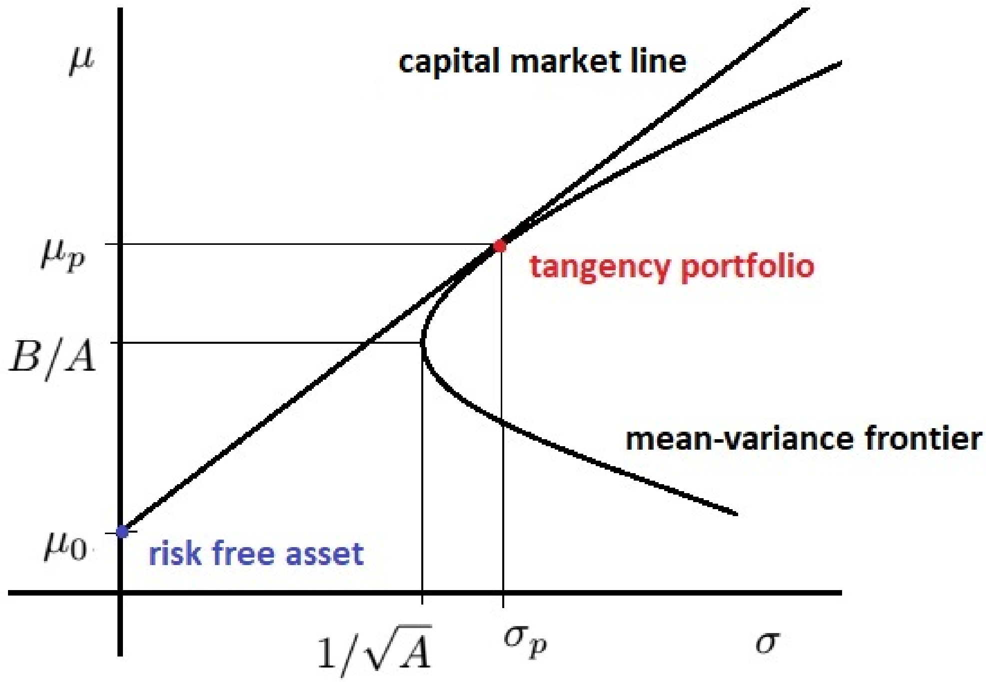

Figure 2: The Mean-Variance Frontier and the Capital Market Line.

4.4 Security Market Line

Finally, we will have a look at the return of the individual assets, and their covariances with the tangency portfolio. Let us denote

$$\sigma_{k,T} = \text{cov}(r_k, r_T),$$

where $r_T = \mathbf{r}^T \mathbf{w}_T$. From there, we can express

$$\begin{aligned} \sigma_{\cdot,T}^T = (\sigma_{1,T}, \ldots, \sigma_{K,T})^T &= \text{cov}(\mathbf{r}, r_T), \\ &= \text{cov}\left(\mathbf{r}, \frac{\mathbf{r}^T \Sigma^{-1}(\boldsymbol{\mu} - \mu_0 \mathbf{1})}{(B - \mu_0 A)}\right), \\ &= \frac{\boldsymbol{\mu} - \mu_0 \mathbf{1}}{B - \mu_0 A}. \end{aligned} \tag{11}$$

Next we can also calculate the variance of tangency portfolio as

$$\sigma_T^2 = \text{var}(\mathbf{r}^T \mathbf{w}_T) = \mathbf{w}_T^T \Sigma \mathbf{w}_T = \frac{\mu_0^2 A - 2\mu_0 B + C}{(B - \mu_0 A)^2},$$

from (9), we can express $\mu_0^2 A - 2\mu_0 B + C = (\mu_T - \mu_0)^2 / \sigma_T^2$ and obtain

$$\sigma_T^2 = \frac{\mu_T - \mu_0}{B - \mu_0 A}.$$

Combining with (11), we get

$$\sigma_{\cdot,T} = \frac{\sigma_T^2}{\mu_T - \mu_0}(\boldsymbol{\mu} - \mu_0 \mathbf{1}).$$

Rewriting such an equation component-wise, we obtain

$$\mu_k - \mu_0 = \frac{\sigma_{k,T}}{\sigma_T^2}(\mu_T - \mu_0), \quad k = 1, \dots, K. \tag{12}$$

Equation (12) depicts the so called *Security Market Line*, which describes how the extra mean return of individual assets is bound to the extra tangency portfolio return. The key object here is the ratio of the covariance of returns of the individual asset and the tangency portfolio and the variance of the tangency portfolio. This ratio is called the asset's *beta*. We set

$$\beta_k = \frac{\sigma_{k,T}}{\sigma_T^2}.$$

Such value measures how much the stock moves correspond to the market moves. You can read about more properties and relationship in here `https://www.investopedia.com/investing/beta-know-risk/`.

There is a tight relationship between asset's beta and the systematic risk of the asset. Basically, beta determines by how much the asset price moves if the market, or the industry, moves. Some more information about systematic risk can be found here `https://www.investopedia.com/ask/answers/031715/how-does-beta-reflect-systematic-risk.asp`.

Finally, let us mention, how we estimate such an asset's beta. In practice, we look at daily/monthly returns of the given stock $r_{k,t}$ of the risk free asset $r_{0,t}$ and of the tangency portfolio $r_{T,t}, t = 1, \dots, T$. (Here, be aware, that we have subscript T for tangency portfolio and parameter T for the time horizon, it should be clear from the context which T is what.. :)). Then, from the security market line, we have

$$r_{k,t} - r_{0,t} = \alpha_k + \beta_k(r_{T,t} - r_{0,t}) + \varepsilon_t, \quad t = l, \dots, T,$$

where the second term in the regression $\beta_k(r_{T,t} - r_{0,t})$ describes the systematic risk and the error term ε_t the unsystematic risk (not correlated to the market moves). Theoretically, the intercept α_k should be zero (from the security market line relationship). This is in practice, however, rarely true, and when α_k is significantly greater than 0, we call the asset to perform better than expected, while for $\alpha_k < 0$, we say the asset is under-performing. One can estimate the parameters by OLS and use classical hypothesis testing to make valid statistical conclusions on this problem. The above model is the famous *Capital Asset Pricing Model*. Some more information and discussion on the systematic v unsystematic risk phenomena can be found here `https://www.investopedia.com/articles/06/capm.asp`.

4.5 Examples and Exercises

Exercise 4.7. Calculate the minimum risk which an investor should expect, if he buys an asset with expected return 10%, is there is a risk-free asset with return 3% and the expected return of market portfolio is estimated via the market index to be 8% with risk 5%?

Hint: use capital market line, consider risk to be the standard deviation of return.

Exercise 4.8. Decide on the expected return of a stock, if its market beta is estimated to be 1.5, while the market portfolio expected return is estimated to be 10% with risk 5% and the risk-free asset yields 4%?

Exercise 4.9. Derive the shape of the security market line, if you observe the following stocks and that you believe that their returns are in-line with the market.

- Expected return $6\%, \beta = 0.5$
- Expected return $12\%, \beta = 1.5$.

What is the expected return of a stock with $\beta = 2$?

Exercise 4.10. The general theory of the Capital Asset Pricing Model is developed under the condition that $\sum_k w_k = 1$. Assume now, that a condition that $w_k \geq 0 \quad \forall k$ is also added.

(a) Formulate the new optimization problem.

(b) Write down the Lagrange function and the first order (Karush-Kuhn-Tucker, or local optimality conditions from subject Úvod do optimalizace) optimality conditions.

(c) Can you propose a way how such a model could be solved numerically?

(d) Can you estimate, what would happen to the new mean-variance frontier and the capital market line if the no short selling condition would be in place?

Hint: (c): consider what class of programming model we have obtained, for (d) use formulas in (8) *and discuss under which conditions the new constraints are violated and where would you expect the solution to move.*

Exercise 4.11. There are three shares available at the market with the expected return and variance-covariance matrix as follows:

$$\boldsymbol{\mu} = \begin{pmatrix} 0.14 \\ 0.08 \\ 0.2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0.06 & 0.0009 & 0.0018 \\ 0.0009 & 0.03 & 0.0018 \\ 0.0018 & 0.0018 & 0.15 \end{pmatrix}.$$

Assume that an investor is targeting expected profit of 15% with the least possible risk. Assume that short-selling is not allowed.

(a) Formulate an optimization model he needs to solve for this purpose.

(b) Write down the Lagrange function and the KKT conditions.

(c) Find the optimal solution of such a model.

Hint: for part (c), you are allowed to use any computational software.

Exercise 4.12. Assume, that there are three stocks, A, B and C and three scenarios of possible return of these stocks. These are given as follows

		return [%]		
scenario	probability	A	B	C
optimistic	0.25	24	28	4
normal	0.5	12	12	8
pessimistic	0.25	0	4	28

Assume that an investor is targeting expected profit of 14% with the least possible risk.

(a) Formulate an optimization model when short selling is allowed.

(b) Write down the formula for the optimal solution, describe all objects involved.

(c) Find the optimal solution of such a model.

Hint: for part (c), you are allowed to use any computational software.

5 Utility Functions

Utility functions represent a theoretical concept within economics. The concept is explained in great detail in subject Mathematical Economics (NMEK531), hence we will give here just very concise overview and show some applications in the portfolio selection problem.

In general, we have a utility function $u : \boldsymbol{X} \to \mathbb{R}$, where $\boldsymbol{X}$ is the set of consumption possibilities. The function $u(\boldsymbol{x}), \boldsymbol{x} \in \boldsymbol{X}$ assigns utility $u(\boldsymbol{x}) \in \mathbb{R}$ the consumer has from consuming the goods $\boldsymbol{x}$. In other words, $u(\boldsymbol{x})$ ranks all possible consumption combinations by preference of the consumer.

In the portfolio selection problem, the consumption set is going to be the set of all possible returns of the portfolio. In general, we want to select the portfolio composition so we maximise the expected utility of the consumer. In other words, we tackle the problem

$$\max_{\mathbf{w}} \quad \mathbb{E}u(\mathbf{w}^T\mathbf{r}) \qquad \text{s.t.} \quad \mathbf{w}^T\mathbf{1} = 1.$$

Example 5.1. Show that

$$
\begin{aligned}
\max_{\mathbf{w}} \quad & \mathbb{E}u(\mathbf{w}^T\mathbf{r}) \\
\text{s.t.} \quad & \mathbf{w}^T\mathbf{1} = 1, \\
& \mathbf{w} \geq 0,
\end{aligned} \tag{13}
$$

where $u(x) = -e^{-\alpha x}, \alpha > 0$ is an exponential utility function with parameter α and $\mathbf{r} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ is a random vector which depicts the distribution of returns is equivalent to a mean-variance problem (4).

Proof. First, let us manipulate with the objective function

$$
\begin{aligned}
\mathbb{E}u(\mathbf{w}^T\mathbf{r}) &= \int_{-\infty}^{\infty} -e^{-\alpha\mathbf{w}^T\mathbf{r}} \frac{1}{(2\pi|\Sigma|)^{n/2}} \exp\left\{-\frac{(\mathbf{r}-\boldsymbol{\mu})^T\Sigma^{-1}(\mathbf{r}-\boldsymbol{\mu})}{2}\right\} d\mathbf{r} \\
&= -\int_{-\infty}^{\infty} \frac{1}{(2\pi|\Sigma|)^{n/2}} \exp\left\{-\frac{(\mathbf{r}-\boldsymbol{\mu})^T\Sigma^{-1}(\mathbf{r}-\boldsymbol{\mu}) + 2\alpha\mathbf{w}^T\mathbf{r}}{2}\right\} d\mathbf{r} \\
&= -\int_{-\infty}^{\infty} \frac{1}{(2\pi|\Sigma|)^{n/2}} \exp\left\{-\frac{(\mathbf{r}-\boldsymbol{\mu}+\alpha\Sigma\mathbf{w})^T\Sigma^{-1}(\mathbf{r}-\boldsymbol{\mu}+\alpha\Sigma\mathbf{w}) + 2\alpha\mathbf{w}^T\boldsymbol{\mu} - \alpha^2\mathbf{w}^T\Sigma\mathbf{w}}{2}\right\} d\mathbf{r} \\
&= -\exp\left\{-\frac{2\alpha\mathbf{w}^T\boldsymbol{\mu} - \alpha^2\mathbf{w}^T\Sigma\mathbf{w}}{2}\right\} \\
&= -\exp\left\{-\alpha\left(\mathbf{w}^T\boldsymbol{\mu} - \frac{\alpha}{2}\mathbf{w}^T\Sigma\mathbf{w}\right)\right\}
\end{aligned}
$$

In the third equation, we used the completing the square method and in the fourth equation we used the fact that the integral of normal density is 1. Consequently, due to the fact, that $-\exp(-\alpha x)$ is for $\alpha > 0$ an increasing function in x, then the problem (13) can be equivalently rewritten (so the optimal solution will not change) to a problem

$$
\begin{aligned}
\max_{\mathbf{w}} \quad & \mathbf{w}^T\boldsymbol{\mu} - \frac{\alpha}{2}\mathbf{w}^T\Sigma\mathbf{w} \\
\text{s.t.} \quad & \mathbf{w}^T\mathbf{1} = 1, \\
& \mathbf{w} \geq 0.
\end{aligned} \tag{14}
$$

Hence if we set $\theta = \frac{2}{\alpha}, \theta > 0$, we can see that this utility maximization problem is equivalent to the mean-variance problem (4), which we discussed at the beginning of Section 4.2. ☐

Exercise 5.2. Solve analytically the problem (13), when

$$
\mathbf{r} \sim \mathcal{N}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 9 \end{pmatrix}\right)
$$

Exercise 5.3. Solve analytically

$$
\begin{aligned}
\max_{\mathbf{w}} \quad & \mathbb{E}u(\mathbf{w}^T\mathbf{r}) \\
\text{s.t.} \quad & \mathbf{w}^T\mathbf{1} = 1, \\
& \mathbf{w} \geq 0,
\end{aligned} \tag{15}
$$

where $u(x) = \log(x+2)$ is a logarithmic utility function and $\mathbf{r}$ is discrete with two scenarios s_1, s_2 with equal probabilities of three stocks a_1, a_2, a_3 and returns

	a_1	a_2	a_3
s_1	2	-1	0
s_2	1	3	4

6 Risk Measures

The mean-variance model is quite simple to interpret. The "mean" depicts the expected return, profit which we expect from the strategy. On the other hand "variance" measures risk, it measures how much we can expect the realization to deviate from the expected profit. One could argue, why we use this "variance" as the measure of risk. For example, we can penalize, or limit, or control only the losses, or returns lower than the expected profit. Or we want to use different penalizations for different level of losses. In general, there are plenty of measures of risk which make sense.

We define these for loss function L (in our case usually negative profit). The most common risk measures are

- Variance
$$\mathrm{var}(L) = \mathbb{E}\left[(L - \mathbb{E}L)^2\right]$$
- Semi-Variance
$$\gamma(L) = \mathbb{E}\left[\left((L - \mathbb{E}L)^+\right)^2\right]$$
- Mean-Absolute Deviation
$$\mathrm{MAD}(L) = \mathbb{E}|L - \mathbb{E}L|$$
- Value-at-Risk at level α
$$\mathrm{VaR}_\alpha(L) = \inf\{l \in \mathbb{R}, P(L > l) \le 1 - \alpha\}$$
- Conditional Value-at-Risk at level α
$$\mathrm{CVaR}_\alpha(L) = \min_{a \in \mathbb{R}} \left\{ a + \frac{1}{1-\alpha} \mathbb{E}[L - a]^+ \right\}$$

Note that for VaR and CVaR, α is usually set to be close to 1, most frequently 0.95 or 0.99. So that in the case of VaR, we control the probability that a loss $L > \mathrm{VaR}_\alpha(L)$ happens with low probability $1 - \alpha$. CVaR is then the average of such losses.

Exercise 6.1. Calculate Semi-Variance, Mean-Absolute Deviation, VaR_α and CVaR_α for $L \sim \mathcal{N}(\mu, \sigma^2)$.

Hint: for CVaR, use another equivalent formula from lecture.

7 Answers

Exercise 1.2: Continuous compounding is always the most profitable. Simple compounding is better than periodic compounding only for time periods shorter than the period of the periodic compounding.

Exercise 1.3:

(a) 72/360, 74/365, 74/360.

(b) 361/360, 366/365, 366/360.

(c) 1466/360, 1487/365, 1487/360.

Exercise 1.5: Loan of 40000 CZK with interest rate 15.75% [simple compounding].

Exercise 1.7: 136 157 CZK (120 000 our contributions, 12 000 state contribution, 4 157 interest). IRR = 3.38% — calculate this as FV = 0 (which is equivalent to PV = 0).

Exercise 1.8: b) is better (FV 1.069 mil. vs 1.063 mil. without considering tax) Nominal return of b) after adjusting for taxes is 2.89%, real return then 0.49%.

Exercise 1.9: a) 26.67 mil. USD, b) 20.24 mil. USD.

Exercise 1.10:

(a) 665071 USD.

(b) 107854 USD.

(c) 7631 USD.

Exercise 2.2: We buy 1 bond A, sell 28/27 of B and sell $10/(11 \cdot 27)$ of C, by this we fix cash-flows in first and second year, Market price of such a portfolio is 9.596, which means we earn free money and that is arbitrage. No-arbitrage price of bond A is 1039.6.

Exercise 3.4:

(a) A: $\mathrm{PV}_A = 1000, D_A = 1$, B: $\mathrm{PV}_B = 1001.084, D_B = 1.942$, C: $\mathrm{PV}_C = 978.996, D_C = 2$.

(b) $\mathrm{PV} = 3 \cdot \mathrm{PV}_A + \mathrm{PV}_B, D = 1.236$,

(c) Duration by formula: 1.246, true duration is also 1.246.

Exercise 3.5: The ratio is 0.448 of A to 1 B.

Exercise 4.2: Liabilities: leasing, mortgage, loan, taxes, the rest are assets.

Exercise 4.4: a) 45 USD, b) 65.7 USD

Exercise 4.7: 0.07

Exercise 4.8: 0.13

Exercise 4.9: $\mu_k = 0.03 + 0.06 \cdot \beta$, 0.15

Exercise 5.2: Use formula (14), plug in $w_1 = 1 - w_2$ and find maximum of a quadratic function in w_1 with constraints. You should arrive to a solution $w_1 = \max\Big(0, 5/6 - 1/(12\alpha)\Big), w_2 = 1 - w_1$.

Exercise 5.3: Formulate (15), plug in $w_3 = 1 - w_1 - w_2$ and modify the problem statement so you arrive to an equivalent problem

$$\begin{aligned} \max_{\mathbf{w}} \quad & -6w_1^2 + w_2^2 + w_1 w_2 + 6w_1 - 8w_2 + 12 \\ \text{s.t.} \quad & w_1 \geq 0, w_2 \geq 0, w_1 + w_2 \leq 1, \end{aligned}$$

Solve this by Lagrange multipliers to obtain solution $w_1 = 1/2, w_2 = 0$ and hence $w_3 = 1/2$.

Exercise 6.1: semi-variance: $\sigma^2/2$, MAD: $\sigma\sqrt{2/\pi}$, VaR and CVaR see calculations for example here `http://blog.smaga.ch/expected-shortfall-closed-form-for-normal-distribution/`