In: Calculus of Variations, Applications and Computations. Proc. 2nd European Conf. on Elliptic and Parabolic Problems, Pont-‘a-Mousson, 1994. (Eds.: C.Bandle et al.) Pitman Res. Notes in Math. Sci. 326, Longmann, 1995, pp. 208-214.

TOMÁŠ ROUBÍČEK

A note about relaxation of vectorial variational problems

1. Introduction; the original variational problem and its relaxation.

The contribution deals with a vectorial variational problem

$$\text{(VP)} \qquad \int_\Omega \varphi(x, y(x), \nabla y(x))\mathrm{d}x \ \to \ \inf, \qquad y \in W^{1,p}(\Omega; \mathbb{R}^m) \ ,$$

where $\Omega \subset \mathbb{R}^n$ is a Lipschitz bounded domain and $\varphi : \Omega \times \mathbb{R}^m \times \mathbb{R}^{n\times m} \to \mathbb{R}$ is a potential density, supposed to be coercive Carathéodory function with a p-growth, $1 < p < +\infty$. The pecularity of the problem is that $\varphi(x, r, \cdot) : \mathbb{R}^{n\times m} \to \mathbb{R}$ is not supposed to be quasiconvex so that (VP) need not possess any solution and its extension (=relaxation) must be done.

We will basically use a continuous extension like the Young-measure setting, cf. [2, 3, 4]. We employ a suitable convex locally compact hull of $L^p(\Omega; \mathbb{R}^{n\times m})$, constructed as follows: Let us define the linear space of integrands $\mathrm{Car}^p(\Omega; \mathbb{R}^{n\times m}) = \{h : \Omega\times\mathbb{R}^{n\times m} \to \mathbb{R}$ Carathéodory; $|h(x, A)| \leq a(x) + b|A|^p,\ a \in L^1(\Omega), b \in \mathbb{R}\}$, equiped with the natural norm $\|h\| = \inf\{\|a\|_{L^1(\Omega)} + b;\ |h(x, A)| \leq a(x) + b|A|^p\}$. Furthermore, let us take a suitable (typically separable) subspace H of $\mathrm{Car}^p(\Omega; \mathbb{R}^{n\times m})$, and define the (norm,weak*)-continuous imbedding $i_H : L^p(\Omega; \mathbb{R}^{n\times m}) \to H^* : u \mapsto (h \mapsto \int_\Omega h(x, u(x))\mathrm{d}x)$. Supposing, as we may, that H contains a coercive integrand (i.e. $h = |A|^p \in H$), we may define

$$Y^p_H(\Omega; \mathbb{R}^{n\times m}) \ = \ \text{w*-cl } i_H(L^p(\Omega; \mathbb{R}^{n\times m})) \ ,$$

which is a convex, closed, locally compact, σ-compact hull of $L^p(\Omega; \mathbb{R}^{n\times m})$ if H^* is considered in its weak* topology. It is natural to address the elements of $Y^p_H(\Omega; \mathbb{R}^{n\times m})$ as “generalized Young functionals” because, for $H = L^1(\Omega; C_0(\mathbb{R}^{n\times m}))$ with C_0 denoting continuous functions vanishing at infinity, this set contains basically the funtionals introduced by L.C.Young in [20], which can be in this case identified with special elements of $L^\infty_{\mathrm{w}}(\Omega; M(\mathbb{R}^{n\times m})) \cong L^1(\Omega; C_0(\mathbb{R}^{n\times m}))^*$, called the Young measures; cf. [19] or also [1, 7, 17]. Nevertheless, the choice $H = L^1(\Omega; C_0(\mathbb{R}^{n\times m}))$ is not much suitable because such H cannot not contain coercive integrands. A more suitable example is rather the measures developed by DiPerna and Majda [8].

Now we can made readily a continuous extension of (VP). For $\eta \in H^*$ and $h \in H$ let us define $h \bullet \eta \in M(\bar{\Omega}) \cong C(\bar{\Omega})^*$ by $\langle h \bullet \eta, g\rangle = \langle \eta, g \cdot h\rangle$ to be valid for all $g \in C(\bar{\Omega})$. Furthemore, let us define the set of "gradient generalized Young functionals" by

$$G_H^p(\Omega;\mathbb{R}^{n\times m}) \;=\; \{\eta \in Y_H^p(\Omega;\mathbb{R}^{n\times m});\ \exists y_\xi \in W^{1,p}(\Omega;\mathbb{R}^m):\ i_H(\nabla y_\xi) \to \eta \text{ weakly*}\}\ .$$

We will suppose that φ is coercive in the sense

$$\varphi(x,r,s) \geq a(x) + b|s|^p \tag{1}$$

with some $a \in L^1(\Omega)$ and b positive, and that φ satisfies

$$\forall y \in L^q(\Omega;\mathbb{R}^m): \qquad \varphi \circ y \in H\ , \tag{2}$$

$$\exists a_1 \in L^1(\Omega)\ \exists b_1, c_1 \in \mathbb{R}^+: \quad |\varphi(x,r,s)| \;\leq\; a_1(x) + b_1|r|^q + c_1|s|^p\ , \tag{3}$$

$$\left.\begin{array}{l} \exists a_2 \in L^{q/(q-1)}(\Omega)\ \exists b_2, c_2 \in \mathbb{R}^+: \\ |\varphi(x,r_1,s) - \varphi(x,r_2,s)| \leq (a_2(x) + b_2|r_1|^{q-1} + b_2|r_2|^{q-1} + c_2|s|^{p(q-1)/q})|r_1 - r_2| \end{array}\right\} \tag{4}$$

with some $1 \leq q < np/(n-p)$ or $1 \leq q$ arbitrary provided $p \geq n$ (so that $W^{1,p}(\Omega;\mathbb{R}^m)$ is compactly imbedded into $L^q(\Omega;\mathbb{R}^m)$). Note that the assumptions (2)–(4) quarantees that the mapping $(y,\eta) \mapsto \langle \eta, \varphi \circ y\rangle$ is (weak$\times$weak*)-continuous. The relaxed problem will look like:

$$\text{(RP)} \qquad \left\{\begin{array}{ll} \text{minimize} & \langle \eta, \varphi \circ y\rangle \\ \text{subject to} & (1 \otimes \mathrm{id}) \bullet \eta = \nabla y\ , \\ & y \in W^{1,p}(\Omega;\mathbb{R}^m)\ , \quad \eta \in G_H^p(\Omega;\mathbb{R}^{n\times m}), \end{array}\right.$$

where we suppose $\varphi \circ y \in H$, with $[\varphi \circ y](x,A) = \varphi(x,y(x),A)$, and id: $\mathbb{R}^{n\times m} \to \mathbb{R}^{n\times m}$ being the identity. It can be shown (see [14]) that (RP) is actually a proper relaxation of (VP) in the sense that it has always a solution, inf(VP) = min(RP), every solution to (RP) is attainable by a minimizing net for (VP) and, conversely, every minimizing net (esp. sequence) for (VP) has a cluster point and each such a cluster point solves (RP).

The aim of this short note is to discuss various possibilities of a numerical approximation of the relaxed problem (RP) which has been also tested by computer experiments, though the results will not be presented here.

2. A finite-element approximation.

A first step in a numerical approximation of (RP) which we can certainly do quite easily consists in a finite-element approximation of (RP). Let us take a triangulation $\mathcal{T}_d$ of Ω, $d > 0$ being a mesh size, and put $V_d = \{y \in W^{1,p}(\Omega;\mathbb{R}^m);\ y \text{ piecewise affine on } \mathcal{T}_d\}$ and $Y_d \subset Y_H^p(\Omega;\mathbb{R}^{n\times m})$ defined by $Y_d = A_d^*(Y_H^p(\Omega;\mathbb{R}^{n\times m}))$ where $P_d : \mathrm{Car}^p(\Omega;\mathbb{R}^{n\times m}) \to \mathrm{Car}^p(\Omega;\mathbb{R}^{n\times m}) : h \mapsto h_d$ with

$h_d(x,A) = \int_\triangle h(\tilde{x},A)\mathrm{d}\tilde{x}/\mathrm{meas}(\triangle)$ for $x \in \triangle \in \mathcal{T}_d$; cf. also [13] for details. It allows us to define the approximate relaxed problem:

$$(\mathsf{RP}_d) \qquad \begin{cases} \text{minimize} & \langle \eta, \varphi \circ y \rangle \\ \text{subject to} & (1 \otimes \mathrm{id}) \bullet \eta = \nabla y \ , \\ & y \in V_d \ , \quad \eta \in G^p_H(\Omega; \mathbb{R}^{n\times m}) \cap Y_d \ , \end{cases}$$

Proposition 1. *Let φ satisfy (1)–(4). Then* (RP_d) *converges to* (RP) *in the sense:*

$$\lim_{d\to 0} \min(\mathsf{RP}_d) = \min(\mathsf{RP}) \ , \qquad \operatorname*{Limsup}_{d\to 0} \ \mathrm{Argmin}(\mathsf{RP}_d) \subset \mathrm{Argmin}(\mathsf{RP}) \ ,$$

where "Limsup" *(i.e. the upper Kuratowski limit) denotes the set of all cluster points of selected nets, and* "Argmin" *stands for the set of solutions to the problem indicated.*

Sketch of the proof. Since apparently the set of admissible pairs for (RP_d) is smaller than for (RP), we have $\min(\mathsf{RP}_d) \geq \min(\mathsf{RP})$.

Now we want to prove that every (y,η) admissible for (RP) can be approximated by suitable admissible pairs for (RP_d) when $d \to 0$. We can easily see that there is a net $\{y_\iota\}_{\iota\in I} \in W^{1,p}(\Omega;\mathbb{R}^m)$ such that $y_\iota \to y$ weakly in $W^{1,p}(\Omega;\mathbb{R}^m)$ and $i_H(\nabla y_\iota) \to \eta$ weakly* in H^*. Moreover, mollifing (if necessary) suitably this net, we can even suppose that $y_\iota \in C^\infty(\bar{\Omega})$. Let $\Pi_d y_\iota \in V_d$ be the linear interpolant of y_ι on the triangulation $\mathcal{T}_d$. For ι fixed and $d \to 0$, we have $\Pi_d y_\iota \to y_\iota$ strongly in $W^{1,p}(\Omega;\mathbb{R}^m)$ because of the regularity of y_ι. Therefore $i_H(\nabla \Pi_d y_\iota) \to i_H(\nabla y_\iota)$ weakly* in H^*. At the same time, the pair $(\Pi_d y_\iota, i_H(\nabla\Pi_d y_\iota))$ is admissible for (RP_d), and therefore

$$\langle i_H(\nabla \Pi_d y_\iota), \varphi \circ \Pi_d y_\iota \rangle \ \geq \ \min(\mathsf{RP}_d) \ \geq \ \min(\mathsf{RP}) \ .$$

Supposing that (y,η) is a solution of (RP), we get by the continuity argument (because (2)–(4) makes the mapping $(y,\eta) \mapsto \langle \eta, \varphi \circ y \rangle$ weakly$\times$weakly* continuous; cf. [17, Lemma 4.3.6]) that

$$\lim_{\iota\in I} \lim_{d\to 0} \langle i_H(\nabla \Pi_d y_\iota), \varphi \circ \Pi_d y_\iota \rangle \ = \ \langle y, \eta \circ y \rangle \ = \ \min(\mathsf{RP}) \ ,$$

so that $\min(\mathsf{RP}_d) \to \min(\mathsf{RP})$ for $d \to 0$.

The rest of the assertion follows immediately by the standard compactness arguments, taking into account the coercivity of the problem. $\square$

3. A further approximation of (RP_d).

The essential problem is that an explicit description of $G^p_H(\Omega;\mathbb{R}^{n\times m})$ is generally not known, which implies that the admissible domain of (RP_d), i.e.

$$\mathcal{D}(\mathsf{RP}_d) \ = \ \{(y,\eta) \in V_d \times (G^p_H(\Omega;\mathbb{R}^{n\times m}) \cap Y_d); \ (1 \otimes \mathrm{id}) \bullet \eta = \nabla y\} \ ,$$

is not effectively defined. A certain way to handle this problem is to confine ourselves to approximations of $\mathcal{D}(\mathsf{RP}_d)$. In principle, one can think either of an inner or of an outer approximation of it.

An inner approximations of $\mathcal{D}(\mathsf{RP}_d)$ has been recently proposed by Nicolaides and Walkington [11]. Namely, for $k \in \mathbb{N}$ they defined

$$\begin{aligned}
\mathcal{D}_k(\mathsf{RP}_d) \;=\; & \{(y,\eta) \in V_d \times Y_d;\ (1 \otimes \mathrm{id}) \bullet \eta = \nabla y, \\
& \langle \eta, h \rangle = \sum_{i=1}^{2^k} \int_\Omega \lambda_i(x) h(x, u_i(x)) \mathrm{d}x, \qquad \lambda_i,\ u_i \text{ elementwise constant}, \\
& \lambda_i = \prod_{j=1}^{k} \lambda_{[i/j]+1,j}, \qquad u_i = A_{ik}, \qquad \lambda_{2i,j} A_{2i,j} + \lambda_{2i-1,j} A_{2i-1,j} = A_{i,j-1}, \\
& \lambda_{2i,j} + \lambda_{2i-1,j} = 1, \qquad \lambda_{2i,j}, \lambda_{2i-1,j} \geq 0, \quad \mathrm{Rank}(A_{2i,j} - A_{2i-1,j}) \leq 1, \\
& i = 1, ..., 2^{j-1}, \quad j = 1, ..., k, \qquad A_{i,1} = \nabla y\} \ .
\end{aligned}$$

In other words, the approximation $\mathcal{D}_k(\mathsf{RP}_d) \subset \mathcal{D}(\mathsf{RP}_d)$ consists of "element-wise constant" generalized Young functionals composed, on each element, from 2^k pairwise rank-1 connected matrices, which can be certainly reached by gradients. The essential theoretical disadvantage of this approximation is that, in general, $\overline{\bigcup_{k \in N} \mathcal{D}_k(\mathsf{RP}_d)} \neq \mathcal{D}(\mathsf{RP}_d)$ because otherwise the minimum of the functional $(y, \eta) \mapsto \langle \eta, \varphi \circ y \rangle$ over $\mathcal{D}_k(\mathsf{RP}_d)$ would have to approach $\min(\mathsf{RP})$ for $k \to \infty$ but, as shown by Dacorogna [7, Sect.5.1.1.2], it converges from above only to Rank-1 envelope of (VP). Therefore, if $\varphi(x, r, \cdot)$ has a quasiconvex envelope which is not rank-1 convex, then the approximation proposed by Nicolaides and Walkington cannot converge.

Nevertheless, we can also use an outer approximation of $\mathcal{D}(\mathsf{RP}_d)$. Let us put $\Xi = \{\xi = (v_1, ..., v_k);\ k \in \mathbb{N},\ v_j : \mathbb{R}^{n \times m} \to \mathbb{R}$ quasiconvex, $|v_j(s)| \leq o(|s|^p)\}$, ordered by the inclusion. This makes Ξ a directed set so that we can use it to index generalized sequences (=nets). For any $\xi \in \Xi$ we put

$$\mathcal{D}^\xi(\mathsf{RP}_d) \;=\; \{(y,\eta) \in V_d \times Y_d;\ \forall v \in \xi :\ (1 \otimes v) \bullet \eta \geq v(\nabla y)\} \ .$$

Always, $\mathcal{D}^\xi(\mathsf{RP}_d) \supset \mathcal{D}(\mathsf{RP}_d)$ and, as a consequence of recent results by Kinderlehrer and Pedregal [10], also $\mathcal{D}(\mathsf{RP}_d) \supset \bigcap_{\xi \in \Xi} \mathcal{D}^\xi(\mathsf{RP}_d) \cap \{\eta \in Y_H^p(\Omega; \mathbb{R}^{n \times m})$ is p-nonconcentrating$\}$, where "p-nonconcentrating" means that $\eta \in Y_H^p(\Omega; \mathbb{R}^{n \times m})$ is attainable by a sequence $\{i_H(u_k)\}_{k \in \mathbb{N}}$ such that the set $\{|u_k|\}_{k \in \mathbb{N}}$ is relatively weakly compact in $L^1(\Omega)$. This result suggests the following approximate problem:

$$(\mathsf{RP}_d^\xi) \qquad\qquad \text{Minimize } \ \langle \eta, \varphi \circ y \rangle \ \text{ s.t. } (y, \eta) \in \mathcal{D}^\xi(\mathsf{RP}_d) \ .$$

Proposition 2. *Let φ satisfy (1)–(4), H be separable and there is $G \otimes V$ dense in H with $G \subset L^\infty(\Omega)$ and $V \subset C(\mathbb{R}^{n \times m})$, and $\forall v \in V\ \exists v_l \in V$ with a growth strictly less than p such that*

$v_l \to v$ uniformly on bounded subsets of $\mathbb{R}^{n\times m}$. Then

$$\lim_{\xi\in\Xi} \min(\mathsf{RP}_d^\xi) = \min(\mathsf{RP}_d) , \qquad \operatorname*{Limsup}_{\xi\in\Xi} \ \mathrm{Argmin}(\mathsf{RP}_d^\xi) \subset \mathrm{Argmin}(\mathsf{RP}_d) .$$

In other words, if (y_ξ, η_ξ) solves (RP_d^ξ), then the net $\{(y_\xi, \eta_\xi)\}_{\xi\in\Xi}$ has a (weak$\times$weak)-cluster point (y, η) in $W^{1,p}(\Omega; \mathbb{R}^m) \times H^*$ and each such a cluster point solves (RP_d).*

Sketch of the proof. As $\min(\mathsf{RP}_d^\xi)$ is certainly bounded from above (e.g. by $\langle i_H(0), \varphi\circ 0\rangle < +\infty$) and the coercivity (1) is assumed, the net in question is bounded and therefore it has a (weak$\times$weak*)-cluster point (y, η). We want to show that (y, η) must solve the auxiliary problem

$$(\mathsf{AP}_d) \qquad \begin{cases} \text{minimize} & \langle \eta, \varphi\circ y\rangle \\ \text{subject to} & (1\otimes v)\bullet\eta \geq v(\nabla y) \quad \forall v \text{ quasiconvex with a growth } < p, \\ & y \in V_d , \quad \eta \in Y_d . \end{cases}$$

As certainly $\min(\mathsf{RP}_d^\xi) \leq \min(\mathsf{AP}_d)$ and the mapping $\xi \mapsto \min(\mathsf{RP}_d^\xi)$ is nondecreasing, we have guaranteed $\lim_{\xi\in\Xi} \min(\mathsf{RP}_d^\xi) \leq \min(\mathsf{AP}_d)$. Supposing $\lim_{\xi\in\Xi} \min(\mathsf{RP}_d^\xi) < \min(\mathsf{AP}_d)$, by the coercivity of the problem we could choose a finer net than $\{(y_\xi, \eta_\xi)\}_{\xi\in\Xi}$ converging to some (y_0, η_0) satisfying all the constraints involved in (AP) but such that $\langle \eta_0, \varphi\circ y_0\rangle < \min(\mathsf{AP}_d)$, which is a contradiction. Therefore, $\lim_{\xi\in\Xi} \min(\mathsf{RP}_d^\xi) = \min(\mathsf{AP}_d)$ and then also (y, η) must solve (AP_d).

Then η must be p-nonconcentrating in the sense that there is a net $\{u_\alpha\}$ bounded in $L^p(\Omega; \mathbb{R}^{n\times m})$ such that $i_H(u_\alpha) \to \eta$ and the set $\{|u_\alpha|^p\}$ is relatively weakly compact in $L^1(\Omega)$. Indeed, if it would not be the case, the p-nonconcentrating modification of η would reach a strictly lower cost than η and all the constraints of (AP_d) would by satisfied as well, which is a contradiction; we refer to [16] for details.

As $p > 1$ is supposed, the inequality constraints of (AP_d) include, in particular, also the constraint $(1\otimes \mathrm{id})\bullet\eta = \nabla y$ involved in (RP_d). Therefore, to prove that (y, η) solves also (RP_d), it suffices to show that $\min(\mathsf{AP}_d) \leq \min(\mathsf{RP}_d)$ (which, however, follows immediately from $\mathcal{D}(\mathsf{AP}_d) \supset \mathcal{D}(\mathsf{RP}_d)$) and that $\eta \in G_H^p(\Omega; \mathbb{R}^{n\times m})$.

First, we can localize our considerations on a current element so that it suffices to investigate only homogeneous Young functionals. As in [13], one can show that η cannot be separated from the set $M_y = \{\eta \in G_H^p(\Omega; \mathbb{R}^{n\times m});\ (1\otimes\mathrm{id})\bullet\eta = \nabla y\}$ by any test function with the growth strictly less than p. However, taking a general $1\otimes v$ and v^l with growth strictly less than p and such that $v^l \to v$ uniformly on bounded sets in $\mathbb{R}^{n\times m}$, then one can show that $\langle \eta, 1\otimes v^l\rangle \to \langle \eta, 1\otimes v\rangle$ for any $\eta \in Y_H^p(\Omega; \mathbb{R}^{n\times m})$ p-nonconcentrating; cf. [13, Example 3.1]. This shows that η in question cannot be separated from the closed convex set M_y by any test function of the form $1\otimes v$, and therefore also by any $\sum_{\text{finite}} g_i \otimes v_i$, so that it must belong to $G_H^p(\Omega; \mathbb{R}^{n\times m})$ which is closed. $\square$

Though having a convergence guaranteed, the fatal disadvantage of the previous method is that the index set Ξ is very large and not effectively defined, so that it has a theoretical significance only.

Anyhow, both approximate methods mentioned above certainly give a general two-side estimate:

$$\min_{(y,\eta)\in\mathcal{D}^{\xi}(\mathsf{RP}_d)}\langle\eta,\varphi\circ y\rangle\ \le\ \min(\mathsf{RP}_d)\ \le\ \min_{(y,\eta)\in\mathcal{D}_k(\mathsf{RP}_d)}\langle\eta,\varphi\circ y\rangle\ . \tag{5}$$

Nevertheless, for practically reasonable indices $\xi\in\Xi$ and $k\in\mathbb{N}$, this estimate might be still very rough. Therefore, it is reasonable to inquire special situations where possibly the equalities for ξ or k large enough can appear.

As an example let us mention the case $\xi=\{\pm\mathrm{adj}_l;\ l=1,...,\min(n,m)\}$. Then the first equality in (5) takes place provided $\varphi(x,r,\cdot):\mathbb{R}^{n\times m}\to\mathbb{R}$ has a polyconvex quasiconvexification. In this case, one can even derive (cf. [15]) optimality conditions for (RP_d^{ξ}), which takes the form

$$\mathcal{H}_{y,\lambda}\bullet\eta\ =\ \max_{A\in\mathbb{R}^{n\times m}}\mathcal{H}_{y,\lambda}(x,A)$$

with the "discrete Hamiltonian" $\mathcal{H}_{y,\lambda}(x,A)=-P_d\varphi(x,y(x),A)+\sum_{l=1}^{\min(n,m)}\lambda_l(x)\cdot\mathrm{adj}_lA$ and with $\lambda_l\in L^{p/l}(\Omega;\mathbb{R}^{\sigma(l)})$, $\sigma(l)=\binom{m}{l}\binom{n}{l}$, satisfying

$$\mathrm{div}\left(\sum_{l=1}^{\min(n,m)}\lambda_l\cdot\frac{\partial\mathrm{adj}_l}{\partial A}(\nabla y)\right)\ =\ (\frac{\partial\varphi}{\partial y}\circ y)\bullet\eta\ .$$

From these conditions we can deduce that there always exists a minimizer in the form of a convex combinations of at most $\prod_{l=1}^{\min(n,m)}\sigma(l)+2$ atoms on each element, which eventually allows an effective computer implementation of (RP_d^{ξ}).

Remark 1. The requirement on $\varphi(x,r,\cdot)$ to have a polyconvex quasiconvexification is certainly not realistic in a general case so that one is forced to try to take larger indices ξ. Each such a choice gives some problem whose minimum is in between the polyconvexified problem and $\inf(\mathsf{VP})$. This is philosophically similar to the approach by Firoozye [9] who also proposed some envelope with such property.

Remark 2. A general feature of the resulted approximate problems is that they admit a partial decomposition, having always the form of a co-operative Stackelberg game. Namely, the leader controls the displacement y, seeking the minimum of the total energy, while the followers (one on each element) seek the minimum of the deformation energy on a current element for ∇y set up by the leader. Thus each follower is to solve repeatedly convex problems parametrized by ∇y.

Remark 3. If H is small enough it may happen that, for every solution (y,η) to (RP_d^{ξ}), η belongs to $G_H^p(\Omega;\mathbb{R}^{n\times m})$. It immediately implies that (y,η) solves also (RP_d) provided ξ contains at least linear functions adj_1. For example, if $\varphi(x,r,s)=v_0(s)$ with v_0 having a polyconvex quasiconvexification, then this feature takes place if $H=G\otimes V$ with V being contained in the linear hull of all minors and v_0; cf. [13].

References

[1] Ball, J.M.: A version of the fundamental theorem for Young measures. In: *PDEs and Continuum Models of Phase Transition.* (Eds. M.Rascle, D.Serre, M.Slemrod.) Lecture Notes in Physics **344**, Springer, Berlin, 1989, pp.207–215.

[2] Ball, J.M., James, R.D.: Fine phase mixtures as minimizers of energy. *Archive Rat. Mech. Anal.* **100** (1988), 13–52.

[3] Ball, J.M., James, R.D.: Proposed experimental tests of a theory of fine microstructure and the two-well problem. *Phil. Trans. Royal Soc. London* A **338** (1992), 389–450.

[4] Chipot, M., Kinderlehrer, D.: Equilibrium configurations of crystals. *Arch. Rational Mech. Anal.* **103** (1988), 237–277.

[5] Collins, C., Luskin, M.: Numerical modelling of the microstructure of crystals with symmetry-related variants. In: *US-Japan Workshop on Smart/Inteligent Materials and Systems* (Eds. I.Ahmad et al.), Technomic Publ. Comp., Lancaster, 1990, pp.309–318.

[6] Collins, C., Luskin, M., Riordan J.: Computational results for a two-dimensional model of crystalline microstructure. In: IMA Vol. in Math. and Applications *Microstructure and Phase Transitions* (Eds. R.James, D.Kinderlehrer, M.Luskin), Springer, New York, to appear.

[7] Dacorogna, B.: *Direct Methods in the Calculus of Variations*, Springer, Berlin, 1989.

[8] DiPerna, R.J., Majda, A.J.: Oscillations and concentrations in weak solutions of the incompressible fluid equations. *Comm. Math. Physics* **108** (1987), pp.667–689.

[9] Firoozye, N.B.: Optimal use of the translation method and relaxations of variational problems. *Comm. Pure Appl. Math.* **44** (1991), 643–678.

[10] Kinderlehrer, D., Pedregal, P.: Remarks about gradient Young measures generated by sequences in Sobolev spaces. *Carnegie-Mellon Research Report* No. 92-NA-007, March 1992.

[11] Nicolaides, R.A., Walkington, N.J.: Computation of microstructure utilizing Young measure representations. Res. Report 92-NA-001, Center for Nonlinear Analysis, Carnegie Mellon University, 1992.

[12] Roubíček, T.: Convex compactifications and special extensions of optimization problems. *Nonlinear Analysis, Th., Methods, Appl.* **16** (1991), 1117–1126.

[13] Roubíček, T.: Effective characterization of generalized Young measures generated by gradients. *Boll. Unione Mat. Italiana* (accepted)

[14] Roubíček, T.: Relaxation of vectorial variational problems. Rapport de Recherche UMPA-93-no.88, ENS Lyon, 1993 (submitted)

[15] Roubíček, T.: Numerical approximation of relaxed variational problems. (submitted)

[16] Roubíček, T.: Nonconcentrating generalized Young functionals. (submitted)

[17] Roubíček, T.: *Relaxation in Optimization Theory and Variational Calculus.* W. de Gruyter, Berlin (in preparation)

[18] Roubíček, T., Hoffmann, K.-H.: Theory of convex local compactifications with applications to Lebesgue spaces. *Nonlinear Analysis, Th., Methods, Appl.* (accepted)

[19] Young, L.C.: Generalized curves and the existence of an attained absolute minimum in the calculus of variations. *Comptes Rendus de la Société des Sciences et des Lettres de Varsovie*, Classe III **30** (1937), 212–234.

[20] Young, L.C.: Generalized surfaces in the calculus of variations. *Ann. Math.* **43** (1942), part I: 84–103, part II: 530–544.

Author's address:
Institute of Information Theory and Automation,
Academy of Sciences of the Czech Republic,
Pod vodárenskou věží 4,
CZ-182 08 Praha 8, Czech Republic.