OPTIMIZATION OF STEADY FLOWS FOR INCOMPRESSIBLE VISCOUS FLUIDS

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Abstract: An optimal-control problem for the stationary Navier-Stokes system are investigated. The maximum principle is derived by a suitable relaxation. Its sufficiency is shown provided data involved in the control problem are small enough (depending on the Reynolds number). Regularity of the Navier-Stokes system and its adjoint problem is used.

Keywords: Navier-Stokes equations, regularity, optimal control, existence, relaxation, maximum principle, sufficiency.

Introduction

In this paper we deal with optimization of steady two- and three-dimensional fluid flows governed by the Navier-Stokes system. Analysis of the various problems of optimal control of viscous flows enjoys recently significant attention within mathematical comunity. Optimal control problem of this sort was already studied in [1, 2, 3, 7, 18, 19, 20] and [25, Section III.11]. For an optimal shape design problem see [21]. Besides, optimal control of evolutionary Navier-Stokes system was treated in [4, 5, 8, 9, 10, 11, 12, 13, 14, 22, 29, 32, 33, 34, 35] and also in [25, Section I.18].

Our main goal is to adapt the relaxation method by convex compactification [30] for Navier-Stokes equations and to exploit (quite standard) regularity results for the stationary (linearized) Navier-Stokes system to derive nontrivial results concerning sufficiency of the maximum principle.

The scheme of the paper is the following. In Section 1, we specify an optimal-control problem $(\mathcal{P})$ we will deal with, and in Section 2 we pose the relaxed problem to $(\mathcal{P})$ and show its correctness under the assumption that a driving force is sufficiently small so that the state response is uniquely defined. In Section 3, we confine ourselves to a special form of the data and derive the corresponding maximum principle (i.e. necessary condition of optimality), and show that this maximum principle forms a sufficient condition provided that the desired velocity profile and the driving force are small enough (depending on the Reynolds number), which ensures that the relaxed cost functional is "enough" uniformly convex with respect to the state; cf. Remark 4 below. For this purpose, L^q-regularity results for a dual (adjoint) equation to the linearized Navier-Stokes system are exploited.

We wish to remark that the result presented here can be extended to the power-law-like fluids in two dimensions (at least for $p > \frac{3}{2}$, where p denotes the power-law exponent, see [26] for more details). While the regularity results applied here to the Navier-Stokes system are well traced in the literature, their extension to the power-law fluids is possible because of recent nontrivial $C^{1,\alpha}$-regularity results for this class of fluids performed in [24] (see also contribution of the same authors in this volume); the method is based on the approach introduced in [27] and [28] but will not be presented in this paper.

1. AN OPTIMAL-CONTROL PROBLEM

We will confine ourselves to steady flows of an incompressible fluid in a two- or three-dimensional bounded domain $\Omega \subset \mathbb{R}^n$ (i.e. $n = 2$ or 3) with no-slip (i.e. homogeneous Dirichlet) boundary condition.

We will first deal with the following optimal control problem for flows governed by the Navier-Stokes system:

$$(\mathcal{P}) \qquad \begin{cases} \text{Minimize} & J(z,u) := \displaystyle\int_\Omega h(x,u(x),z(x))\,\mathrm{d}x \\ \text{subject to} & (u\cdot\nabla)u - \mu\Delta u + \nabla p = f(\cdot,u,z) \qquad \text{on } \Omega, \\ & \operatorname{div} u = 0 \qquad \text{on } \Omega, \\ & z(x) \in S(x) \qquad \text{for a.a. } x \in \Omega, \\ & u \in W_0^{1,2}(\Omega;\mathbb{R}^n), \quad p \in L_0^2(\Omega), \quad z \in L^q(\Omega;\mathbb{R}^m). \end{cases}$$

Here, z denotes the control, u represents the velocity field and p is the pressure. Not completely rigorous but frequently used notation $(u\cdot\nabla)u$ means $\sum_{k=1}^n u_k \frac{\partial u}{\partial x_k}$. By $\mu > 0$ we denote a fluid viscosity which is indirectly proportional to the Reynolds number. Further, $h : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are given Carathéodory functions. Finally, $S : \Omega \rightrightarrows \mathbb{R}^m$ is a given multi-valued function forming the control constraints.

We use standard notation of function spaces: $C(\cdot)$ for the spaces of bounded continuous functions; $C_0(\mathbb{R}^m)$ for the space of continuous functions on $\mathbb{R}^m$ vanishing at infinity; $L^q(\Omega)$, $q \in [1,\infty]$, for the

Lebesgue spaces and $W_0^{1,q}(\Omega)$ for the Sobolev spaces having zero trace at the boundary $\partial\Omega$. The corresponding vector-valued spaces are denoted by $L^q(\Omega;\mathbb{R}^n)$ and $W_0^{1,q}(\Omega;\mathbb{R}^n)$, respectively. By (g,f) we mean $\int_\Omega g(x)\cdot f(x)\mathrm{d}x$. Finally we use the shorthand notation $L_0^q(\Omega)$ and $W_{0,\mathrm{DIV}}^{1,q}(\Omega;\mathbb{R}^n)$ for subspaces of zero-mean-value functions in $L^q(\Omega)$ and divergence-free functions in $W_0^{1,q}(\Omega;\mathbb{R}^n)$, respectively, i.e.

$$L_0^q(\Omega) := \{p \in L^q(\Omega);\ \int_\Omega p\,\mathrm{d}x = 0\}\,, \tag{1.1a}$$

$$W_{0,\mathrm{DIV}}^{1,q}(\Omega;\mathbb{R}^n) := \{u \in W_0^{1,q}(\Omega;\mathbb{R}^n); \operatorname{div} u = 0\}\,. \tag{1.1b}$$

Of course, the solution (u,p) to the Navier-Stokes system in $(\mathcal{P})$ is understood in the weak sense, which means that $u \in W_{0,\mathrm{DIV}}^{1,2}(\Omega;\mathbb{R}^n)$ and, for a given z,

$$((u\cdot\nabla)u,v) + \mu(\nabla u,\nabla v) = (f(u,z),v) \quad \forall v \in W_{0,\mathrm{DIV}}^{1,2}(\Omega;\mathbb{R}^n)\,. \tag{1.2}$$

The basic data qualification we will need are the following:

$$|h(x,r,s)| \le a_1(x) + \beta(|r|) + c|s|^q, \tag{1.3a}$$

$$|h(x,r_1,s) - h(x,r_2,s)| \le (\tilde{a}_1(x) + \beta(\max(|r_1|,|r_2|)) + c|s|^q)|r_1 - r_2|, \tag{1.3b}$$

$$h(x,r,s) \ge c_0|s|^q, \tag{1.3c}$$

$$|f(x,r,s)| \le a_2(x), \tag{1.3d}$$

$$|f(x,r_1,s) - f(x,r_2,s)| \le (\tilde{a}_2(x) + \beta(\max(|r_1|,|r_2|)))|r_1 - r_2|, \tag{1.3e}$$

S admits a measurable q-integrable selection, (1.3f)

where $\beta : \mathbb{R}^+ \to \mathbb{R}^+ := [0,\infty)$ is a continuous increasing function with $\beta(0) = 0$, $a_1, \tilde{a}_1 \in L^1(\Omega)$, $a_2, \tilde{a}_2 \in L^2(\Omega)$, $c \in \mathbb{R}^+$, and $c_0 > 0$.

Remark 1. Let us recall that, having a solution $u \in W_{0,\mathrm{DIV}}^{1,2}(\Omega;\mathbb{R}^n)$ satisfying (1.2) and assuming that f fulfills (1.3d), it is standard (see [35]) to construct the corresponding pressure $p \in L_0^2(\Omega)$ such that

$$((u\cdot\nabla)u,v) + \mu(\nabla u,\nabla v) - (p,\operatorname{div} v) = (f(u,z),v) \tag{1.4}$$

for all $v \in W_0^{1,2}(\Omega;\mathbb{R}^n)$. We will involve the pressure only in the formulations of the theorems and lemmas but not in the proofs, because p can always be reconstructed uniquely if one knows that $u \in W_{0,\mathrm{DIV}}^{1,2}(\Omega;\mathbb{R}^n)$ satisfies (1.2),

Remark 2. Note that (1.3d) leads, just by taking $v := u$ in (1.2), to the energy estimate

$$\|\nabla u\|_{L^2(\Omega;\mathbb{R}^{n\times n})} \le \frac{k_0}{\mu}\|a_2\|_{L^2(\Omega)}\ , \tag{1.5}$$

where the constant k_0 comes from the Poincaré inequality.

2. RELAXED PROBLEM

We will extend continuously the problem $(\mathcal{P})$ on a suitable convex locally (sequentially) compact envelope of the set of admissible controls

$$Z_{\rm ad} := \{z \in L^q(\Omega;\mathbb{R}^m);\ z(x) \in S(x) \text{ for a.a. } x \in \Omega\}; \tag{2.1}$$

note that (1.3f) just means that $Z_{\rm ad} \neq \emptyset$. To do this, we take a suitable linear space of Carathéodory integrands containing all possible nonlinearities occurring in the problem $(\mathcal{P})$, e.g.

$$H := \operatorname{span}\Big\{ g_1 \cdot (h \circ u) + g_2 \cdot (f \circ u);\\ g_1 \in C(\bar\Omega),\quad g_2 \in L^2(\Omega;\mathbb{R}^n),\ u \in W^{1,2}(\Omega;\mathbb{R}^n)\Big\}, \tag{2.2}$$

where $[g_1 \cdot (h \circ u)](x,s) := g_1(x)h(x,u(x),s)$, and similarly $[g_2 \cdot (f \circ u)](x,s) := g_2(x) \cdot f(x,u(x),s)$. It is natural to equip H by

$$\|h\|_H := \inf_{\substack{\{a \in L^1(\Omega),\, c \in \mathbb{R};\ \forall x \in \Omega,\ s \in \mathbb{R}^m \\ |h(x,s)| \le a(x) + c|s|^q\ \}}} \|a\|_{L^1(\Omega)} + c\,, \tag{2.3}$$

which is a norm (see [30, Example 3.4.13]) making H separable (see [31, Lemma 1]). Then we imbed $L^q(\Omega;\mathbb{R}^m)$ (norm,weak*)-continuously into H^* by

$$i : z \mapsto \left(h \mapsto \int_\Omega h(x,z(x))\,\mathrm{d}x\right) \tag{2.4}$$

and define the set of the so-called generalized Young functionals by $Y_H^q(\Omega;\mathbb{R}^m) := \text{w*-cl}\, i(L^q(\Omega;\mathbb{R}^m))$. It is known (cf. [30]) that, as a consequence of (2.2) with (1.3a–e), $Y_H^q(\Omega;\mathbb{R}^m)$ is a convex locally (sequentially) compact envelope of $L^q(\Omega;\mathbb{R}^m)$. The set of admissible relaxed controls is then defined by

$$\bar Z_{\rm ad} := \text{w*-cl}\, i(Z_{\rm ad}). \tag{2.5}$$

Thanks to the special form (2.1), also the set $\bar Z_{\rm ad}$ is convex and locally compact in H^* if the weak* topology on H^* is considered.

We will need a continuous extension of the Nemytskiĭ mapping $z \mapsto f_0(x,z(x)) : L^q(\Omega;\mathbb{R}^m) \to L^1(\Omega;\mathbb{R}^{m_1})$ with some $f_0 : \Omega \times \mathbb{R}^m \to \mathbb{R}^{m_1}$ satisfying $|f_0(x,s)| \le a(x) + c|s|^q$ for some $a \in L^1(\Omega)$ and $c \in \mathbb{R}$. This extension is defined by

$$f_0 \bullet \eta \in \operatorname{rca}(\bar\Omega;\mathbb{R}^{m_1}) \cong C(\bar\Omega;\mathbb{R}^{m_1})^* : \tag{2.6}$$
$$\int_{\bar\Omega} g(x)[f_0 \bullet \eta](\mathrm{d}x) \equiv \langle f_0 \bullet \eta, g\rangle = \langle \eta, g \cdot f_0\rangle$$

for any $g \in C(\bar\Omega;\mathbb{R}^{m_1})$, where $[g \cdot f_0](x,s) := \sum_{k=1}^{m_1} g_k(x) f_{0k}(x,s)$. Note that, due to (2.2), $g \mapsto g \cdot f_0 : C(\bar\Omega;\mathbb{R}^{m_1}) \to H$ is continuous because $\|g \cdot f_0\|_H \le \|g\|_{C(\bar\Omega;\mathbb{R}^{m_1})}\|f_0\|_H$ and $\eta \mapsto f_0 \bullet \eta$ is linear; cf. [30, Example 3.6.3]. Obviously, $f_0 \bullet i(z) = f_0(z)$. We will use this extension for $f_0 := h \circ u$ (and $m_1 := 1$) and also for $f_0 := f \circ u$ (and $m_1 := n$). In the later case, we always have $(f \circ u) \bullet \eta \in L^2(\Omega)$ due to (1.3d).

Then the continuous extension of the original problem $(\mathcal{P})$ looks naturally as follows:

$$(\mathcal{RP}) \quad \begin{cases} \text{Minimize} & \bar J(\eta, u) := \int_{\bar\Omega} [(h \circ u) \bullet \eta](\mathrm{d}x) \\ \text{subject to} & (u \cdot \nabla)u - \mu\Delta u + \nabla p = (f \circ u) \bullet \eta \ , \\ & \operatorname{div} u = 0, \\ & u \in W_0^{1,2}(\Omega;\mathbb{R}^n),\ p \in L_0^2(\Omega),\ \eta \in \bar Z_{\mathrm{ad}} \subset Y_H^q(\Omega;\mathbb{R}^m). \end{cases}$$

Again, by (u,p) we understand a weak solution, which means analogously to (1.2) that $u \in W^{1,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n)$, $p \in L_0^2(\Omega)$, and, for a given $\eta \in Y_H^q(\Omega;\mathbb{R}^m)$, the following identity holds:

$$((u \cdot \nabla)u, v) + \mu(\nabla u, \nabla v) = \langle \eta, v \cdot (f \circ u)\rangle \quad \forall v \in W^{1,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n), \tag{2.7}$$

or, in accord with Remark 1, for all $v \in W^{1,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n)$,

$$((u \cdot \nabla)u, v) + \mu(\nabla u, \nabla v) - (p, \operatorname{div} v) = \langle \eta, v \cdot (f \circ u)\rangle. \tag{2.8}$$

Next lemma shows that, in fact, the solution to (2.7) is regular and the Navier-Stokes system in $(\mathcal{RP})$ holds almost everywhere. Moreover, assuming certain condition on the smallness of a_2, $\tilde a_2$ and β occurring at (1.3d,e) we obtain the continuous dependence of u on η. Note that $W^{2,2}$-regularity of the velocity will be used in assumption (1.3b,e) (because β may have an arbitrary growth) and in other places, too.

Lemma 1. *Let Ω be a C^2-domain and (1.3d) hold. Let $u \in W^{1,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n)$ denote the solution to the Navier-Stokes system with the relaxed control $\eta \in \bar Z_{\mathrm{ad}}$. Then*

$$\forall \eta \in \bar Z_{\mathrm{ad}}:\ \|u\|_{L^\infty(\Omega;\mathbb{R}^n)} \le c\,\|u\|_{W^{2,2}(\Omega;\mathbb{R}^n)} \le C \equiv C(\Omega, \|a_2\|_{L^2(\Omega)}). \tag{2.9}$$

Moreover, let also (1.3e) hold and u^1 and $u^2 \in W^{1,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n)$ be two solutions to the Navier-Stokes system with relaxed controls η^1 and $\eta^2 \in \bar Z_{\mathrm{ad}}$, respectively. Then

$$\|u^1 - u^2\|_{W^{1,2}(\Omega;\mathbb{R}^n)} \le C_0\|\eta^1 - \eta^2\|_{H^*} \tag{2.10}$$

provided that a_2, β and c occurring at (1.3d,e) satisfy

$$k_1\|\tilde a_2\|_{L^2(\Omega)} + k_0^2\beta(C(\|a_2\|_{L^2(\Omega)})) + \frac{k_0k_1}{\mu}\|a_2\|_{L^2(\Omega)} < \mu\,, \tag{2.11}$$

where k_0 comes from the Poincaré inequality (cf. Remark 2) and k_1 comes from the inequality

$$\|\omega\|_{L^4(\Omega;\mathbb{R}^n)} \leq \sqrt{k_1}\|\nabla\omega\|_{L^2(\Omega;\mathbb{R}^{n\times n})}\,. \tag{2.12}$$

Proof. Due to (1.3d), the right-hand side $(f\circ y)\bullet\eta$ of the relaxed Navier-Stokes equation (2.7) is bounded in $L^2(\Omega,\mathbb{R}^n)$ if η ranges $Y_H^q(\Omega;\mathbb{R}^m)$. Then we can directly use nowdays standard regularity approach to the stationary Navier-Stokes equations (cf. [16], or [6]) to obtain (2.9).

Next, let u^1 and u^2 solve the identity (2.7) with $\eta := \eta^1$ and $\eta := \eta^2$, respectively. Subtracting these identities and putting $v := u^1 - u^2$ gives

$$\begin{aligned}\mu\|\nabla u^1-\nabla u^2\|^2_{L^2(\Omega;\mathbb{R}^{n\times n})} &= \langle\eta^1,(u^1-u^2)\cdot(f\circ u^1)\rangle-\langle\eta^2,(u^1-u^2)\cdot(f\circ u^2)\rangle\\ &\quad+((u^2\cdot\nabla)u^2,u^1-u^2)-((u^1\cdot\nabla)u^1,u^1-u^2)\\ &= \langle\eta^1,(u^1-u^2)\cdot[(f\circ u^1)-(f\circ u^2)]\rangle+\langle\eta^1-\eta^2,(u^1-u^2)\cdot(f\circ u^2)\rangle\\ &\quad+((u^2-u^1)\cdot\nabla)u^2,u^1-u^2)+((u^1\cdot\nabla)(u^2-u^1),u^1-u^2).\end{aligned}$$

Due to divergence-free constraint, the last term vanishes. The other terms are estimated by means of (1.3d,e). Thus, we obtain

$$\begin{aligned}\mu\|\nabla u^1-\nabla u^2\|^2_{L^2(\Omega;\mathbb{R}^{n\times n})} &\leq \|\tilde a_2\|_{L^2(\Omega)}\|u^1-u^2\|^2_{L^4(\Omega;\mathbb{R}^n)}\\ &\quad+\beta(\max(\|u^1\|_{L^\infty(\Omega;\mathbb{R}^n)},\|u^2\|_{L^\infty(\Omega;\mathbb{R}^n)}))\|u^1-u^2\|^2_{L^2(\Omega;\mathbb{R}^n)}\\ &\quad+\|\eta^1-\eta^2\|_{H^*}\|u^1-u^2\|_{L^2(\Omega;\mathbb{R}^n)}\|a_2\|_{L^2(\Omega)}\\ &\quad+\|\nabla u^2\|_{L^2(\Omega;\mathbb{R}^{n\times n})}\|u^1-u^2\|^2_{L^4(\Omega;\mathbb{R}^n)}\ ,\end{aligned}$$

where we used also the estimate

$$\begin{aligned}\Big\|(u^1-u^2)\cdot(f\circ u^2)\Big\|_H &\leq \Big\||u^1-u^2|\,a_2\Big\|_{L^1(\Omega)}\\ &\leq \|u^1-u^2\|_{L^2(\Omega;\mathbb{R}^n)}\|a_2\|_{L^2(\Omega)},\end{aligned}$$

which follows by the Hölder inequality from (1.3d) and (2.3). By (2.9), $\|u^i\|_{L^\infty(\Omega,\mathbb{R}^n)} \leq C = C(\|a_2\|_{L^2(\Omega)})$ for $i=1,2$. As β is increasing we see that $\beta(\max(\|u^1\|_{L^\infty(\Omega;\mathbb{R}^n)},\|u^2\|_{L^\infty(\Omega;\mathbb{R}^n)})) \leq \beta(C)$. Using (2.12), Poincaré inequality (see (1.5)) and Young inequality, we have

$$\begin{aligned}\mu\|\nabla u^1-\nabla u^2\|^2_{L^2(\Omega;\mathbb{R}^{n\times n})} &\leq (k_1\|\tilde a_2\|_{L^2(\Omega)}\\ &\quad+k_0^2\beta(C))\|\nabla u^1-\nabla u^2\|^2_{L^2(\Omega;\mathbb{R}^{n\times n})}+\delta\|\nabla u^1-\nabla u^2\|^2_{L^2(\Omega;\mathbb{R}^{n\times n})}\\ &\quad+\frac{k_0^2\|a_2\|^2_{L^2(\Omega)}}{4\delta}\|\eta^1-\eta^2\|^2_{H^*}+\frac{k_0k_1}{\mu}\|a_2\|_{L^2(\Omega)}\|\nabla u^1-\nabla u^2\|^2_{L^2(\Omega;\mathbb{R}^{n\times n})}\end{aligned}$$

for arbitrary $\delta > 0$, which implies (2.10) if (2.11) holds. ∎

In particular, (2.11) ensures a unique response $u = u(\eta)$ for $(\mathcal{RP})$ to a given generalized control η. Hencefore, we can then put

$$\Phi(z) := J(z, u(z)) \quad \& \quad \bar{\Phi}(\eta) := \bar{J}(\eta, u(\eta)) \,. \tag{2.13}$$

Although the unique response is desirable, some results, as e.g. Proposition 1 below, hold even without this assumption.

Now, we can state the existence of a solution to the relaxed problem and relations between this problem and the original one, see also [30] for such a kind of results.

Proposition 1. *Let Ω be a C^2-domain, and let (1.3) and (2.10) be satisfied. Then*

1. *$(\mathcal{RP})$ possesses at least one optimal control.*
2. *Moreover, $\inf(\mathcal{P}) = \min(\mathcal{RP})$.*
3. *For any optimal solution $(\eta, u, p) \in Y_H^q(\Omega; \mathbb{R}^m) \times W_0^{1,2}(\Omega; \mathbb{R}^n) \times L_0^2(\Omega)$ to $(\mathcal{RP})$ and any sequence $\{(z^k, u^k, p^k)\}_{k\in\mathbb{N}}$ such that $z^k \in Z_{\rm ad}$ and (u^k, p^k) solves the Navier-Stokes system in $(\mathcal{P})$ with $z := z^k$ and $i(z^k) \to \eta$ weakly* in H^*, it holds $\Phi(z^k) \to \inf(\mathcal{P})$, so that this sequence is minimizing for $(\mathcal{P})$.*
4. *Conversely, having a minimizing sequence $\{(z^k, u^k, p^k)\}_{k\in\mathbb{N}}$ for $(\mathcal{P})$, there exists a subsequence of $\{(i(z^k), u^k, p^k)\}_{k\in\mathbb{N}}$ converging weakly* in $H^* \times W^{1,2}(\Omega; \mathbb{R}^n) \times L^2(\Omega)$ and the limit of any such subsequence solves $(\mathcal{RP})$.*

Sketch of the proof. In accord with Remark 1, we will omit p's in this proof. Always, there is a minimizing sequence $\{(z^k, u^k)\}_{k\in\mathbb{N}}$ for $(\mathcal{P})$. In view of (1.2), it holds

$$((u^k \cdot \nabla) u^k, v) + \mu(\nabla u^k, \nabla v) = (f(u^k, z^k), v) \tag{2.14}$$

for all $v \in W^{1,2}_{0,\rm DIV}(\Omega; \mathbb{R}^n)$. By (1.3a,d,f), $\inf(\mathcal{P}) < +\infty$. Then, by (1.3c), the following apriori estimate holds:

$$\limsup_{k\to\infty} \int_\Omega c_0 |z^k|^q \mathrm{d}x \le \lim_{k\to\infty} \int_\Omega h(x, u^k(x), z^k(x)) \mathrm{d}x = \inf(\mathcal{P}) < +\infty. \tag{2.15}$$

This implies $\{z^k\}_{k\in\mathbb{N}}$ bounded in $L^q(\Omega; \mathbb{R}^m)$. Then $i(z^k)$ converges weakly* to some $\eta \in \bar{Z}_{\rm ad}$, if a suitable subsequence is selected. Due to (1.5) and the Poincaré inequality, $\{u^k\}_{k\in\mathbb{N}}$ is bounded in $W^{1,2}(\Omega; \mathbb{R}^n)$. Thus, taking another subsequence if necessary, we obtain that $u^k \to$

u weakly in $W_0^{1,2}(\Omega;\mathbb{R}^n)$, which implies that $u^k \to u$ strongly in $L^4(\Omega;\mathbb{R}^n)$. Thus, for any $v \in W^{1,2}_{0,\text{DIV}}(\Omega;\mathbb{R}^n)$,

$$((u^k\cdot\nabla)u^k,v) = -((u^k\cdot\nabla)v,u^k) \to -((u\cdot\nabla)v,u) = ((u\cdot\nabla)u,v)\,.$$

Moreover, by (1.3d), $\{f(u^k,z^k)\}_{k\in\mathbb{N}}$ is bounded in $L^2(\Omega;\mathbb{R}^n)$ and, by using also (1.3e), $f(u^k,z^k) \to (f\circ u)\bullet\eta$ weakly in $L^2(\Omega;\mathbb{R}^n)$; cf. [30, Lemma 3.6.7]. Similarly, by (1.3a,b) we can see that $\{h(u^k,z^k)\}_{k\in\mathbb{N}}$ is bounded in $L^1(\Omega)$ and converges to $(h\circ u)\bullet\eta$ weakly* in $\mathrm{rca}(\bar\Omega)$.

Altogether, it enables us to pass to the limit in the integral identity (2.14), which gives just (2.7). Thus u satisfies (2.7), i.e. (η,u) is admissible for $(\mathcal{RP})$.

Moreover,

$$\begin{aligned}\lim_{k\to\infty} J(z^k,u^k) &= \lim_{k\to\infty}\int_\Omega h(x,u^k(x),z^k(x))\,\mathrm{d}x\\ &= \int_{\bar\Omega}[(h\circ u)\bullet\eta](\mathrm{d}x) = \bar J(\eta,u).\end{aligned}\tag{2.16}$$

As $J(z^k,u^k) \to \inf(\mathcal{P})$, we showed $\bar J(\eta,u) = \inf(\mathcal{P})$ so that certainly $\inf(\mathcal{RP}) \le \inf(\mathcal{P})$.

Taking a minimizing sequence $\{(\eta^k,u^k)\}_{k\in\mathbb{N}}$ for $(\mathcal{RP})$, we can prove similarly as above that $\{\eta^k\}_{k\in\mathbb{N}}$ converges (after taking possibly a subsequence) weakly* in H^* and the limit solves $(\mathcal{RP})$, as claimed in 1.

Taking (η,u) a solution to $(\mathcal{RP})$, there is a sequence $\{z^k\}_{k\in\mathbb{N}} \subset Z_{\rm ad}$ bounded in $L^q(\Omega;\mathbb{R}^m)$ such that w*-$\lim_{k\to\infty} i(z^k) = \eta$. Then one can prove similarly as above that $\Phi(z^k) = \bar\Phi(i(z^k)) \to \bar\Phi(\eta) = \min(\mathcal{RP})$, so that $\min(\mathcal{RP}) \ge \inf(\mathcal{P})$.

Thus 2 was proved, justifying also the points 3–4 as a side effect. ∎

Remark 3. If Ω were only a Lipschitz domain, we do not know whether (2.9) holds; then the growth of β in (1.3) would have to be specified appropriately.

Remark 4. As H is separable and $h(x,r,\cdot)$ has a q-growth while $f(x,r,\cdot)$ is bounded (see (1.3c,d)), by using [30, Lemmas 4.2.3–4] one can see that any optimal relaxed control $\eta \in Y^q_H(\Omega;\mathbb{R}^m)$ is q-nonconcetrating in the sense that there is a sequence of controls $\{z^k\}_{k\in\mathbb{N}}$ such that w*-$\lim_{k\to\infty} i(z^k) = \eta$ and the set $\{|z^k|^q;\ k\in\mathbb{N}\}$ is relatively weakly compact in $L^1(\Omega)$. Every such η has a so-called L^q-Young-measure representation $\nu \in \mathcal{Y}^q(\Omega;\mathbb{R}^m)$ (possibly not determined uniquely) satisfying

$$\forall h\in H: \qquad \langle\eta,h\rangle = \int_\Omega\int_{\mathbb{R}^m} h(x,s)\nu_x(\mathrm{d}s)\,\mathrm{d}x, \tag{2.17}$$

where $\mathcal{Y}^q(\Omega;\mathbb{R}^m)$ denotes the set of all L^q-Young measures, i.e. weakly measurable families $\nu := \{\nu_x\}_{x\in\Omega}$ of probability Radon measures on $\mathbb{R}^m$ satisfying $\int_\Omega \int_{\mathbb{R}^m} |s|^q \nu_x(\mathrm{d}s)\,\mathrm{d}x < +\infty$; the adjective "weakly measurable" means that for any $v \in C_0(\mathbb{R}^m)$ the mapping $\Omega \to \mathbb{R}$: $x \mapsto \langle \nu_x, v\rangle := \int_{\mathbb{R}^m} v(s)\nu_x(\mathrm{d}s)$ is measurable in the usual sense.

If S is measurable and closed-valued, the relaxed problem $(\mathcal{RP})$ can be rewritten in terms of L^q-Young measure into the following form:

$$(\mathcal{RP}') \quad \begin{cases} \text{Minimize} & \bar{J}(\eta, u) := \displaystyle\int_\Omega \int_{\mathbb{R}^m} h(x, u(x), s)\nu_x(\mathrm{d}s)(\mathrm{d}x) \\ \text{subject to} & (u\cdot\nabla)u - \mu\Delta u + \nabla p = \displaystyle\int_{\mathbb{R}^m} f(x, u(x), s)\nu_x(\mathrm{d}s)\ , \\ & \operatorname{div} u = 0, \\ & \operatorname{supp}(\nu_x) \subset S(x) \quad \text{for a.a. } x \in \Omega, \\ & u \in W_0^{1,2}(\Omega; \mathbb{R}^n), \quad p \in L_0^2(\Omega), \quad \nu \in \mathcal{Y}^q(\Omega; \mathbb{R}^m). \end{cases}$$

For an extension in terms of classical relaxed controls (i.e. L^∞-Young measures) we refer also [8, 12, 13, 33, 34].

An example for usage of $(\mathcal{RP}')$ is the following existence result.

Proposition 2. *Let Ω be a C^2-domain, let (1.3) and (2.11) hold, let S be measurable and closed-valued. Denote by $h \times f$ the mapping of $\Omega\times\mathbb{R}^n\times\mathbb{R}^n$ onto $\mathbb{R}\times\mathbb{R}^n$ such that $[h\times f](x,r,s) = (h(x,r,s), f(x,r,s))$. Assume that for all $r \in \mathbb{R}^n$ and a.a. $x \in \Omega$*

$$\overline{\mathrm{co}}\ [h \times f](x, r, S(x)) \subset Q(x, r), \tag{2.18}$$

where the "orientor field" Q is defined by

$$Q(x, r) := \{(a, y) \in \mathbb{R} \times \mathbb{R}^n;\ \ a \geq h(x, r, s),\ y = f(x, r, s),\ s \in S(x)\}. \tag{2.19}$$

Then $(\mathcal{P})$ has a solution.

Sketch of the proof. (For more details see [31, Lemma 2].) Take a solution η which does exist by Proposition 1(i). By Remark 4, η is q-nonconcentrating and (every) its L^q-Young-measure representation ν solves $(\mathcal{RP}')$. For any $x \in \Omega$ for which $\int_{\mathbb{R}^m} |s|^q \nu_x(\mathrm{d}s) < \infty$ we have

$$\int_{\mathbb{R}^m} [h \times f](x, u(x), s)\nu_x(\mathrm{d}s) \in \overline{\mathrm{co}}\ [h \times f](x, u(x), S(x)) \subset Q(x, u(x))\ , \tag{2.20}$$

where we used also (2.18). Let us put

$$R(x) := \Big\{ s \in S(x);\ h(x, u(x), s) \leq \int_{\mathbb{R}^m} h(x, u(x), \sigma)\nu_x(\mathrm{d}\sigma), \tag{2.21}$$
$$f(x, u(x), s) = \int_{\mathbb{R}^m} f(x, u(x), \sigma)\nu_x(\mathrm{d}\sigma) \Big\}\ .$$

By (2.19), for any $(a,y) \in Q(x,u(x))$ there is $s \in S(x)$ such that $a \geq h(x,u(x),s)$ and $y = f(x,u(x),s)$. Hence, for the particular choice

$$(a,y) = (a(x),y(x)) := \int_{\mathbb{R}^m} [h \times f](x,u(x),s)\nu_x(\mathrm{d}s), \tag{2.22}$$

the inclusion (2.20) implies that $a(x) \geq h(x,u(x),s)$ and $y(x) = f(x,u(x),s)$ for some $s \in S(x)$, hence $R(x) \neq \emptyset$. Besides, the multi-valued mapping $R : \Omega \rightrightarrows \mathbb{R}^m$ defined by (2.21) is measurable and closed-valued, thus it possesses a measurable selection $z(x) \in R(x)$. In particular, $z(x) \in S(x)$. Moreover, in view of (3.1) with (3.3),

$$f(x,u(x),z(x)) = y(x) = \int_{\mathbb{R}^m} f(x,u(x),s)\nu_x(\mathrm{d}s) \tag{2.23}$$

for a.a. $x \in \Omega$, so that z and ν give the same response u, i.e. $u(z) = u(\nu) := u(\eta)$ with η given by (2.17). Hence the pair (z,u) is admissible for $(\mathcal{P})$. Moreover, by using also Proposition 1(ii), we get $\int_\Omega h(x,u(x),z(x))\,\mathrm{d}x \leq \int_\Omega a(x)\,\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^m} h(x,u(x),s)\nu_x(\mathrm{d}s)\mathrm{d}x = \min(\mathcal{RP}') = \min(\mathcal{RP}) = \inf(\mathcal{P})$. In particular, the coercivity (1.3c) implies $c_0 \int_\Omega |z(x)|^q\,\mathrm{d}x \leq \int_\Omega h(x,u(x),z(x))\,\mathrm{d}x \leq \inf(\mathcal{P}) < +\infty$; note that (1.3a,d,f) makes $\inf(\mathcal{P})$ indeed finite. Therefore, $z \in L^q(\Omega;\mathbb{R}^m)$, which completes the proof that z solves $(\mathcal{P})$. ∎

Remark 5. Note that (2.18) is fulfilled if for example $Q(x,r)$ is convex and compact. This is ensured if $S(x)$ is compact for a.a. x (as h, f are Carathéodory functions) and $Q(x,r)$ is convex, which is a slightly generalized variant of the Filippov–Roxin condition. A very special case that can be however handled by a direct method occurs if $S(x)$ is convex, $f(x,r,\cdot)$ is affine and $h(x,r,\cdot)$ is convex on $S(x)$ for a.a. $x \in \Omega$; cf. e.g. [19] for such a type of existence result.

3. MAXIMUM PRINCIPLE

In this section we formulate first-order necessary optimality conditions for $(\mathcal{RP})$ in terms of a maximum principle. For maximum principle for Navier-Stokes optimal control problems, we refer also to [4, 13, 33] or for other type of first-order optimality conditions also to [3, 5, 7, 19, 20, 22, 25, 29, 35]. To give as simple proofs as possible, we confine ourselves to the special case

$$h(x,r,s) := \frac{1}{2}|r - u_\mathrm{d}(x)|^2 + \hat{h}(x,s) \ , \qquad f(x,r,s) := \hat{f}(x,s) \ , \tag{3.1}$$

where $u_\mathrm{d} \in L^{q_0}(\Omega;\mathbb{R}^n)$ is a desired (given) velocity profile, and

$$q_0 > n. \tag{3.2}$$

The first term in (3.1) realizes the so-called flow tracking often used in literature, cf. [5, 2, 18, 19, 22, 23, 25, 29].

To formulate the maximum principle we will need the so-called adjoint state $w \in W^{1,2}_{0,\text{DIV}}(\Omega;\mathbb{R}^n)$ satisfying the integral identity

$$\mu(\nabla w,\nabla v)-((u\cdot\nabla)w,v)+(w,(v\cdot\nabla)u) = (u_{\mathrm{d}}-u,v) \quad \forall v\in W^{1,2}_{0,\text{DIV}}(\Omega;\mathbb{R}^n)\,. \tag{3.3}$$

It is worth mentioning that w in (3.3) is a weak solution of the adjoint system to the linearized Navier-Stokes equations, i.e.

$$-\mu\Delta w_i+\frac{\partial\pi}{\partial x_i} \;=\; (u_{\mathrm{d}}-u)_i-\sum_{k=1}^{n}(w_k\frac{\partial u_k}{\partial x_i}-u_k\frac{\partial w_k}{\partial x_i})\,, \quad i=1,\dots,n, \tag{3.4a}$$

$$\operatorname{div} w \;=\; 0\,, \tag{3.4b}$$

where $\pi \in L^2_0(\Omega)$, cf. Remark 1. The following regularity of the adjoint state, higher than e.g. in [19, Theorem 3.2], will be essential for (3.18) below. Let us remark that, in context of fluid control, condition (3.5) was already used by Bilič [1].

Lemma 2. *Let Ω be a C^2-domain, let (1.3) with (3.1) with $u_{\mathrm{d}} \in L^{q_0}(\Omega;\mathbb{R}^n)$ hold, and let a_2 from (1.3d) satisfy*

$$\frac{k_0k_1}{\mu^2}\|a_2\|_{L^2(\Omega)} < 1. \tag{3.5}$$

Then there is C_1 depending on Ω, μ and $\|a_2\|_{L^2(\Omega)}$ such that for arbitrary small ϵ

$$\forall \eta\in \bar{Z}_{\mathrm{ad}}: \quad \|\nabla w(\eta)\|_{L^\infty(\Omega;\mathbb{R}^{n\times n})} \le C_1\|u-u_{\mathrm{d}}\|_{L^{n+\epsilon}(\Omega;\mathbb{R}^n)}\,, \tag{3.6}$$

where $w = w(\eta) \in W^{1,2}_{0,\text{DIV}}(\Omega;\mathbb{R}^n)$ solves (3.3) with $u = u(\eta)$.

Proof. Let us first observe that (3.5) implies the existence of C depending on the above mentioned quantities such that

$$\|\nabla w\|_{L^2(\Omega;\mathbb{R}^{n\times n})} \le C\,. \tag{3.7}$$

Indeed, testing in (3.3) by $v := w$, and using the Hölder inequality and (1.5) we obtain (notice that $((u\cdot\nabla)w,w) = 0$)

$$\begin{aligned}\mu\|\nabla w\|^2_{L^2(\Omega;\mathbb{R}^{n\times n})} &= ((u\cdot\nabla)w,w)-(w,(w\cdot\nabla)u)+(u_{\mathrm{d}}-u,w)\\ &\le \|w\|^2_{L^4(\Omega;\mathbb{R}^n)}\|\nabla u\|_{L^2(\Omega;\mathbb{R}^{n\times n})}+\|u-u_{\mathrm{d}}\|_{L^2(\Omega;\mathbb{R}^n)}\|w\|_{L^2(\Omega;\mathbb{R}^n)}\\ &\le \frac{k_0k_1}{\mu}\|a_2\|_{L^2(\Omega)}\|\nabla w\|^2_{L^2(\Omega;\mathbb{R}^n)}+k_0\|u-u_{\mathrm{d}}\|_{L^2(\Omega;\mathbb{R}^n)}\|\nabla w\|_{L^2(\Omega;\mathbb{R}^{n\times n})}.\end{aligned}$$

Therefore

$$\left(\mu - \frac{k_0 k_1}{\mu}\|a_2\|_{L^2(\Omega)}\right)\|\nabla w\|_{L^2(\Omega;\mathbb{R}^{n\times n})} \le k_0\|u-u_{\rm d}\|_{L^2(\Omega;\mathbb{R}^n)}\,, \quad (3.8)$$

and (3.7) follows due to (3.5).

Now, using the facts that $u \in W^{2,2}(\Omega;\mathbb{R}^n)$ and $w \in W^{1,2}(\Omega;\mathbb{R}^n)$, we can view (3.4) as the Stokes system with the right-hand side belonging at least to $L^2(\Omega;\mathbb{R}^n)$ (the restriction comes from the term $\sum_{k=1}^n u_k \frac{\partial w_k}{\partial x_i}$). Then applying standard L^2-regularity result for the Stokes system one obtains $w \in W^{2,2}(\Omega;\mathbb{R}^n)$ with

$$\|w\|_{W^{2,2}(\Omega;\mathbb{R}^n)} \le c\|u-u_{\rm d}\|_{L^2(\Omega;\mathbb{R}^n)}\,.$$

However, using this we easily observe that the right-hand side of (3.4a) belongs now to $L^{n+\epsilon}(\Omega;\mathbb{R}^n)$, $\epsilon > 0$, $\epsilon \le \min(q_0, \frac{2n}{n-2})$. The L^q-regularity theory for the Stokes system (cf. [16] for example) then implies

$$\|w\|_{W^{2,n+\epsilon}(\Omega;\mathbb{R}^n)} \le \tilde{C}_1(\Omega,\mu,\|a_2\|_{L^2(\Omega)})\,\|u-u_{\rm d}\|_{L^{n+\epsilon}(\Omega;\mathbb{R}^n)}\,. \quad (3.9)$$

The assertion then follows from the imbedding $W^{2,n+\epsilon}(\Omega)$ into $W^{1,\infty}(\Omega)$. Then C_1 is $\tilde{C}_1$ multiplied by the norm of the imbedding $W^{2,n+\epsilon}(\Omega) \subset W^{1,\infty}(\Omega)$. ■

Lemma 3. *Defining the so-called Hamiltonian $\mathcal{H}_w : \Omega \times \mathbb{R}^m \to \mathbb{R}$ by*

$$\mathcal{H}_w(x,s) := w(x)\cdot \hat{f}(x,s) - \hat{h}(x,s)\ , \quad (3.10)$$

the following increment formula holds

$$\bar{\Phi}(\tilde{\eta}) - \bar{\Phi}(\eta) + \int_{\bar{\Omega}} \mathcal{H}_w \bullet (\tilde{\eta}-\eta)\,{\rm d}x = \int_\Omega \frac{1}{2}|\tilde{u}-u|^2{\rm d}x - \Big(((\tilde{u}-u)\cdot\nabla)w, \tilde{u}-u\Big) \quad (3.11)$$

provided $\eta, \tilde{\eta} \in Y_H^q(\Omega;\mathbb{R}^m)$, $u = u(\eta)$, $\tilde{u} = u(\tilde{\eta})$, and the adjoint state $w \in W^{1,2}_{0,\rm DIV}(\Omega;\mathbb{R}^n)$ solves (3.3).

Proof. We use successively the formula for the Hamiltonian (3.10), the weak formulation (2.7) both for $u = u(\eta)$ and for $\tilde{u} = u(\tilde{\eta})$ with $v := w$, the adjoint equation (3.3) with $v := \tilde{u} - u$, the algebraic identity $\frac{1}{2}|\tilde{u}-u_{\rm d}|^2 - \frac{1}{2}|u-u_{\rm d}|^2 - (u-u_{\rm d})\cdot(\tilde{u}-u) = \frac{1}{2}|\tilde{u}-u|^2$, and the Green theorem. Thus we can obtain:

$$\bar{\Phi}(\tilde{\eta}) - \bar{\Phi}(\eta) + \int_{\bar{\Omega}} \mathcal{H}_w \bullet (\tilde{\eta}-\eta){\rm d}x = \frac{1}{2}\int_\Omega |\tilde{u}-u_{\rm d}|^2 - |u-u_{\rm d}|^2{\rm d}x + \langle \tilde{\eta}-\eta, w\cdot\hat{f}\rangle$$

$$= \frac{1}{2}\int_\Omega |\tilde{u}-u_{\rm d}|^2 - |u-u_{\rm d}|^2{\rm d}x + \mu(\nabla\tilde{u}-\nabla u, \nabla w) - ((u\cdot\nabla)u, w) + ((\tilde{u}\cdot\nabla)\tilde{u}, w)$$

$$= \int_\Omega \frac{1}{2}|\tilde{u}-u_{\rm d}|^2 - \frac{1}{2}|u-u_{\rm d}|^2 + (u_{\rm d}-u)\cdot(\tilde{u}-u)\,{\rm d}x - ((u\cdot\nabla)u, w)$$

$$+((\tilde{u}\cdot\nabla)\tilde{u},w)+((u\cdot\nabla)w,\tilde{u}-u)+(w,((u-\tilde{u})\cdot\nabla)u)$$
$$=\int_\Omega\frac{1}{2}|\tilde{u}-u|^2\mathrm{d}x+((\tilde{u}\cdot\nabla)\tilde{u},w)-(w,(\tilde{u}\cdot\nabla)u)+((u\cdot\nabla)w,\tilde{u}-u)$$
$$=\int_\Omega\frac{1}{2}|\tilde{u}-u|^2\mathrm{d}x-\Big(((\tilde{u}-u)\cdot\nabla)w,\tilde{u}-u\Big)\,. \qquad \blacksquare$$

As a simple consequence we can now get the integral maximum principle for the relaxed problem as the first-order necessary optimality condition.

Proposition 3. *Let the assumptions of Lemma 2 hold, and let* $(\eta,u)\in Y_H^q(\Omega;\mathbb{R}^m)\times W_0^{1,2}(\Omega;\mathbb{R}^n)$ *be an optimal solution for* $(\mathcal{RP})$. *Then there is* $w\in W_{0,\mathrm{DIV}}^{1,2}(\Omega;\mathbb{R}^n)$ *solving (3.3) such that, for the Hamiltonian* $\mathcal{H}_w$ *from (3.10), the following maximum principle holds:*

$$\int_{\bar{\Omega}}\mathcal{H}_w\bullet\eta\,\mathrm{d}x=\sup_{z\in Z_{\mathrm{ad}}}\int_\Omega\mathcal{H}_w(x,z(x))\,\mathrm{d}x\ . \tag{3.12}$$

Sketch of the proof. Let us calculate the directional derivative of $\bar{\Phi}$, which is by definition:

$$\mathrm{D}\bar{\Phi}(\eta,\tilde{\eta}-\eta):=\lim_{\varepsilon\searrow0}\frac{\bar{\Phi}(\eta+\varepsilon(\tilde{\eta}-\eta))-\bar{\Phi}(\eta)}{\varepsilon}\ =\int_{\bar{\Omega}}(\hat{h}-w\cdot\hat{f})\bullet(\tilde{\eta}-\eta)\mathrm{d}x$$
$$+\ \lim_{\varepsilon\searrow0}\frac{1}{\varepsilon}\int_\Omega\frac{1}{2}|u_\varepsilon-u|^2-(((u_\varepsilon-u)\cdot\nabla)w)\cdot(u_\varepsilon-u)\,\mathrm{d}x$$

where we used also (3.11) with $\tilde{u}:=u_\varepsilon$ denoting the solution of the relaxed Navier-Stokes equation (2.7) but with $\eta_\varepsilon:=\eta+\varepsilon(\tilde{\eta}-\eta)$ in place of η. Let us agree to consider only $\tilde{\eta}\in\bar{Z}_{\mathrm{ad}}$ and $0<\varepsilon\le1$, which will be sufficient for usage in (3.14) and which will guarantee $\eta_\varepsilon\in\bar{Z}_{\mathrm{ad}}$.

Note that (3.5) now implies (2.11) because (1.3e) now holds with with $\tilde{a}_2=0$ and $\beta=0$. Hencefore we have Lemma 1 at our disposal, so that (2.10) gives

$$\|u_\varepsilon-u\|_{L^2(\Omega;\mathbb{R}^n)}\le\|u_\varepsilon-u\|_{W^{1,2}(\Omega;\mathbb{R}^n)}\le C_0\|\eta_\varepsilon-\eta\|_{H^*}=\varepsilon C_0\|\tilde{\eta}-\eta\|_{H^*}\ .$$

This yields $\int_\Omega|u_\varepsilon-u|^2\mathrm{d}x=\mathcal{O}(\varepsilon^2)$. Similarly, the term

$$\left|\int_\Omega(((u_\varepsilon-u)\cdot\nabla)w)\cdot(u_\varepsilon-u)\,\mathrm{d}x\right|\le\|\nabla w\|_{L^\infty(\Omega;\mathbb{R}^{n\times n})}\|u_\varepsilon-u\|^2_{L^2(\Omega;\mathbb{R}^n)}$$

is $\mathcal{O}(\varepsilon^2)$ because ∇w is bounded in $L^\infty(\Omega;\mathbb{R}^{n\times n})$ due to (3.6) with (1.5). Altogether, we have proved the expression for the directional derivative, which apparently depends linearly and continuously on the direction as soon as $\tilde{\eta} \in \bar{Z}_{\rm ad}$. Thus $\bar{\Phi}$ has a Gâteaux differential $\nabla\bar{\Phi}$ given by

$$\langle \nabla\bar{\Phi}(\eta), \tilde{\eta} - \eta \rangle = \langle \eta - \tilde{\eta}, \mathcal{H}_w \rangle, \quad w \text{ solves (3.3) with } u = u(\eta). \tag{3.13}$$

Then (η, u) solves $(\mathcal{P})$, which means that η minimizes $\bar{\Phi}$ on $\bar{Z}_{\rm ad}$, implies that $-\nabla\bar{\Phi}(\eta)$ belongs to the normal cone to the convex set $\bar{Z}_{\rm ad}$ at η, which is just equivalent to

$$\forall \tilde{\eta} \in \bar{Z}_{\rm ad}: \qquad \langle \nabla\bar{\Phi}(\eta), \tilde{\eta} - \eta \rangle \le 0 \ . \tag{3.14}$$

This means precisely

$$\begin{aligned} \int_{\bar{\Omega}} \mathcal{H}_w \bullet \eta \,\mathrm{d}x &= \langle -\nabla\bar{\Phi}(\eta), \eta \rangle = \max_{\tilde{\eta} \in \bar{Z}_{\rm ad}} \langle -\nabla\bar{\Phi}(\eta), \tilde{\eta} \rangle \\ &= \max_{\tilde{\eta} \in \bar{Z}_{\rm ad}} \int_{\bar{\Omega}} \mathcal{H}_w \bullet \tilde{\eta} \,\mathrm{d}x = \sup_{z \in Z_{\rm ad}} \int_{\Omega} \mathcal{H}_w(x, z(x)) \,\mathrm{d}x \ . \end{aligned} \tag{3.15}$$

∎

As in [30, Theorem 4.2.2], one can modify the integral maximum principle (3.12) to a pointwise (sometimes called Pontryagin's) maximum principle (cf. also [4, 13, 33]):

Corollary 1. *Let the assumptions of Lemma 2 hold, and let S be measurable and closed-valued. Then for any solution (η, u) to $(\mathcal{RP})$ it holds*

$$[\mathcal{H}_w \bullet \eta](x) = \max_{s \in S(x)} \mathcal{H}_w(x, s) \qquad \text{for a.a. } \ x \in \Omega \tag{3.16}$$

with the Hamiltonian $\mathcal{H}_w$ from (3.10) with $w \in W^{1,2}_{0,\mathrm{DIV}}(\Omega; \mathbb{R}^n)$ solving (3.3).

Having an L^q-Young-measure representation ν of an optimal relaxed control η, (3.16) says that ν_x is supported on the set where $\mathcal{H}_w(x, \cdot)$ attains its maximum. By Lemma 2, we have in particular an L^∞-regularity of the multiplier w, which then gives the following assertion:

Corollary 2. *If $\hat{h}$ and $\hat{f}$ are independent of $x \in \Omega$, then any optimal relaxed control η for $(\mathcal{RP})$ has an L^∞-Young measure representation ν, i.e. ν_x is compactly supported independently of $x \in \Omega$.*

The following assertion states an important global property of $\bar{\Phi}$ if the Reynolds number is small, see Remark 6 below.

Lemma 4. *Let the assumptions of Lemma 2 hold, and let a_2 from (1.3d) satisfy (3.5) and also*

$$C_{n,q_0} \frac{k_0}{\mu} \|a_2\|_{L^2(\Omega)} + c \|u_{\rm d}\|_{L^{q_0}(\Omega; \mathbb{R}^n)} \le \frac{1}{2C_1} \tag{3.17}$$

with $C_1 = C_1(\Omega, \mu, \|a_2\|_{L^2(\Omega)})$ from Lemma 2 and C_{n,q_0} denoting the norm of the imbedding $W^{1,2}(\Omega) \subset L^{q_0}(\Omega)$. Then the extended cost functional $\bar{\Phi} : \bar{Z}_{\rm ad} \to \mathbb{R}$ is convex with respect to the geometry of the space H^*.

Proof. Using (1.5), Lemma 2 and (3.17), the second-order term in (3.11) in nonnegative because of the following estimate:

$$\begin{aligned}
&\int_\Omega \frac{1}{2}|\tilde{u}-u|^2 \mathrm{d}x - \Big(((\tilde{u}-u)\cdot\nabla)w, \tilde{u}-u\Big) \qquad (3.18)\\
&\geq \left(\frac{1}{2} - \|\nabla w\|_{L^\infty(\Omega;\mathbb{R}^{n\times n})}\right)\|\tilde{u}-u\|^2_{L^2(\Omega;\mathbb{R}^n)}\\
&\geq \left(\frac{1}{2} - C_1\|u-u_{\rm d}\|_{L^{n+\epsilon}(\Omega;\mathbb{R}^n)}\right)\|\tilde{u}-u\|^2_{L^2(\Omega;\mathbb{R}^n)}\\
&\geq \left(\frac{1}{2} - C_1\|u\|_{L^{n+\epsilon}(\Omega;\mathbb{R}^n)} - C_1\|u_{\rm d}\|_{L^{n+\epsilon}(\Omega;\mathbb{R}^n)}\right)\|\tilde{u}-u\|^2_{L^2(\Omega;\mathbb{R}^n)}\\
&\geq \left(\frac{1}{2} - C_1 C_{n,q_0}\frac{k_0}{\mu}\|a_2\|_{L^2(\Omega)} - C_1\ c\|u_{\rm d}\|_{L^{q_0}(\Omega;\mathbb{R}^n)}\right)\|\tilde{u}-u\|^2_{L^2(\Omega;\mathbb{R}^n)}.
\end{aligned}$$

By (3.17) and the proof of Proposition 3, we have just obtained $\bar{\Phi}(\tilde{\eta}) - \bar{\Phi}(\eta) - [\nabla\bar{\Phi}(\eta)](\tilde{\eta}-\eta) \geq 0$ and, replacing the roles of η and $\tilde{\eta}$, also $\bar{\Phi}(\eta) - \bar{\Phi}(\tilde{\eta}) - [\nabla\bar{\Phi}(\tilde{\eta})](\eta-\tilde{\eta}) \geq 0$. Therefore, by addition, we obtain $[\nabla\bar{\Phi}(\eta) - \nabla\bar{\Phi}(\tilde{\eta})](\tilde{\eta}-\eta) \geq 0$, which just says that $\nabla\bar{\Phi}$ is monotone, from which the convexity of $\bar{\Phi}$ follows by well-known arguments. ∎

We are now ready to state also the sufficiency of the maximum principle (3.12).

Proposition 4. *Let condition (3.17) be satisfy. Then the maximum principle consisting of (3.3), (3.10), and (3.12) is sufficient in the sense that, having a triple $(\eta, u, w) \in \bar{Z}_{\rm ad} \times W^{1,2}_{0,\rm DIV}(\Omega;\mathbb{R}^n)^2$ such that u solves the Navier-Stokes system (2.7), and w solves the adjoint problem to the linearized Navier-Stokes system (3.3), and the maximum principle (3.12) holds, then (η, u) is the optimal solution to $(\mathcal{RP})$.*

Proof. By Lemma 4, $\bar{\Phi}$ is convex, so that (3.14) is also a sufficient optimality condition. Yet, (3.14) is equivalent with (3.15). ∎

Remark 6. The constant C_1 from (3.6) depends on μ as $\mathcal{O}(\mu^{-1})$. Then, for given $u_{\rm d}$ and a_2, the condition (3.17) requires μ sufficiently large. Hencefore, (3.17) needs a sufficiently small Reynolds number. As the fluid (and its viscosity μ) is usually given, we rather need a sufficiently small driving force and desired velocity profile, as expressed in (3.17), indeed.

Acknowledgments

This research has been partly covered by the grant 201/96/0228 (Grant agency of the Czech Republic). Moreover, the second author has been partly supported also by the grant A 107 5707 (Grant agency of the Academy of Sciences of the Czech Republic).

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