Continuum Mechanics and Thermodynamics manuscript No.
(will be inserted by the editor)

Incompressible ionized fluid mixtures

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Received: Jan. 2004 / Revised version: Oct. 2005

Abstract The model combining Navier-Stokes' equation for barycentric velocity together with Nernst-Planck's equation for concentrations of particular mutually reacting constituents, the heat equation, and the Poisson equation for self-induced quasistatic electric field is formulated and its thermodynamics is discussed. Then, existence of a weak solution to an initial-boundary-value problem for this system is proved in two special cases: zero Reynolds' number and constant temperature.

Keywords: Chemically reacting fluids, Navier-Stokes, Nernst-Planck, Poisson equations, heat equation.

Mathematics Subject Classification (2000): 35Q35, 76T30, 80A32.

Physics and Astronomy Classification Scheme (2001): 47.27Ak, 47.70Fw.

1 Introduction

Chemically reacting mixtures represent a framework for modelling various complicated processes in biology and chemistry. The main ambitions I had in mind are as much thermodynamic consistency as possible and simultaneously amenability for rigorous mathematical analysis, and also a high complexity of the model which would not restrict potential biological applications. This led to a choice of incompressible Newtonian framework with barycentric balancing of the impulse. The incompressibility refers here both to each particular constituent and, through volume-additivity hypothesis as in e.g. [18,28], also to the overall mixture. The electro-neutrality hypothesis, often (explicitly or not) assumed to simplify the task, is not assumed here so that the self-induced electrostatic field ought to be considered; let us remind that very large intensity of electric field exist on each cell membrane (about 10-100 MV/m), i.e. e.g. inside each ionic channel, although intensities inside fluid media e.g. inside cells or in intercellular space are certainly smaller. Beside biological modelling, the applications are, however, broader and expectedly cover, e.g., chemical reactors operating on electrolytes under varying temperature. Of course, in specific applications the generality of the model can be reduced, cf. Remark 4.3 below; e.g. biological application on a cellular level can well be considered both isothermal and with Reynolds number zero.

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On the other hand, it should be emphasized that many simplifications are adopted in the presented model, too. In particular, we consider small electrical currents (i.e. magnetic field is neglected), adopt the mentioned volume-additivity assumption, assume the diffusion fluxes independent of other constituent's gradients (cross-effects are neglected) as well as of the temperature gradient (i.e. Soret's effect is neglected) and (in agreement with Onsager's reciprocity principle [23]) also heat flux independent of the concentration gradients (i.e. Dufour's effect is neglected), see Samohýl [35] for more detailed discussion. Finally, the temperature-independent diffusion and mobility coefficients and mass densities are are considered the same for each constituents, cf. Remark 4.4 for the more general case outlined. Besides, mathematical analysis (i.e. here existence of solutions to the respective initial-value problems) will be performed only in certain special cases: anisothermal Stokes flow (in Section 3.1) and isothermal general Navier-Stokes flow (in Section 3.2). Existence of a solution to a fully coupled system was done in [30] if one consider a certain shear-thickening power-law dependence of the viscosity coefficient.

The "barycentric" (also called Eckart-Prigogine's [9,24]) concept, which balances the impulse of barycenter only, is known to yield difficulties with a definition of an entropy that would satisfy the Clausius-Duhem inequality. This seems to be reflected here, too; cf. Remark 2.3. In the compressible case, this barycentric concept has been developed in particular in Andrej, Dvořák and Maršík [1], Balescu [3], deGroot and Mazur [7], and Giovangigli [12]. A newer and more rational (also called Truesdell's) description of mixtures balances impulses for each constituent separately instead of postulating phenomenological fluxes. It has been proposed in Truesdell and Toupin [40], and further developed in particular by Drumheller [8], Mills [18], Müller [19] and Ruggeri [20], Rajagopal and Tao [27], Rajagopal, Wineman, and Gandhi [28], Samohýl [32–34], Samohýl and Šilhavý [36]. Involvement of, in concrete problems usually unknown, interaction terms between the particular constituents in Truesdell's model is compensated by more rigor and less phenomenology but, on the other hand, richer investigations can be done rather in two-component mixtures only, cf. [18] and [27, Chapter 7]. Therefore, as already said, we chose the more phenomenological but expectedly more applicable "barycentric" concept. The derivation of our model from Truesdell's one under specific simplifying assumptions was made by Samohýl [35].

2 The model and its thermodynamics

We consider L mutually reacting chemical constituents occupying a bounded domain $\Omega \subset \mathbb{R}^3$ with a Lipschitz (or, for Sect. 3.1, smooth) boundary $\Gamma := \partial\Omega$. Our model consists in a system of $3 + L + 2$ differential equations combining the Navier-Stokes system (2.1a), the Nernst-Planck equation generalized for moving media (2.1b), the Poisson equation (2.1c), and the heat equation (2.1d):

$$\varrho\frac{\partial v}{\partial t} + \varrho(v\cdot\nabla)v - \nu\Delta v + \nabla p = \sum_{\ell=1}^{L} c_\ell f_\ell\,, \qquad \mathrm{div}(v) = 0\,, \qquad f_\ell = -e_\ell\nabla\phi, \tag{2.1a}$$

$$\frac{\partial c_\ell}{\partial t} + \mathrm{div}\big(j_\ell + c_\ell v\big) = r_\ell(c_1, ..., c_L, \theta)\,, \qquad j_\ell = -d(\theta)\nabla c_\ell - mc_\ell(e_\ell - q)\nabla\phi, \quad \ell = 1, ..., L\,, \tag{2.1b}$$

$$\varepsilon\Delta\phi = -q, \qquad q = \sum_{\ell=1}^{L} e_\ell c_\ell\,, \tag{2.1c}$$

$$c_{\rm v}\frac{\partial\theta}{\partial t} - \mathrm{div}\big(\kappa\nabla\theta - c_{\rm v}v\theta\big) = \nu|\nabla v|^2 + \sum_{\ell=1}^{L}\big(f_\ell\cdot j_\ell - h_\ell(\theta)r_\ell(c_1, ..., c_L, \theta)\big) \tag{2.1d}$$

with the initial conditions

$$v(0,\cdot) = v_0\,, \quad c_\ell(0,\cdot) = c_{0\ell}\,, \quad \theta(0,\cdot) = \theta_0 \;\text{ on }\; \Omega. \tag{2.2}$$

The notation "·" means the scalar product between vectors. The meaning of the variables is:

v barycentric velocity,

p pressure,

c_ℓ concentration of ℓ-constituent, presumably to satisfy $\sum_{\ell=1}^L c_\ell = 1$, $c_\ell \geq 0$,
ϕ electrostatic potential,
θ temperature,
q the total electric charge,

and of the data is:

$\varrho > 0$ mass density both of the mixture and of the constituents,
$\nu > 0$ viscosity,
e_ℓ valence (i.e. electric charge) of ℓ-constituent,
$\varepsilon > 0$ permitivity,
$r_\ell(c_1, ..., c_L, \theta)$ production rate of the ℓ-constituent by chemical reactions,
$h_\ell(\theta)$ the enthalpy contained in the ℓth constituent,
f_ℓ body force acting on ℓ-constituent: $f_\ell = -e_\ell \nabla\phi$,
j_ℓ phenomenological flux of ℓ-constituent given in (2.1b),
$d = d(\theta)$, $m > 0$ diffusion and mobility coefficients, respectively,
$c_{\rm v} > 0$ specific heat (within constant volume),
$\kappa > 0$ heat conductivity.

Due to the constraint $c_\ell \geq 0$ and the *volume-additivity* constraint (i.e. Amagat's law)

$$\sum_{\ell=1}^{L} c_\ell = 1 \tag{2.3}$$

(implicitly contained in (2.1) if the initial and boundary conditions are compatible with it), the variables $c = (c_1, ..., c_L)$ can also be called *volume fractions*; as all constituents are assumed incompressible, c are simultaneously mass fractions.

Derivation of the model is briefly motivated as follows: The equation (2.1a) is based on Hamilton's dissipation principle generalized for dissipative systems, cf. [8]; the body force f_ℓ comes from Lorenz' force acting on a charge e_ℓ moving in the electromagnetic field (E, B), i.e. $f_\ell = e_\ell(E + v_\ell \times B)$ after the simplification that $E = -\nabla\phi$ and $B = 0$. The equation (2.1b) balances concentration of the particular constituents as usual in Nernst-Plank equations but here completed with the advection term ${\rm div}(c_\ell v)$ related with moving medium in Eulerian coordinates, while (2.1c) is the rest from the full electro-magnetic Maxwell's system which remains if assuming relatively slow movements of electric charges and small electric currents which do not create fast changes of electric fields and substantial magnetic field, and eventually (2.1d) is the usual balance of energy again in moving medium in Eulerian coordinates, see e.g. [12, 1] and Remark 2.1. The only peculiarity is the term $q\nabla\phi$ in the diffusive flux j_ℓ in (2.1b). The interpretation of this term is as a *reaction force* keeping the natural requirement

$$\sum_{\ell=1}^{L} j_\ell = 0 \tag{2.4}$$

satisfied, which eventually fixes also the mentioned volume-additivity constraint (2.3), cf. the argument (3.18) below. This volume-additivity assumption is often accepted in the theory of mixtures, although it should be emphasized that it is only a certain approximation of reality; cf. the discussion in [27, Sect. 2.8]. The condition (2.4) itself is routinely assumed even for compressible mixtures, see [12, Formula (2.5.9)]. One can derive the expression of this reaction force, let us denote it for a moment by $f_{\rm R}$, if assuming it *to act equally on each constituent*: indeed, considering the flux j_ℓ in a general form $j_\ell = -d(\theta)\nabla c_\ell - mc_\ell e_\ell \nabla\phi + mc_\ell f_{\rm R}$, by summing it and requiring (2.4) as well as assuming (2.3), we obtain

$$0 =: \sum_{\ell=1}^{L} j_\ell = -d(\theta)\nabla\Big(\sum_{\ell=1}^{L} c_\ell\Big) - m\Big(\sum_{\ell=1}^{L} c_\ell e_\ell\Big)\nabla\phi + m\sum_{\ell=1}^{L} c_\ell f_{\rm R} = m\Big(-q\nabla\phi + f_{\rm R}\Big), \tag{2.5}$$

hence we obtain $f_{\rm R} = q\nabla\phi$ as indeed used (2.1b). Introducing this force is perhaps the most novelty in the model, although in special cases this seems not to be entirely surprising, cf. Remark 2.5. Note also that $f_{\rm R}$ is the right-hand side of (2.1a) with the negative sign. Usually, $f_{\rm R}$ is small because $|q|$ is small in comparison with $\max_{\ell=1,...,L}|e_\ell|$. Often, the electro-neutrality assumption $q=0$ is even postulated for simplicity, which obviously makes this reaction force zero.

We have still to consider some boundary conditions, e.g. a closed thermally isolated container which in some simplified version leads to:

$$v=0, \qquad c_\ell=c_\ell^\Gamma, \quad \varepsilon\frac{\partial\phi}{\partial\mathfrak{n}}=\alpha(\phi_\Gamma-\phi), \qquad \frac{\partial\theta}{\partial\mathfrak{n}}=0 \quad \text{on } \Sigma, \tag{2.6}$$

where $\mathfrak{n}$ denotes the unit outward normal to the boundary Γ and the coefficient α can be interpreted as a "surface permitivity" of the boundary and ϕ_Γ is an outer potential. Fixing concentrations on Γ is certainly rather simplifying and some nonlinear conditions Newton-type conditions are often used to describe chemical reactions on possible electrodes on Γ, cf. [31].

Considering a fixed time horizon $T>0$, we use the notation $I:=[0,T]$, $Q:=I\times\Omega$, and $\Sigma:=I\times\partial\Omega$. Besides, we naturally assume $r_\ell:\mathbb{R}^{L+1}\to\mathbb{R}$ continuous and the mass and electric charge conservation in all chemical reactions and nonnegative production of ℓth constituent if there is none, and the initial and boundary conditions satisfy the volume-additivity constraints, i.e.

$$\sum_{\ell=1}^L r_\ell(c_1,...,c_L,\theta)=0=\sum_{\ell=1}^L e_\ell r_\ell(c_1,...,c_L,\theta)\ , \tag{2.7a}$$

$$c_\ell=0 \quad\Rightarrow\quad r_\ell(c_1,...,c_L,\theta)\ge 0, \tag{2.7b}$$

$$\sum_{\ell=1}^L c_{0\ell}=1, \quad c_{0\ell}\ge 0, \tag{2.7c}$$

$$\sum_{\ell=1}^L c_\ell^\Gamma=1, \quad c_\ell^\Gamma\ge 0. \tag{2.7d}$$

Remark 2.1 (*Energy balance.*) To show conservation of the total energy, let us assume, for simplicity, $\phi_\Gamma=\phi_\Gamma(x)$ time independent and then calculate the rate of electrostatic energy:

$$\begin{aligned}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega\varepsilon|\nabla\phi|^2\,\mathrm{d}x+\int_\Gamma\alpha|\phi-\phi_\Gamma|^2\,\mathrm{d}S\Big)&=\int_\Omega\varepsilon\nabla\phi\cdot\nabla\frac{\partial\phi}{\partial t}\,\mathrm{d}x+\int_\Gamma\alpha\frac{\partial\phi}{\partial t}(\phi-\phi_\Gamma)\,\mathrm{d}S\\
&=\int_\Omega\varepsilon\nabla\phi\cdot\nabla\frac{\partial\phi}{\partial t}\,\mathrm{d}x-\int_\Gamma\varepsilon\phi\frac{\partial}{\partial t}\Big(\frac{\partial\phi}{\partial\mathfrak{n}}\Big)\,\mathrm{d}S=-\int_\Omega\varepsilon\phi\Delta\frac{\partial\phi}{\partial t}\,\mathrm{d}x\\
&=\int_\Omega\phi\sum_{\ell=1}^L e_\ell\frac{\partial c_\ell}{\partial t}\,\mathrm{d}x=\int_\Omega\phi\sum_{\ell=1}^L e_\ell\big(r_\ell(c,\theta)-\,\mathrm{div}(j_\ell+c_\ell v)\big)\,\mathrm{d}x\\
&=-\int_\Omega\phi\sum_{\ell=1}^L e_\ell\mathrm{div}(j_\ell+c_\ell v)\,\mathrm{d}x\\
&=\int_\Omega\nabla\phi\cdot\sum_{\ell=1}^L e_\ell(j_\ell+c_\ell v)\,\mathrm{d}x-\int_\Gamma\phi\sum_{\ell=1}^L e_\ell j_\ell\cdot\mathfrak{n}\,\mathrm{d}S\end{aligned} \tag{2.8}$$

where (2.1c) and (2.1b) have been used together with the electric-charge-preservation assumption (2.7a) and twice Green's formula counting also with the boundary conditions (2.6). Testing (2.1a) by v, we obtain rate of kinetic energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\varrho\frac{|v|^2}{2}\,\mathrm{d}x=\int_\Omega\sum_{\ell=1}^L c_\ell(f_\ell\cdot v)-\varrho\big((v\cdot\nabla)v\big)\cdot v-\nu|\nabla v|^2\,\mathrm{d}x=-\int_\Omega\nu|\nabla v|^2+\sum_{\ell=1}^L c_\ell e_\ell\nabla\phi\cdot v\,\mathrm{d}x. \tag{2.9}$$

The rate of internal energy can be obtained simply by integration of (2.1d) over Ω and using Green's theorem with the considered boundary conditions $\partial\theta/\partial\mathfrak{n} = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega c_\mathrm{v}\,\theta\,\mathrm{d}x = \int_\Omega \nu|\nabla v|^2 - \sum_{\ell=1}^{L}\big(e_\ell j_\ell\nabla\phi + h_\ell(\theta)r_\ell(c,\theta)\big)\,\mathrm{d}x. \tag{2.10}$$

Altogether, summing (2.8)–(2.10) and using also (2.1b) integrated over Ω and Green's formula, we obtain the following balance:

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_\Omega\Big(\varrho\frac{|v|^2}{2} + \varepsilon\frac{|\nabla\phi|^2}{2} + c_\mathrm{v}\theta\Big)\,\mathrm{d}x + \int_\Gamma \alpha\frac{|\phi-\phi_\Gamma|^2}{2}\,\mathrm{d}S\bigg)$$
$$= -\int_\Omega\sum_{\ell=1}^{L}h_\ell(\theta)r_\ell(c,\theta)\,\mathrm{d}x - \int_\Gamma\phi\sum_{\ell=1}^{L}e_\ell j_\ell\cdot\mathfrak{n}\,\mathrm{d}S, \tag{2.11}$$

where we used the boundary conditions (2.6). Hence, (2.11) just says that the total energy rate, i.e. the rate of the sum of kinetic, electrostatic, and internal energy $\frac{1}{2}\varrho|v|^2 + \frac{1}{2}\varepsilon|\nabla\phi|^2 + c_\mathrm{v}\theta$ over Ω and the electrostatic energy $\frac{1}{2}\alpha|\phi-\phi_\Gamma|^2$ deposited on Γ, is balanced with the enthalpy production rate $\sum_{\ell=1}^{L}h_\ell r_\ell$ over Ω and the normal flux of electro-energy $\sum_{\ell=1}^{L}\phi e_\ell j_\ell\cdot\mathfrak{n}$ through the boundary Γ.

Remark 2.2 (*Sources of heat.*) When substituting f_ℓ and j_ℓ from (2.1a,b), the right-hand side of (2.1d) equals

$$f(v,c,\phi,\theta) := \nu|\nabla v|^2 + d(\theta)\nabla q\cdot\nabla\phi + \sum_{\ell=1}^{L}mc_\ell e_\ell^2|\nabla\phi|^2 - mq^2|\nabla\phi|^2 - \sum_{\ell=1}^{L}h_\ell(\theta)r_\ell(c,\theta). \tag{2.12}$$

Hence the particular source terms in f represent respectively the heat production due to loss of kinetic energy by viscosity, the power (per unit volume) of the electric current arising by the diffusion flux, the power of *Joule heat* produced by the electric currents j_ℓ, the rate of cooling by the force which balances the volume-additivity constraint, and the heat produced or consumed by chemical reactions. The influence of the cooling term $-mq^2|\nabla\phi|^2$ is presumably very small as usually $|q| << \max_{\ell=1,\dots,L}|e_\ell|$. Besides, Joule's heat always dominates this cooling effect because $\sum_{\ell=1}^{L}c_\ell e_\ell^2 \ge \big(\sum_{\ell=1}^{L}c_\ell e_\ell\big)^2$ if $\sum_{\ell=1}^{L}c_\ell = 1$ and all c_ℓ's are non-negative just by Jensen's inequality. The effective specific *electric conductivity* is obviously $m(\sum_{\ell=1}^{L}c_\ell e_\ell^2 - q^2)$. The term $d(\theta)\nabla q\cdot\nabla\phi$ has an indefinite sign in general and may create local cooling effects via diffusive flux of the electric charge against the gradient of the electrostatic field, which is related with the so-called *Peltier effect* in the lines of, e.g., deGroot and Mazur [7].

Remark 2.3 (*Entropy.*) A relation with standard thermodynamic concepts is through specific Helmholtz' *free energy* taking the form

$$\psi(v,\phi,c,\theta) = \frac{\varepsilon}{2}|\nabla\phi|^2 - c_\mathrm{v}\theta\ln(\theta). \tag{2.13}$$

The specific *entropy* s is then defined by the Gibbs' relation $s := -\partial\psi/\partial\theta = c_\mathrm{v}(1+\ln(\theta))$, and the *internal energy* is $e := \psi + \theta s = c_\mathrm{v}\theta + \frac{1}{2}\varepsilon|\nabla\phi|^2$. The requirement of preservation of total energy (i.e. the sum of the kinetic and the internal ones) leads to the energy balance

$$\theta\Big[\frac{\partial}{\partial t} + v\cdot\nabla\Big](s) + \mathrm{div}\,j = f \tag{2.14}$$

where the heat flux j is subjected to Fourier's law $j = -\kappa\nabla\theta$ and $f = f(v,c,\phi,\theta)$ is the dissipation rate identified in (2.12); note that (2.14) is just (2.1d). The thermodynamic consistency of this model can formally be claimed only if one assumes the diffusion coefficient $d = d(\theta)$ approaching zero for $\theta \searrow 0$. This, physically acceptable assumption is to "switch off" the indefinite term $d(\theta)\nabla q\cdot\nabla\phi$ if temperature θ approaches zero but brings essential mathematical troubles in obtaining a-priori estimates because one has to prove that the temperature is away from zero. This needs very sophisticated techniques and is always difficult, if possible at all; see Feireisl [6] who showed a "weak positivity" of θ (in the sense that $\ln\theta$ belongs to $L^2(Q)$) in the compressible context. Yet, one should realize that, due to phase transitions and other effects, validity of the model ends in reality much sooner than θ approaches the absolute zero. Anyhow, at least formally, the assumption $\lim_{\theta\searrow 0} d(\theta)=0$ allows for claiming non-negativity of θ at least if also a natural assumption that reaction rates $r_\ell(c,\theta)$ vanishes for $\theta\searrow 0$ is accepted. It seems acceptable to assume still that the chemical-reaction rates are designed naturally (=by "nature") not to consume entropy, i.e.

$$\sum_{\ell=1}^{L}\frac{h_\ell(\theta)\,r_\ell(c,\theta)}{\theta}\geq 0. \tag{2.15}$$

Under the mentioned positivity of temperature, this would allow us to claim the *Clausius-Duhem inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega s\,\mathrm{d}x=\int_\Omega\Big(\frac{f(v,c,\phi,\theta)}{\theta}+\mathrm{div}\big(\frac{\kappa\nabla\theta}{\theta}\big)+\kappa\frac{|\nabla\theta|^2}{\theta^2}\Big)\,\mathrm{d}x\geq 0 \tag{2.16}$$

if one would prove still non-negativity of the "Peltier-effect" term $\int_\Omega d(\theta)\nabla q{\cdot}\nabla\phi/\theta\,\mathrm{d}x$; let us note that $\int_\Omega\mathrm{div}\big(\kappa\nabla\theta/\theta\big)\,\mathrm{d}x=-\int_\Gamma\kappa\theta^{-1}\partial\theta/\partial\mathfrak{n}\,\mathrm{d}S=0$ due to the isolation on the boundary (2.6). As standard option for d and m is

$$d(\theta)=R\,M\,\theta\quad\text{and}\quad m=F\,M \tag{2.17}$$

where R is the universal gas constant, F is Faraday's constant, and M is the actual mobility, see e.g. [10, Sect.3.3.2] or [25, Sect.3.4]. The mentioned non-negativity of the Peltier-like term then holds: indeed, by using Green's formula twice, we get

$$\begin{aligned}\int_\Omega\frac{d(\theta)\nabla q{\cdot}\nabla\phi}{\theta}\,\mathrm{d}x&=\int_\Omega RM\nabla q{\cdot}\nabla\phi\,\mathrm{d}x=-\varepsilon RM\int_\Omega\nabla(\Delta\phi){\cdot}\nabla\phi\,\mathrm{d}x\\&=\varepsilon RM\int_\Omega|\Delta\phi|^2\,\mathrm{d}x-\varepsilon RM\int_\Gamma\Delta\phi\frac{\partial\phi}{\partial\mathfrak{n}}\,\mathrm{d}S\geq RM\int_\Gamma q\alpha\big(\phi_\Gamma{-}\phi\big)\,\mathrm{d}S,\end{aligned} \tag{2.18}$$

so that the overall entropy production by the term $d(\theta)\nabla q\cdot\nabla\phi/\theta$ inside Ω is non-negative if the system is isolated, i.e. $\alpha=0$.

Remark 2.4 (*One simple test.*) Let us test the model on a simple example of an electrolyte composed from two constituents, cations and anions with equal charge (but opposite sign, of course, i.e. $L=2$ and $e_1=-e_2>0$) in a calm initial state (i.e. $v_0=0$) in thermal equilibrium (i.e. $\theta_0=$constant) placed in a container of the length D between two electrodes with voltage U and the constant coefficient $\alpha=\alpha_0$ as indicated on Figure 1. Assume further the electro-neutrality initial and boundary conditions, i.e. $c_{01}=\frac{1}{2}=c_{02}$ and $c_1^\Gamma=\frac{1}{2}=c_2^\Gamma$. The experience related with this virtual experiment ultimately says that the electrolyte will remain calm (i.e. $v=0$) and electro-neutral (i.e. $q=0$) and simultaneously will conduct an electric current which will heat it up.

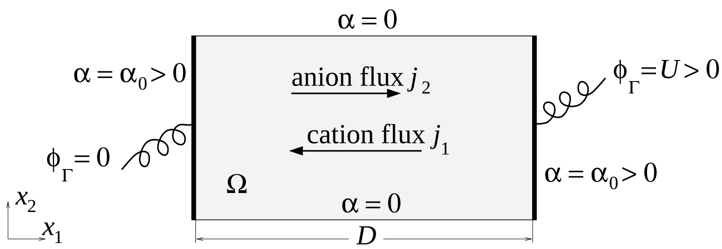

Fig. 1. A virtual experiment with electro-neutral two-component electrolyte placed into an electrostatic field between two electrodes.

Indeed, (2.7a) here says $r_1+r_2=0$ and $r_1-r_2=0$ so that ultimately $r_1=r_2=0$; it says that no chemical reaction can run if the third constituent is not allowed to be created. It is a matter of simple direct calculations to verify that $c_1=c_2=\frac{1}{2}$, $v=0$, φ constant in time and affine in space with $\nabla\phi=\big(\alpha_0U/(\alpha_0D+2\varepsilon),0\big)$, and θ constant in space and increasing linearly in time with the constant rate $\frac{\partial}{\partial t}\theta=c_{\rm v}^{-1}me_1^2\alpha_0^2U^2/(\alpha_0D+2\varepsilon)^2$ consist a solution to the initial-boundary-value problem (2.1), (2.2), and (2.6). The diffusive flux is obviously $j_1=\big(-me_1\alpha_0U/(2\alpha_0D+4\varepsilon),0\big)=-j_2$ and the power of Joule's heat per unit volume is $-e_1j_1\cdot\nabla\phi-e_2j_2\cdot\nabla\phi=me_1^2\alpha_0^2U^2/(\alpha_0D+2\varepsilon)^2$. The specific electric conductivity is me_1^2.

Remark 2.5 (*A special case: diluted water solutions.*) In very diluted water solutions of salts, that typically occur in conventional electro-chemistry or biological applications too, an alternative option is to consider velocity of water as the referential velocity instead of the barycentric one as used here. This is sometimes called Hittorf's referential system. Then, assuming again that diffusivity and mobility coefficients are the same for each constituents and after suitable simplifications relying on small concentrations of non-water constituents, the "reaction force" $f_{\rm R}=q\nabla\phi$ arises simply by transformation from the Hittorf's system to the barycentric one; see [32,35]. This gives a certain light to our arguments in (2.5) which holds exactly for general mixtures being based on the only assumption that $f_{\rm R}$ acts equally on each constituent.

3 Analysis of the model

We use the following standard notation for functions spaces: $L^r(\Omega;\mathbb{R}^3)$ denotes the Lebesgue space of measurable functions $\Omega\to\mathbb{R}^3$ whose r-power is integrable, $W_0^{1,2}(\Omega;\mathbb{R}^3)$ is the Sobolev space of functions whose gradient is in $L^2(\Omega;\mathbb{R}^{n\times n})$ and whose trace on $\partial\Omega$ vanishes, $W_{0,\rm DIV}^{1,2}(\Omega;\mathbb{R}^3)=\{v\in W_0^{1,2}(\Omega;\mathbb{R}^3);\ {\rm div}\,v=0$ in the sense of distributions$\}$, and $W^{-1,2}(\Omega;\mathbb{R}^3)\cong W_0^{1,2}(\Omega;\mathbb{R}^3)^*$. Likewise, $W^{k,2}$ indicates all kth derivatives belonging to the L^2 space. Occasionally, we will use also k non-integer, referring to the Sobolev-Slobodetskiĭ space with fractional derivatives. We will assume the following data qualification:

$$\varepsilon,\ \nu,\ c_{\rm v},\ \varrho,\ \kappa,\ m\ \text{ positive constants},\quad \alpha=\alpha(x)\geq 0, \tag{3.1a}$$

$$v_0\in L^2(\Omega;\mathbb{R}^3),\ \ c_0\in L^2(\Omega;\mathbb{R}^L),\ \ \theta_0\in L^2(\Omega), \tag{3.1b}$$

$$r_\ell:\mathbb{R}^{L+1}\to\mathbb{R}\ \text{continuous},\ \ |r_\ell(c,\theta)|\leq L_0+L_1|\theta|^{1-\eta}, \tag{3.1c}$$

$$h_\ell:\mathbb{R}\to\mathbb{R}\ \text{continuous and bounded}, \tag{3.1d}$$

$$d:\mathbb{R}\to\mathbb{R}\ \text{continuous},\ \ 0<d_0\leq d(\cdot)\leq d_1\,, \tag{3.1e}$$

for some $0<\eta\leq 1$ and some $d_1,d_2\in\mathbb{R}$. The sub-linear growth of reaction rates is certainly not a realistic assumptions because usually even an exponential growth is a typical phenomenon. Likewise, enthalpies $h_\ell(\theta)$ usually growth linearly with temperature so their boundedness is a simplifying assumption, too. Yet, it seems difficult to exclude a blow-up in finite time (i.e. an explosion) via some finer assumptions. Moreover, (3.1) is inconsistent with (2.17) which would require very sophisticated mathematical tricks, as already mentioned in Remark 2.3.

The notion of a weak solution to (3.9) can be defined, except (3.6), standardly as follows:

Definition 3.1 We will call $v\in L^2(I;W_{0,\rm DIV}^{1,2}(\Omega;\mathbb{R}^3))$, $\phi\in L^\infty(I;W_0^{1,2}(\Omega))$, $c\in L^2(I;W^{1,2}(\Omega;\mathbb{R}^L))$, and $\theta\in L^2(I;W^{1,2}(\Omega))$ a weak solution to the system (2.1) with the initial and boundary conditions (2.2) and (2.6) if

$$\int_Q\varrho v\frac{\partial z}{\partial t}-\nu\nabla v{:}\nabla z-\Big(\varrho(v\cdot\nabla)v+\sum_{\ell=1}^L c_\ell e_\ell\nabla\phi\Big)\cdot z\,{\rm d}x\,{\rm d}t=-\varrho\int_\Omega v_0(x)\cdot z(0,x)\,{\rm d}x \tag{3.2}$$

for any $z\in L^2(I;W_{0,\rm DIV}^{1,2}(\Omega;\mathbb{R}^3))\cap W^{1,2}(I;L^{6/5}(\Omega;\mathbb{R}^3))$ with $z(\cdot,T)=0$, where ":" means $[\tau_{ij}]{:}[e_{ij}]=\sum_{i=1}^n\sum_{j=1}^n\tau_{ij}e_{ij}$.

$$\int_Q c\cdot\frac{\partial z}{\partial t}+(j+c\otimes v):\nabla z+r(c,\theta)z\,{\rm d}x\,{\rm d}t=-\int_\Omega c_0\cdot z(0,x)\,{\rm d}x \tag{3.3}$$

satisfying also the boundary conditions $c_\ell|_\Sigma = c_\ell^\Gamma$ with the flux vector $j = (j_1, ..., j_L) \in L^2(Q;\mathbb{R}^{3\times L})$ defined in (2.1b) and $c_0 = (c_{01}, ..., c_{0L})$ from (2.2) and with the test-function $z \in L^2(I;W^{1,2}(\Omega;\mathbb{R}^L)) \cap W^{1,2}(I;W^{6/5}(\Omega;\mathbb{R}^L))$ arbitrary with $z(\cdot,T) = 0$,

$$\int_Q \varepsilon\nabla\phi\cdot\nabla z - qz\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{3.4}$$

for any $z \in L^2(I;W^{1,2}(\Omega;\mathbb{R}^L))$, and

$$\int_Q c_\mathrm{v}\theta\frac{\partial z}{\partial t} - (c_\mathrm{v}v\theta+\kappa\nabla\theta)\cdot\nabla z + fz\,\mathrm{d}x\,\mathrm{d}t = -c_\mathrm{v}\int_\Omega \theta_0 z(0,x)\,\mathrm{d}x \tag{3.5}$$

with $f \in L^1(Q)$ from (2.12) for any z smooth with $z(\cdot,T) = 0$ on Ω and $\frac{\partial}{\partial\mathrm{n}}z = 0$ on Σ. Finally, c satisfies

$$\sum_{\ell=1}^{L} c_\ell = 1 \ \ \& \ \ c_\ell \geq 0 \quad \text{a.e. on } Q. \tag{3.6}$$

Remark 3.2 The volume-additivity constraint and non-negativity of all c_ℓ, i.e. (3.6), which gives the vector $(c_1, ..., c_L)$ the desired sense of concentrations of particular constituents, is not explicitly involved in the equations (2.1) and indeed cannot be read from them. Anyhow, the assumptions (2.7) will impose these additional algebraic constraints in a fine way through the specific structure of the system (2.1).

In what follows, we will confine ourselves to two special cases only because the general case (2.1) seems to bring serious difficulties. This is because to treat the heat equation in the framework of conventional L^2-theory as in Section 3.1 one would need a regularity of the Oseen problem with the "fixed" velocity of the same quality, which is similar as in the Navier-Stokes system but this is recognized as an extremely difficult and so far open problem for general 3-dimensional case with large data. Without this regularity, one can treat the heat equation in the framework of L^1-theory as in [21] but then, beside other technical troubles, the continuity needed for the fixed-point theorem seems difficult due to the advection term. The analysis of the full system (2.1) seems to require some modifications, e.g. power-law shear-thickening non-Newtonian fluids instead of the Newtonian fluid (2.1a) as shown recently in [30].

3.1 Stokes' case.

In this subsection, we will assume that the velocity v is so small that the quadratic term $(v\cdot\nabla)v$ play a role of a 2nd-order perturbation and can be neglected in (2.1a). In other words, we consider a fully laminar flow with Reynolds' number zero that can be described by the Stokes equation instead of the Navier-Stokes equation (2.1a). As we will employ regularity both for the Poisson equation and for the Stokes system, we have additionally to assume

$$\Omega \text{ is of the class } C^{2,\mu},\ \mu>0, \text{ and } \ \phi_\Gamma,\alpha\in L^\infty(\Gamma) \text{ so smooth that} \tag{3.7a}$$

$$q\mapsto\phi : L^2(\Omega)\to W^{2,2}(\Omega) \text{ is bounded with } \phi \text{ solving (2.1c)–(2.6)}, \tag{3.7b}$$

$$v_0 \in W_0^{2,2}(\Omega;\mathbb{R}^3). \tag{3.7c}$$

For analysis, we define a retract $K : \{\xi{\in}\mathbb{R}^L;\ \sum_{\ell=1}^L \xi_\ell = 1\} \to \{\xi{\in}\mathbb{R}^L; \sum_{\ell=1}^L \xi_\ell = 1 \ \&\ \xi_\ell \geq 0,\ \ell = 1, ..., L\}$ by

$$K_\ell(\xi) := \frac{\xi_\ell^+}{\sum_{l=1}^L \xi_l^+}, \qquad \xi_\ell^+ := \max(\xi_\ell, 0). \tag{3.8}$$

Note that K is continuous and bounded. Starting with $\bar{c} \equiv (\bar{c}_\ell)_{\ell=1,..,L}$, $\bar{v}$ and $\bar{\theta}$ given such that $\sum_{\ell=1}^L \bar{c}_\ell = 1$, we solve successively the following auxiliary decoupled system consisting in the Poisson equation, the Stokes equation, the generalized Nernst-Planck equations, and finally the heat equation, i.e.

$$\varepsilon\Delta\phi = -q\,, \qquad q = \sum_{\ell=1}^{L} e_\ell K_\ell(\bar{c})\,, \tag{3.9a}$$

$$\varrho\frac{\partial v}{\partial t} - \nu\Delta v + \nabla p = q\,\nabla\phi, \qquad \mathrm{div}(v) = 0\,, \tag{3.9b}$$

$$\frac{\partial c_\ell}{\partial t} - \mathrm{div}(d(\bar{\theta})\nabla c_\ell - c_\ell v) = r_\ell\big(K(\bar{c}),\bar{\theta}\big) - \mathrm{div}\big(mK_\ell(\bar{c})(e_\ell - q)\nabla\phi\big), \quad \ell = 1,..,L, \tag{3.9c}$$

$$c_{\rm v}\frac{\partial\theta}{\partial t} - \mathrm{div}\big(\kappa\nabla\theta - c_{\rm v}v\theta\big) = \widehat{f}(v, K(\bar{c}), c, \phi, \bar{\theta}) \tag{3.9d}$$

where, similarly as in (2.12), the heat source equals

$$\widehat{f}(v,w,c,\phi,\bar{\theta}) := \nu|\nabla v|^2 + \sum_{\ell=1}^{L}\big(f_\ell\cdot j_\ell - h_\ell(\bar{\theta})r_\ell(w,\bar{\theta})\big), \tag{3.10}$$

$$\text{with } j_\ell = mw_\ell\Big(\sum_{l=1}^{L} e_l w_l - e_\ell\Big)\nabla\phi - d(\bar{\theta})\nabla c_\ell, \qquad f_\ell = -e_\ell\nabla\phi. \tag{3.11}$$

Involving also the initial and the boundary conditions (2.2)–(2.6), the notion of the weak solutions to (3.9) is understood in a way analogous to Definition 3.1.

Lemma 3.3 *Let (2.7a,c,d), (3.1), and (3.7) hold. For any $\bar{c} \in L^2(Q;\mathbb{R}^L)$ satisfying $\sum_{\ell=1}^L \bar{c}_\ell = 1$ and any $\bar{\theta} \in L^2(Q)$, the equations (3.9) have a weak solution (v,ϕ,c,θ) which is unique and satisfies the following a-priori bounds:*

$$\big\|\phi\big\|_{L^\infty(I;W^{2,2}(\Omega))} \le C_0, \tag{3.12a}$$

$$\big\|v\big\|_{L^6(I;W^{2,6}(\Omega;\mathbb{R}^3))\cap L^\infty(I;L^2(\Omega;\mathbb{R}^3))} \le C_0, \qquad \Big\|\frac{\partial v}{\partial t}\Big\|_{L^2(Q;\mathbb{R}^3)} \le C_0, \tag{3.12b}$$

$$\big\|c_\ell\big\|_{L^2(I;W^{1,2}(\Omega))\cap L^\infty(I;L^2(\Omega))} \le C_0 + C_1\|\bar{\theta}\|_{L^2(Q)}^{1-\eta}, \quad \Big\|\frac{\partial c_\ell}{\partial t}\Big\|_{L^2(I;W^{1,2}(\Omega)^*)} \le C_0 + C_1\|\bar{\theta}\|_{L^2(Q)}^{1-\eta}, \tag{3.12c}$$

$$\big\|\theta\big\|_{L^2(I;W^{1,2}(\Omega))\cap L^\infty(I;L^2(\Omega))} \le C_0 + C_1\|\bar{\theta}\|_{L^2(Q)}^{1-\eta}, \quad \Big\|\frac{\partial\theta}{\partial t}\Big\|_{L^2(I;W^{1,2}(\Omega)^*)} \le C_0 + C_1\|\bar{\theta}\|_{L^2(Q)}^{1-\eta}, \tag{3.12d}$$

with the constants C_0 and C_1 independent of $\bar{c}$ and $\bar{\theta}$. Besides, c satisfies the volume-additivity constraint $\sum_{\ell=1}^L c_\ell = 1$ (but not necessarily $c_\ell \ge 0$).

Proof. Existence of weak solutions of the particular decoupled equations (3.9) can be shown by usual methods, e.g. by using Galerkin's approximation; realize that all these equations are linear. The only essential point are the a-priori estimates.

Using the usual $W^{2,2}$-regularity for (3.9a), we obtain the estimate (3.12a); realize the smoothness assumptions (3.7a,b) for Ω, α and ϕ_Γ, and that eventually $K(\bar{c})$ is a-priori bounded even in $L^\infty(Q;\mathbb{R}^L)$ if $\sum_{\ell=1}^L \bar{c}_\ell = 1$ as indeed assumed. For regularity of (3.9b), we use a result for the evolutionary Stokes problem

$$\varrho\frac{\partial v}{\partial t} - \nu\Delta v + \nabla p = g, \qquad \mathrm{div}(v) = 0\,, \tag{3.13}$$

with $g := \sum_{\ell=1}^L K_\ell(\bar{c})e_\ell\nabla\phi$, whose solution satisfies the bound $\|v\|_{L^6(I;W^{2,6}(\Omega;\mathbb{R}^3))} \le C\|g\|_{L^6(I;L^6(\Omega))}$, see Solonnikov [38,39]; even a bit less regularity of v_0 than assumed in (3.7c) is needed for this result.

Due to the a-priori bound (3.12a), we have even better integrability of g, namely $\|g\|_{L^\infty(I;L^6(\Omega;\mathbb{R}^3))} \leq \|\sum_{\ell=1}^L K_\ell(\bar{c})e_\ell\|_{L^\infty(Q)}\|\nabla\phi\|_{L^\infty(I;L^6(\Omega;\mathbb{R}^3))}$ a-priori bounded. The test of (3.13) by $\partial v/\partial t$ yields standardly $\|\partial v/\partial t\|_{L^2(Q;\mathbb{R}^3)}$ a-priori bounded; here $v_0 \in W_0^{1,2}(\Omega)$ is needed but we assumed even more in (3.7c).

Now we test (3.9c) by c_ℓ and use Green's formula for both the left-hand and the right-hand sides and the identities

$$\int_\Omega \operatorname{div}(c_\ell v)c_\ell\,\mathrm{d}x = -\int_\Omega c_\ell v\nabla c_\ell\,\mathrm{d}x = -\frac{1}{2}\int_\Omega v\nabla|c_\ell|^2\,\mathrm{d}x = \frac{1}{2}\int_\Omega \operatorname{div}(v)|c_\ell|^2\,\mathrm{d}x = 0 \tag{3.14}$$

and, when employing the boundary conditions (2.6), also

$$\begin{aligned}\int_\Omega -\operatorname{div}\big(mK_\ell(\bar{c})(e_\ell-q)\nabla\phi\big)c_\ell\,\mathrm{d}x &= \int_\Omega \big(mK_\ell(\bar{c})(e_\ell-q)\nabla\phi\big)\cdot\nabla c_\ell\,\mathrm{d}x\\ &\quad+\int_\Gamma mK_\ell(\bar{c})(e_\ell-q)\alpha(\phi-\phi_\Gamma)c_\ell^\Gamma\,\mathrm{d}S.\end{aligned} \tag{3.15}$$

By this way, we obtain the estimate

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\big\|c_\ell\big\|^2_{L^2(\Omega)} + d_0\big\|\nabla c_\ell\big\|^2_{L^2(\Omega;\mathbb{R}^3)} \leq \int_\Omega r_\ell\big(K(\bar{c}),\bar{\theta}\big)c_\ell\\ &\quad+\big(mK_\ell(\bar{c})(e_\ell-q)\nabla\phi\big)\cdot\nabla c_\ell\,\mathrm{d}x + \int_\Gamma mK_\ell(\bar{c})(e_\ell-q)\alpha(\phi-\phi_G)c_\ell\,\mathrm{d}S\\ &\quad\leq C\big(1+\big\|\bar{\theta}\big\|^{1-\eta}_{L^2(\Omega)}\big)\big(1+\big\|c_\ell\big\|^2_{L^2(\Omega)}\big) + \frac{2m}{d_0}\max_{l=1,\ldots,L} e_l^2\,\big\|\nabla\phi\big\|^2_{L^2(\Omega;\mathbb{R}^3)}\\ &\quad+\frac{d_0}{2}\big\|\nabla c_\ell\big\|^2_{L^2(\Omega;\mathbb{R}^3)} + 2m\alpha\max_{l=1,\ldots,L}|e_l|\Big(N_1\big\|\phi(t,\cdot)\big\|_{W^{1,2}(\Omega)} + N_2\big\|\phi_\Gamma\big\|_{W^{1/2,2}(\Gamma)}\Big)\end{aligned} \tag{3.16}$$

where d_0 is from (3.1f) and $C = C(L_0, L_1, \Omega, \eta)$ is a constant and N_1 and N_2 denote the norm of the trace operator $\phi \mapsto \phi|_\Gamma : W^{1,2}(\Omega) \to L^1(\Gamma)$ and of the embedding $W^{1/2,2}(\Gamma) \subset L^1(\Gamma)$, respectively. Note that we used a trivial estimate $\|e_\ell - q\|_{L^\infty(\Omega)} \leq 2\max_{l=1,\ldots,L}|e_l|$. Altogether, the estimate (3.12c) follows by Gronwall's inequality. To be more precise, (3.15) and thus also (3.16) requires the trace of $\bar{c}$ on Γ to be defined, but eventually the estimate is completely independent of this trace because K_ℓ is bounded, hence this estimate holds for a general $\bar{c} \in L^2(Q;\mathbb{R}^L)$ by a density argument. The second estimate in (3.12c) can be obtained by testing (3.9c) by $z \in L^2(I;W^{1,2}(\Omega))$ as follows:

$$\begin{aligned}\Big\|\frac{\partial c_\ell}{\partial t}\Big\|_{L^2(I;W^{1,2}(\Omega)^*)} &:= \sup_{\|z\|_{L^2(I;W^{1,2}(\Omega))}\leq 1}\Big\langle\frac{\partial c_\ell}{\partial t}, z\Big\rangle\\ &= \sup_{\|z\|_{L^2(I;W^{1,2}(\Omega))}\leq 1}\bigg(\int_Q d(\bar{\theta})\nabla c_\ell\cdot\nabla z - c_\ell v\cdot\nabla z - r_\ell(K(\bar{c},\bar{\theta})z\\ &\qquad -mK_\ell(\bar{c})(e_\ell{-}q)\nabla\phi\cdot\nabla z\,\mathrm{d}x\,\mathrm{d}t + \int_\Sigma m\alpha K_\ell(\bar{c})(e_\ell{-}q)(\phi{-}\phi_\Gamma)z\,\mathrm{d}S\,\mathrm{d}t\bigg)\\ &\leq C\Big(\big\|\nabla c_\ell\big\|_{L^2(Q;\mathbb{R}^3))} + \big\|c_\ell\big\|_{L^\infty(I;L^2(\Omega))}\big\|v\big\|_{L^6(I;L^\infty(\Omega;\mathbb{R}^3))}\\ &\qquad +1+\big\|\bar{\theta}\big\|^{1-\eta}_{L^2(Q)} + \big\|\nabla\phi\big\|_{L^2(Q;\mathbb{R}^3))} + \big\|\phi-\phi_\Gamma\big\|_{L^2(I;W^{1/2}(\Gamma))}\Big)\end{aligned} \tag{3.17}$$

where $C = C\big(\Omega, d_1, m, \alpha, \max_\ell|e_\ell|\big)$ is a constant. Then we use (3.12b) and the already proved part of (3.12c)

To go on to (3.12d), let us now estimate the particular terms in $\widehat{f}(v, K(\bar{c}), c, \phi, \bar{\theta})$ from (3.11). The first term, $\nu|\nabla v|^2$, is a-priori bounded in $L^3(I;L^\infty(\Omega))$ because of the estimate (3.12b). The term $e_\ell\nabla c_\ell{\cdot}\nabla\phi$ can be estimated as $\|e_\ell\nabla c_\ell{\cdot}\nabla\phi\|_{L^2(I;L^{3/2}(\Omega))} \leq |e_\ell|\,\|\nabla c_\ell\|_{L^2(Q;\mathbb{R}^3)}\|\nabla\phi\|_{L^\infty(I;L^6(\Omega;\mathbb{R}^3))}$ hence it is a-priori bounded in $L^2(I;L^{3/2}(\Omega))$ and hence also in $L^2(I;L^{6/5}(\Omega))$ which is a subspace of the "energetic dual” to $L^2(I;W^{1,2}(\Omega))$ in our 3-dimensional case. The next term, i.e. $me_\ell K_\ell(\bar{c})(e_\ell - q)|\nabla\phi|^2$, is a-priori bounded even in $L^\infty(I;L^3(\Omega))$ due to the estimate (3.12a). The last term, $-h_\ell(\bar{\theta})r_\ell(K(\bar{c}),\bar{\theta})$, can be estimated, e.g., in $L^2(Q)$ bounded as $\mathcal{O}(\|\bar{\theta}\|_{L^2(Q)}^{1-\eta})$. Then, testing (3.9d) by θ yields, after using Green’s formula for the left-hand side and the identity (3.14) for θ instead of c_ℓ, the first part of the estimate (3.12d). The second part of (3.12d) can then be got like (3.17).

The uniqueness of the solutions to the auxiliary de-coupled equations (3.9) is trivial when realizing that all those equations are linear and using formulae like (3.14) when testing by the difference of two solutions.

Now, we have to prove that the constraint $\sum_{\ell=1}^L c_\ell = 1$ is satisfied. Let us abbreviate $\sigma(t,\cdot) := \sum_{\ell=1}^L c_\ell(t,\cdot)$ By summing (3.9c) for $\ell = 1, ..., L$, one gets

$$\begin{aligned}\frac{\partial\sigma}{\partial t} &= \sum_{\ell=1}^{L} r_\ell\big(K(\bar{c}),\theta\big) + \mathrm{div}\Big(d(\bar{\theta})\nabla\sigma + v\sigma \\ &\quad - \sum_{\ell=1}^{L} mK_\ell(\bar{c})\Big(e_\ell - \sum_{l=1}^{L} e_l K_l(\bar{c})\Big)\nabla\phi\Big) = \mathrm{div}\big(d(\bar{\theta})\nabla\sigma\big) + v\cdot\nabla\sigma \qquad (3.18)\end{aligned}$$

where (2.7a) has been used. Thus (3.18) results to the linear equation $\frac{\partial}{\partial t}\sigma - v\cdot\nabla\sigma - \mathrm{div}(d(\bar{\theta})\nabla\sigma) = 0$. We assumed $\sigma|_{t=0} = \sum_{\ell=1}^L c_{0\ell} = 1$ and $\sigma|_\Sigma = \sum_{\ell=1}^L c_\ell^\Gamma = 1$ on Σ, cf. (2.2) and (2.6) with (2.7c,d), so that the unique solution to this equation is $\sigma(t,\cdot) \equiv 1$ for any $t > 0$. □

Lemma 3.4 *Let (3.1a), and (3.7a,b) hold. Then the mapping $\bar{c} \mapsto \phi$, $\sum_{\ell=1}^L \bar{c}_\ell = 1$, determined by (3.9a) is continuous as a mapping $L^2(Q;\mathbb{R}^L) \to L^r(I;W^{2,2}(\Omega))$ with $1 \le r < +\infty$ arbitrary.*

Proof. Obvious from the continuity of the Nemytskiĭ mapping $\bar{c} \mapsto K(\bar{c}) : L^2(Q;\mathbb{R}^L) \to L^r(Q;\mathbb{R}^L)$ when restricted on $\{\bar{c} \in L^2(Q;\mathbb{R}^L);\ \sum_{\ell=1}^L \bar{c}_\ell = 1\}$ and by the a-priori estimate (3.12a) and linearity of the equation (3.9a). □

Lemma 3.5 *Let (3.1a,b), and (3.7). Then the mapping $\bar{c} \mapsto v$ determined by (3.9b) with ϕ determined by (3.9a) is continuous as a mapping $L^2(Q;\mathbb{R}^L) \to L^6(I;W^{1,6}(\Omega;\mathbb{R}^3))$ if $\bar{c}$ is again subjected to the constraints $\sum_{\ell=1}^L \bar{c}_\ell = 1$.*

Proof. The mapping $(\bar{c},\phi) \mapsto K_\ell(\bar{c})\nabla\phi : L^2(Q) \times L^r(I;W^{2,2}(\Omega)) \to L^r(I;L^6(\Omega;\mathbb{R}^3))$ is continuous if $\sum_{\ell=1}^L \bar{c}_\ell = 1$ holds. The solution to the Stokes problem depends continuously on the right-hand side from $L^r(I;L^6(\Omega;\mathbb{R}^3))$ to $L^6(I;W^{2,2}(\Omega;\mathbb{R}^3))$; cf. the a-priori estimate (3.12c) and realize the linearity of (3.9b). □

Lemma 3.6 *Let (2.7a,c,d), (3.1), and (3.7) hold. Then the mapping $(\bar{c},\bar{\theta}) \mapsto c$ determined by (3.9c) with ϕ determined by (3.9a) and v determined by (3.9b) is continuous as a mapping $L^2(Q;\mathbb{R}^L) \times L^2(Q) \to L^2(I;W^{1,2}(\Omega;\mathbb{R}^L))$.*

Proof. One can easily prove the continuity to the weak topology of $L^2(I;W^{1,2}(\Omega;\mathbb{R}^L))$, cf. also the a-priori estimate (3.12c). To prove the continuity to the norm topology, let us take a sequence $(\bar{c}^k,\bar{\theta}^k)$ converging to $(\bar{c},\bar{\theta})$ and the corresponding weak solutions c_ℓ^k converging weakly to c_ℓ. Subtracting (3.12c) written for c_ℓ^k from (3.12c) written for c_ℓ and testing the resulting equation by $c_\ell^k - c_\ell$, one can estimate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big\|c_\ell^k - c_\ell\big\|_{L^2(\Omega)}^2 + d_0\big\|\nabla(c_\ell^k - c_\ell)\big\|_{L^2(\Omega;\mathbb{R}^3)}^2 &= \int_\Omega (c_\ell v - c_\ell^k v^k)\nabla(c_\ell^k - c_\ell) \\ &+ \Big(r_\ell\big(K(\bar{c}^k),\bar{\theta}^k\big) - r_\ell\big(K(\bar{c}),\bar{\theta}\big)\Big)(c_\ell^k - c_\ell) \\ &+ m\Big(K_\ell(\bar{c}^k)(e_\ell - q^k)\nabla\phi^k - K_\ell(\bar{c})(e_\ell - q)\nabla\phi\Big)\cdot\nabla(c_\ell^k - c_\ell) \\ &+ \big(d(\bar{\theta}) - d(\bar{\theta}^k)\big)\nabla c_\ell\cdot\nabla(c_\ell^k - c_\ell)\,\mathrm{d}x \\ &+ \int_\Gamma m\alpha\Big(K_\ell(\bar{c}^k)(e_\ell - q^k)\phi c_\ell^k - K_\ell(\bar{c})(e_\ell - q)\phi c_\ell\Big)(c_\ell^k - c_\ell)\,\mathrm{d}S, \qquad (3.19)\end{aligned}$$

where naturally $q^k := \sum_{l=1}^L e_l\bar c_l^k$. By Aubin-Lions theorem (see [2] and [17, Sect.I.5.2]) and the a-priori estimate (3.12c), we know $c_\ell^k \to c_\ell$ strongly in $L^2(I;L^{6-\delta}(\Omega))$ for $\delta>0$ arbitrary. This convergence also holds weakly* in $L^\infty(I;L^2(\Omega))$. By interpolation (e.g. in ratio $\frac12$ and $\frac12$), one can see that

$$\big\|v_k-v\big\|_{L^4(I;L^{3-\varsigma}(\Omega))} \le \big\|v_k-v\big\|^{1/2}_{L^2(I;L^{6-\delta}(\Omega))}\big\|v_k-v\big\|^{1/2}_{L^\infty(I;L^2(\Omega))} \to 0 \tag{3.20}$$

with some $\zeta>0$ arbitrarily small (depending on $\delta>0$), cf. e.g. Lions [17, Sect.III.2.1]. Moreover, from Lemma 3.5, we already know that $v^k \to v$ in $L^6(I;W^{1,6}(\Omega;\mathbb R^3)) \subset L^6(I;L^\infty(\Omega;\mathbb R^3))$. Altogether, $(c_\ell v - c_\ell^k v^k)\nabla(c_\ell^k - c_\ell)$ converges to zero weakly in $L^{12/11}(I;L^{(6-2\zeta)/(5-\zeta)}(\Omega)) \subset L^1(Q)$. The next term converges to zero weakly in $L^1(I;L^{3/2}(\Omega))$ because $r_\ell\big(K(\bar c^k),\bar\theta^k\big) \to r_\ell\big(K(\bar c),\bar\theta\big)$ in $L^{2/(1-\eta)}(Q)$ due to the assumption (3.1d) and the standard Nemytskiĭ-mapping theorem and because $c_k\to c$ in $L^2(I;L^6(\Omega))$. The further term converges to zero weakly in $L^{2-\delta}(I;L^{3/2}(\Omega))$ for any $\delta>0$ because $K_\ell(\bar c^k)(e_\ell - q^k)\nabla\phi^k \to K_\ell(\bar c)(e_\ell-q)\nabla\phi$ in $L^r(I;L^6(\Omega;\mathbb R^3))$ and $\nabla c_k\to\nabla c$ weakly in $L^2(Q;\mathbb R^3)$. Taking $c_\delta \in L^\infty(I;W^{1,\infty}(\Omega))$ such that $\|\nabla c_\delta - \nabla c_\ell\|_{L^2(Q;\mathbb R^3)} \le \delta$, we can estimate

$$\int_0^t\!\!\int_\Omega \big(d(\bar\theta)-d(\bar\theta^k)\big)\nabla c_\ell\cdot\nabla(c_\ell^k - c_\ell)\,\mathrm dx\,\mathrm dt \le \int_0^t\!\!\int_\Omega \big(d(\bar\theta)-d(\bar\theta^k)\big)\nabla c_\delta\cdot\nabla(c_\ell^k - c_\ell)\,\mathrm dx\,\mathrm dt + \delta\big\|d(\bar\theta)-d(\bar\theta^k)\big\|_{L^\infty(Q)}\big\|\nabla(c_\ell^k-c_\ell)\big\|_{L^2(Q)}, \tag{3.21}$$

where the right-hand-side integral converges to zero because $\nabla c_\ell^k \to \nabla c_\ell$ weakly in $L^2(Q;\mathbb R^3)$ and $d(\bar\theta)\to d(\bar\theta^k)$ strongly in $L^2(Q)$, and therefore we can see that the left-hand-side integral converges to zero because $\delta>0$ can be taken arbitrarily small. Eventually, the boundary term in (3.19) simply vanishes because $c_\ell^k - c_\ell = c_\ell^\Gamma - c_\ell^\Gamma = 0$ on Γ. Altogether, from (3.19) by Gronwall's inequality, we get the strong convergence $c_\ell^k \to c_\ell$ in $L^2(I;W^{1,2}(\Omega))$, as claimed, and also in $L^\infty(I;L^2(\Omega))$. □

Lemma 3.7 *Let (2.7a,c,d), (3.1), and (3.7). Then the mapping $(\bar c,\bar\theta)\mapsto\theta$ determined by (3.9d) with c determined by (3.9c) with ϕ determined by (3.9a) and v determined by (3.9b) is continuous as a mapping $L^2(Q;\mathbb R^L)\times L^2(Q)\to L^2(Q)$.*

Proof. We start with proving continuity of $(v,\bar c,c,\phi,\bar\theta)\mapsto \widehat f(v,K(\bar c),c,\phi,\bar\theta)$ with $\widehat f$ from (3.10) as a mapping from $L^6(I;W^{1,6}(\Omega;\mathbb R^3))\times L^2(Q;\mathbb R^L)\times L^2(I;W^{1,2}(\Omega;\mathbb R^L))\times L^r(I;W^{2,2}(\Omega))\times L^2(Q)$ to the weak topology of $L^2(I;L^{6/5}(\Omega))$, which is a subset of the natural "energetic dual" $L^2(I;W^{1,2}(\Omega)^*)$, so that the standard L^2-theory for the heat-transfer equation will apply. Let us go through the particular terms in $\widehat f$.

By Lemma 3.5, $\bar v\mapsto|\nabla v|^2$ is continuous to the norm topology of $L^3(Q;\mathbb R^3))$ which is certainly a subset of $L^2(I;L^{6/5}(\Omega))$. As to $(c_\ell,\phi)\mapsto\nabla c_\ell\cdot\nabla\phi$, by Lemma 3.6 we know continuity in ∇c_ℓ in the norm topology of $L^2(Q)$ and by the a-priori estimate (3.12a) we know also the continuity in $\nabla\phi$ in the weak* topology of $L^\infty(I;L^6(\Omega))$, hence altogether we have continuity in $\nabla c_\ell\cdot\nabla\phi$ in the weak topology of $L^2(I;L^{3/2}(\Omega))$ which is again a subset of $L^2(I;L^{6/5}(\Omega))$. By Lemma 3.4 and by continuity of the Nemytskiĭ mappings, the continuity in the term $K_\ell(\bar c)(e_\ell - \sum_{l=1}^L K_l(\bar c))|\nabla\phi|^2$ is into the norm topology $L^{r/2}(I;L^3(\Omega))$ which is again a subset of $L^2(I;L^{6/5}(\Omega))$ if $r\ge4$ is considered. Eventually, the continuity in $r_\ell(K(\bar c),\bar\theta)$ in the norm topology of $L^{2/(1-\eta)}(Q)$ is a consequence of (3.1d).

Then, we get the continuity in θ in the weak topology of $L^2(I;W^{1,2}(\Omega))\cap W^{1,2}(\Omega;W^{1,2}(\Omega)^*)$, cf. the a-priori estimate (3.12d) and realize that the limit passage in the convective term $\mathrm{div}(v\theta)=v\cdot\nabla\theta$ is simply due to strong convergence in v. Eventually, the continuity in θ in the norm topology of $L^2(Q)$ is by the Aubin-Lions theorem. □

Proposition 3.8 *Let (2.7), (3.1), and (3.7) hold and let $R>0$ be so large that $R\ge\sqrt T(C_0+C_1R^{1-\eta})$ with C_0 and C_1 from Lemma 3.3 and η from (3.1d). Then the mapping $(\bar c,\bar\theta)\mapsto(c,\theta)$ has a fixed point (c,θ) on the set*

$$\Big\{(c,\theta)\in L^2(Q;\mathbb R^{L+1});\ \ \big\|c\big\|_{L^2(Q;\mathbb R^L)}\le R,\ \ \big\|\theta\big\|_{L^2(Q)}\le R,\ \ \sum_{\ell=1}^L c_\ell = 1\Big\}, \tag{3.22}$$

and moreover every such a fixed point satisfies also $c_\ell \geq 0$ *for any* ℓ*. Thus, considering also* ϕ *and* v *related with this fixed point* (c,θ)*, the quadruple* (ϕ,v,c,θ) *is a weak solution (in the sense of Definition 3.1) to the system (2.1) with the convective term* $(v\cdot\nabla)v$ *in (2.1a) omitted.*

Proof. By the a-priori estimate (3.12d), it holds $\|\theta\|_{L^2(Q)} \leq \sqrt{T}\|\theta\|_{L^\infty(I;L^2(\Omega))} \leq \sqrt{T}(C_0 + C_1\|\bar\theta\|^{1-\eta}_{L^2(Q)}) \leq R$ provided $\|\bar\theta\|_{L^2(Q)} \leq R$. By (3.12c), it then also holds $\|c\|_{L^2(Q;\mathbb{R}^L)} \leq R$. The continuity of $(\bar c,\bar\theta) \mapsto (c,\theta)$ in $L^2(Q;\mathbb{R}^{L+1})$ has been proved in previous Lemmas. By a-priori estimates (3.12c,d) and Aubin-Lions' theorem, the image of the convex set (3.22) is compact in $L^2(Q;\mathbb{R}^L)$. By Schauder's theorem, this mapping has a fixed point, say (c,v). Thus we get also ϕ, and θ, and the quadruple (ϕ,v,c,θ) is a weak solution to (3.9) provided we also prove (3.6).

The constraint $\sum_{\ell=1}^L c_\ell = 1$ is, as proved in (3.18), satisfied and, at this fixed point, we have additionally also $c_\ell(t,\cdot) \geq 0$ satisfied for any t. To see this, test (3.9c) written with $c_\ell = \bar c_\ell$ by the negative part c_ℓ^- of c_ℓ. Realizing $K_\ell(c)\nabla c_\ell^- = 0$ because, for a.a. $(t,x) \in Q$, either $K_\ell(c(t,x)) = 0$ (if $c_\ell(t,x) \leq 0$) or $\nabla c_\ell(t,x)^- = 0$ (if $c_\ell(t,x) > 0$), and $r_\ell(\cdot)c_\ell^- \geq 0$ because of (2.7b), we obtain $c_\ell^- = 0$ a.e. on Q. To be more precise, we can assume, for a moment, that r_ℓ is defined on the whole $\mathbb{R}^L$ in such a way that $r_\ell(c_1,...,c_L) \geq 0$ for $c_\ell < 0$. As we are just proving that $c_\ell \geq 0$, the values of r_ℓ for negative concentrations are eventually irrelevant.

The non-negativity of c_ℓ together with $\sum_{\ell=1}^L c_\ell = 1$ ensures that $c(t,x) \in \mathrm{Range}(K)$ for a.a. $(t,x) \in Q$ so that $c_\ell = K_\ell(c)$ and thus the quadruple (ϕ,v,c,θ) is a weak solution not only to (3.9) with $\bar v = v$ and $\bar c = c$ but even to the original system (2.1). $\square$

3.2 Isothermal case.

A lot of applications run essentially on constant temperature because of the negligible heat production and/or a sufficiently fast transfer of the produced heat outside the considered domain Ω. In such cases, we can consider the production rate $r_\ell = r_\ell(c)$ independent of θ, the diffusion coefficient d constant, and kick the heat equation (2.1d) out. This enables us to analyze the remaining system (2.1a-c) without any need of regularity of the Navier-Stokes system (2.1a) so that we can consider the convective term $(v\cdot\nabla)v$ in (2.1a), i.e. arbitrary Reynolds' numbers. Moreover, no regularity for the Poisson equation (2.1c) is needed, either, so we do not need the data qualification (3.7) at all. Even a more constructive analysis through the Galerkin method instead of the fixed-point approach used here is possible, as shown recently in [29].

For analysis, we will use again the retract K defined in (3.8) and design the fixed-point procedure as follows: starting with $\bar c \equiv (\bar c_\ell)_{\ell=1,..,L}$ and $\bar v$ given such that $\sum_{\ell=1}^L \bar c_\ell = 1$, we solve successively the following auxiliary decoupled system consisting in the Poisson, the approximate Navier-Stokes (so-called Oseen) equation, and finally the generalized Nernst-Planck equations, i.e.

$$\varepsilon\Delta\phi = -q\,, \quad q = \sum_{\ell=1}^L e_\ell K_\ell(\bar c)\,, \tag{3.23a}$$

$$\varrho\frac{\partial v}{\partial t} + \varrho(\bar v\cdot\nabla)v - \nu\Delta v + \nabla p = q\,\nabla\phi, \qquad \mathrm{div}(v) = 0\,, \tag{3.23b}$$

$$\frac{\partial c_\ell}{\partial t} - \mathrm{div}\big(d\nabla c_\ell - c_\ell\bar v\big) = r_\ell\big(K(\bar c)\big) - \mathrm{div}\big(mK_\ell(\bar c)(e_\ell - q)\nabla\phi\big), \quad \ell = 1,..,L. \tag{3.23c}$$

The notion of the weak solutions to (3.23) with the boundary and the initial conditions (2.2) and (2.6) is understood in a way analogous to Definition 3.1 with the heat equation (3.5) omitted, of course.

Lemma 3.9 *Let (2.7a,c,d) and (3.1) hold. For any* $\bar c \in L^2(Q;\mathbb{R}^L)$ *satisfying* $\sum_{\ell=1}^L \bar c_\ell = 1$ *and for any* $\bar v \in L^2(I;W^{1,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3)) \cap L^\infty(I;L^2(\Omega;\mathbb{R}^3))$*, the equations (3.23) have a weak solution* (v,ϕ,c) *which and satisfies the following a-priori bounds:*

$$\big\|\phi\big\|_{L^\infty(I;W^{1,2}(\Omega))} \leq C_0, \tag{3.24a}$$

$$\|v\|_{L^2(I;W^{1,2}(\Omega;\mathbb{R}^3))\cap L^\infty(I;L^2(\Omega;\mathbb{R}^3))} \le C_0, \tag{3.24b}$$

$$\Big\|\frac{\partial v}{\partial t}\Big\|_{L^{4/3}(I;W^{1,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3)^*)} \le C_0 + C_1\|\bar v\|_{L^2(I;W^{1,2}(\Omega;\mathbb{R}^3))\cap L^\infty(I;L^2(\Omega;\mathbb{R}^3))}, \tag{3.24c}$$

$$\|c_\ell\|_{L^2(I;W^{1,2}(\Omega))\cap L^\infty(I;L^2(\Omega))} \le C_0, \qquad \Big\|\frac{\partial c_\ell}{\partial t}\Big\|_{L^{4/3}(I;W^{1,2}(\Omega)^*)} \le C_0, \tag{3.24d}$$

with the constants C_0 and C_1 independent of $\bar c$ and $\bar v$. Besides, c always satisfies the volume-additivity constraint $\sum_{\ell=1}^L c_\ell = 1$ (but not necessarily $c_\ell \ge 0$).

Proof. It mostly simplifies the proof of Lemma 3.3 above. As to (3.24a), it just suffices to test (3.23a) by ϕ itself; note that no regularity is used now, unlike in Lemma 3.3 before. The estimate (3.24b) can be obtained by testing (3.23b) by v itself and using the usual trick that $\int_\Omega \nabla p\cdot v\,\mathrm{d}x = -\int_\Omega p\,\mathrm{div}(v)\,\mathrm{d}x = 0$ as well as $\int_\Omega(\bar v\cdot\nabla)v\cdot v\,\mathrm{d}x = 0$ so that the bound in (3.24b) is completely independent of $\bar v$. The estimate (3.24c) can be obtained by testing (3.23b) by a suitable z as follows:

$$\begin{aligned}\varrho\Big\|\frac{\partial v}{\partial t}\Big\|_{L^{4/3}(I;W^{1,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3)^*)} &:= \sup_{\|z\|_{L^4(I;W^{1,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3))}\le 1} \Big\langle \varrho\frac{\partial v}{\partial t}, z\Big\rangle \\ &= \sup_{\|z\|_{L^4(I;W^{1,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3))}\le 1} \int_Q \nu\nabla v : \nabla z + \varrho(\bar v\cdot\nabla)v\cdot z - q\,\nabla\phi\cdot z\,\mathrm{d}x\,\mathrm{d}t \\ &\le \|\nabla v\|_{L^2(Q;\mathbb{R}^{3\times3})}T^{1/4}\Big(\nu + \varrho N^{3/2}\|\bar v\|^{1/2}_{L^2(I;W^{1,2}(\Omega;\mathbb{R}^3))}\|\bar v\|^{1/2}_{L^\infty(I;L^2(\Omega;\mathbb{R}^3))}\Big) \\ &\qquad + 2N\max_{\ell=1,..,L}|e_\ell|\,\|\nabla\phi\|_{L^{4/3}(I;L^{6/5}(\Omega))}\end{aligned} \tag{3.25}$$

where we used the Hölder inequality and the interpolation as in (3.20) to estimate the convective term

$$\begin{aligned}\int_Q(\bar v\cdot\nabla)v\cdot z\,\mathrm{d}x\,\mathrm{d}t &\le \|\bar v\|_{L^4(I;L^3(\Omega;\mathbb{R}^3))}\|\nabla v\|_{L^2(Q;\mathbb{R}^{3\times3})}\|z\|_{L^4(I;L^6(\Omega;\mathbb{R}^3))} \\ &\le \|\bar v\|^{1/2}_{L^2(I;L^6(\Omega;\mathbb{R}^3))}\|\bar v\|^{1/2}_{L^\infty(I;L^2(\Omega;\mathbb{R}^3))}\|\nabla v\|_{L^2(Q;\mathbb{R}^{3\times3})}\|z\|_{L^4(I;L^6(\Omega;\mathbb{R}^3))}\end{aligned}$$

and where N denotes the norm of the embedding $W^{1,2}(\Omega)\subset L^6(\Omega)$. Using the already obtained estimates (3.24a) and (3.24b), the estimate (3.24c) follows.

The proof of (3.24d) remains essentially the same; note that neither (3.16) nor (3.17) needs any regularity of ϕ, the latter estimate (3.17) requires a modification

$$\begin{aligned}\int_Q c_\ell\bar v\cdot\nabla z\,\mathrm{d}x\,\mathrm{d}t &\le \|c_\ell\|_{L^2(I;L^6(\Omega))}\|\bar v\|_{L^4(I;L^3(\Omega;\mathbb{R}^3))}\|\nabla z\|_{L^4(I;L^2(\Omega;\mathbb{R}^3))} \\ &\le \|c_\ell\|_{L^2(I;L^6(\Omega))}\|\bar v\|^{1/2}_{L^2(I;L^6(\Omega;\mathbb{R}^3))}\|\bar v\|^{1/2}_{L^\infty(I;L^2(\Omega;\mathbb{R}^3))}\|\nabla z\|_{L^4(I;L^2(\Omega;\mathbb{R}^3))}.\end{aligned}$$ □

Let us abbreviate

$$\mathcal{W}_1 := \Big\{c\in L^2\big(I;W^{1,2}(\Omega;\mathbb{R}^L)\big);\ \ \frac{\partial c}{\partial t}\in L^{4/3}\big(I;W^{1,2}(\Omega;\mathbb{R}^L)^*\big)\Big\}, \tag{3.26}$$

$$\mathcal{W}_2 := \Big\{v\in L^2\big(I;W^{1,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3)\big);\ \frac{\partial v}{\partial t}\in L^{4/3}\big(I;W^{1,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3)^*\big)\Big\}. \tag{3.27}$$

Endowed by the respective "$\frac{\partial}{\partial t}$-graph" norms, these spaces become Banach spaces and the already used Aubin-Lions theorem [2,17] gives the compact embeddings $\mathcal{W}_1 \subset L^2(I;L^{6-\delta}(\Omega;\mathbb{R}^L))$ for any $\delta>0$, and similarly $\mathcal{W}_2 \subset L^2(I;L^{6-\delta}(\Omega;\mathbb{R}^3))$. Moreover, we will also use the well-known fact that $\mathcal{W}_2 \subset L^\infty(I;L^2(\Omega;\mathbb{R}^3))$ continuously.

Lemma 3.10 *Let (3.1a,b) hold. Then the set-valued mapping* $(\bar{c},\bar{v}) \mapsto \{v \in \mathcal{W}_2;\ v$ *is a weak solution to (3.23b) with* ϕ *determined by (3.23a)*$\}$ *is (weak,weak) upper semicontinuous convex-valued mapping* $\mathcal{W}_1 \times \mathcal{W}_2 \rightrightarrows \mathcal{W}_2$ *if* $\bar{c}$ *is again subjected to the constraints* $\sum_{\ell=1}^{L} \bar{c}_\ell = 1$.

Proof. Taking a sequence of $\{(\bar{c}^k,\bar{v}^k)\}_{k\in\mathbb{N}}$ converging weakly to $(\bar{c},\bar{v})$ in $\mathcal{W}_1 \times \mathcal{W}_2$, by Aubin-Lions' theorem we have $\bar{c}^k \to \bar{c}$ strongly in $L^2(Q;\mathbb{R}^L)$, hence $\phi^k \to \phi$ in $L^r(I;W^{1,2}(\Omega))$, and also $K_\ell(\bar{c}^k)\nabla\phi^k \to K_\ell(\bar{c})\nabla\phi$ in $L^r(I;L^2(\Omega;\mathbb{R}^3))$ with $r < \infty$ arbitrary. Then the limit passage in (3.23b) is routine; obviously $\int_Q(\bar{v}^k{\cdot}\nabla)v^k{\cdot}z\,\mathrm{d}x \to \int_Q(\bar{v}{\cdot}\nabla)v{\cdot}z\,\mathrm{d}x$ at least for $z \in L^\infty(Q)$ (those functions are densely contained in the set of test functions for (3.2), if they are contained at all) because $\bar{v}^k \to \bar{v}$ strongly in $L^2(Q;\mathbb{R}^3)$ and $\nabla v^k \to \nabla v$ weakly $L^2(Q;\mathbb{R}^{3\times3})$.

As (3.23a,b) is linear for $(\bar{c},\bar{v})$ fixed, the set of v's in question is convex. □

Lemma 3.11 *Let (2.7a,c,d) and (3.1). Then the set-valued mapping* $(\bar{c},\bar{v}) \mapsto \{c{\in}\mathcal{W}_1;\ c$ *is a weak solution to (3.23c) with with* ϕ *determined by (3.23a)*$\}$ *is (weak,weak) upper semicontinuous convex-valued mapping* $\mathcal{W}_1 \times \mathcal{W}_2 \rightrightarrows \mathcal{W}_1$ *if* $\bar{c}$ *is again subjected to the constraints* $\sum_{\ell=1}^{L} \bar{c}_\ell = 1$.

Proof. By a-priori estimates (3.24d), by standard arguments the limit passage in (3.23c) formulated weakly easily follows.

As (3.23a,c) is linear for $(\bar{c},\bar{v})$ fixed, the set of c's in question is convex. □

Proposition 3.12 *Let (2.7) and (3.1) hold. The set-valued mapping* $M : (\bar{c},\bar{v}) \mapsto \{(c,v){\in}\mathcal{W}_1{\times}\mathcal{W}_2;\ (c,v)$ *is a weak solution to (3.23b,c) with* ϕ *determined by (3.23a)*$\}$ *has a fixed point* (c,v) *on the convex closed set*

$$\Big\{(c,v){\in}\mathcal{W}_1{\times}\mathcal{W}_2:\ \ \big\|c\big\|_{L^2(I;W^{1,2}(\Omega;\mathbb{R}^L))} \le C_0,\ \ \Big\|\frac{\partial c}{\partial t}\Big\|_{L^{4/3}(I;W^{1,2}(\Omega;\mathbb{R}^L)^*)} \le C_0,$$
$$\big\|v\big\|_{L^2(I;W^{1,2}(\Omega;\mathbb{R}^3))\cap L^\infty(I;L^2(\Omega;\mathbb{R}^3))} \le C_0,$$
$$\Big\|\frac{\partial v}{\partial t}\Big\|_{L^{4/3}(I;W^{1,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^3)^*)} \le C_0(1{+}C_1),\ \ \sum_{\ell=1}^{L} c_\ell = 1\Big\} \tag{3.28}$$

with C_0 *and* C_1 *from (3.24). Moreover, every such a fixed point satisfies also* $c_\ell \ge 0$ *for any* ℓ*. Thus, considering also* ϕ *related with this fixed point* (c,v)*, the triple* (ϕ,v,c) *is a weak solution (in the sense of Definition 3.1) to the system (2.1) with the heat equation (2.1d) omitted.*

Proof. The (weak,weak) upper semicontinuity of $M : \mathcal{W}_1{\times}\mathcal{W}_2 \rightrightarrows \mathcal{W}_1{\times}\mathcal{W}_2$ has been proved in previous Lemmas 3.10 and 3.11. By a-priori estimates (3.24b-d) and by arguments as (3.18), this mapping maps the convex set (3.28) into itself, and the values of M are nonempty. By Lemmas 3.10 and 3.11, this values are also convex. Both $\mathcal{W}_1$ and $\mathcal{W}_2$ are compact if endowed with the weak topologies; here it is important that the set $\{v \in \mathcal{W}_2;\ \|v\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^3))} \le C_0\}$ is closed in $\mathcal{W}_2$ due to the continuous embedding $\mathcal{W}_2 \subset L^\infty(I;L^2(\Omega;\mathbb{R}^3))$. By the Kakutani fixed-point theorem saying that any upper semi-continuous nonempty-convex-valued mapping on a compact convex set has a fixed point, we obtain existence of a fixed point $(c,v) \in M(c,v)$. The non-negativity of c_ℓ is then to be proved as done Proposition 3.8. □

4 Concluding remarks

Remark 4.1 (*Composition-dependent coefficients.*) Making the coefficients $\varepsilon = \varepsilon(c)$, $d = d(c)$, $m = m(c)$, $c_\mathrm{v} = c_\mathrm{v}(c)$, or $\kappa = \kappa(c)$ dependent on the concentrations brings essentially no problems as far as this dependence is continuous and these coefficients do not degenerate to zero. The auxiliary decoupled systems (3.9) and (3.23) are then to be constructed by replacing c with $K(\bar{c})$ in these coefficients, cf. [29] for the isothermal case. On the other hand, making the mass density ϱ dependent on c would indicate that mass densities of particular constituents differ from each other, and then the whole concept becomes much more complicated because one must distinguish between volume fractions and mass fractions [35].

Remark 4.2 (*Alternative models.*) The dissipative heat, i.e. the first term in (2.12), is to be questioned. Considering only one-component electrically neutral system (i.e. $L = 1$, $e_1 = 0$), there are various models appearing in the literature, cf. e.g. [4,15,26] for a genesis of various possibilities in case of an additional buoyancy. The starting point is always the complete compressible fluid system of $n + 2$ conservation laws for mass, impulse, and energy; n denotes the spatial dimension. Then, the so-called incompressible limit represents a small perturbation around a stationary homogeneous state, i.e. around constant mass density, constant temperature, and zero velocity. E.g., the conventional Oberbeck-Boussinesq model neglects the dissipative heat. It should be emphasized that, though the original full system is thermodynamically consistent, the incompressible limit system of $n + 1$ equations in general violates both the energy conservation law and the Clausius-Duhem inequality. Hence it is certainly interesting that, in our case, we got these properties back.

Remark 4.3 (*Some special cases.*) The general system (2.1) covers also some other special cases studied in literature. Neglecting the heat equation (2.1d) as we did in Section 3.2 and further the Navier-Stokes flow part (2.1a) by considering a fully stationary medium, i.e. $v = 0$ and p constant, (2.1) reduces into the so-called Nernst-Planck-Poisson system, which is a basic model for electro-diffusion of ions in electrolytes formulated by W. Nernst and M. Planck at the end of 19th century, and which has massively been scrutinized in the literature, see Glitzky [11] for its mathematical analysis. Often, the electro-hydro-dynamics (EHD) does not require $\sum_{\ell=1}^{L} c_\ell = 1$, see e.g. [5,16,25,37] where however no mathematical analysis is done, or it is even considered as a constraint and involved through a Lagrange multiplier, see [22] for such an attempt. Neglecting the flow and the electric field (2.1a,c) by putting $v = 0$, $p = 0$, and $\phi = 0$, one gets the model studied by Henri [14] for the special case $r_\ell = \sum_j k_{\ell j} f_j$ where $f_j = f_j(c_1, ..., c_L, \theta)$.

Remark 4.4 (*More general mobility and diffusivity coefficients.*) Some mixtures exhibit markable differences between mobilities of particular constituents (especially if the size of the involved (macro)molecules varies considerably) and also cross-effects may occur. Then the diffusivity and mobility are rather matrices $d_{k\ell}$ and $m_{k\ell}$, respectively. We assume again that the reaction force $f_{\rm R}$ balancing the heat fluxes j_ℓ to zero sum (2.4) acts equally on each constituents, i.e. the previous setting $j_\ell = -d(\theta)\nabla c_\ell - mc_\ell(e_\ell\nabla\phi - f_{\rm R})$ generalizes to

$$j_\ell = \sum_{k=1}^{L} \Big(- d_{k\ell}(\theta)\nabla c_k - m_{k\ell}c_k\big(e_k\nabla\phi - f_{\rm R}\big)\Big). \tag{4.1}$$

The requirement (2.4) then ultimately implies by a simple algebra that $f_{\rm R}$ must take the form

$$f_{\rm R} = \sum_{k=1}^{L}\sum_{\ell=1}^{L} \Big(d_{k\ell}(\theta)\nabla c_k + m_{k\ell}c_k e_k\nabla\phi\Big)\Big/M, \qquad M := \sum_{k=1}^{L}\sum_{\ell=1}^{L} m_{k\ell}c_k. \tag{4.2}$$

By Onsager's principle [23], the matrices $[d_{k\ell}(\theta)]$ and $[m_{k\ell}]$ are symmetric. The former case $f_{\rm R} = q\nabla\phi$ is, of course, a special case of (4.2) for $[d_{k\ell}(\theta)]$ and $[m_{k\ell}]$ diagonal with $d_{\ell\ell}(\theta) = d(\theta)$ and $m_{\ell\ell} = m$ and with (2.3) holding, and it was considered for the sake of lucidity of the explanation not to make the formulas and the analysis too complicated. Let us only mention that, in the case (4.2), the a-prori estimates (3.12c) and (3.24d) must be done for all concentrations $c = (c_1, \cdots, c_L)$ simultaneously by summing the Nernst-Planck equations for c_ℓ tested by c_ℓ, which requires $\sum_{\ell=1}^{L} j_\ell \cdot \nabla c_\ell \geq \delta \sum_{\ell=1}^{L} |\nabla c_\ell|^2$ for some $\delta > 0$, i.e. $[d_{k\ell}(\theta)]$ to be positive definite uniformly with respect to θ. The fixed-point procedure (3.9) must be modified accordingly, i.e. all c_k in (4.2) are to be replaced by $K_k(c)$. The a-priori estimates as well as limit passage bear appropriate modifications, too. The parabolic equation (3.18) modifies to the hyperbolic $\frac{\partial}{\partial t}\sigma + v \cdot \nabla\sigma = 0$ which admits again the unique solution $\sigma = 1$ because of the initial and boundary conditions $\sigma = 1$ and because v and σ are enough regular. Let us finally mention that an attempt for another method to made (2.4) satisfied had been implemented in [12, Sect.2.5.1] without considering electric charges, however.

Acknowledgments: The author is deeply indebted to prof. F. Maršík, dr. M. Pokorný and prof. I. Samohýl for many valuable discussions, and to all 3 referees for many useful comments and suggestions. The work was partly supported also by the grants 201/03/0934 (GA ČR), and MSM 21620839 (MŠMT ČR).

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The second estimate in (3.24d) depends also on $\bar{v}$. More importandly, the non-negativity of c_ℓ in the proof of Proposition 3.12 unfortunately does not seem to be convincing becuase $\frac{\partial}{\partial t}c_\ell$ is not in duality to c_ℓ^- which is thus not a legal test function for the corresponding Nernst-Planck equation. The results in Sect.3.2 remains however true if one applies the following changes:

Instead of $c_\ell\bar{v}$ in (3.23c), put $K_\ell(\bar{c})v$.

Then (3.23c) still remains a system of separated single linear equations, each of them having a unique weak solution. The possible non-uniqueness comes from possible non-uniqueness of v but the convexity of the set-valued mapping in Lemma 3.11 is preserved. The estimate (3.24d) then applies with $L^{4/3}(I;.)$ replaced by $L^2(I;.)$ because the last estimate in the proof of Lemma 3.9 can now be made simply as

$$\int_Q K_\ell(\bar{c})v\cdot\nabla z\,\mathrm{d}x\,\mathrm{d}t \le \|v\|_{L^2(Q;\mathbb{R}^3)}\|\nabla z\|_{L^2(Q;\mathbb{R}^3)} \le T^{1/2}C_0\|\nabla z\|_{L^2(Q;\mathbb{R}^3)}$$

with C_0 referring to (3.24b); moreover, the estimate (3.24d) is now indeed independent of $\bar{v}$. Then (3.18) even simplifies because

$$\sum_{\ell=1}^{L}\mathrm{div}(K_\ell(\bar{c})v) = \mathrm{div}(v\sum_{\ell=1}^{L}K_\ell(\bar{c})) = \mathrm{div}(v) = 0.$$

The non-negativity of c_ℓ in the Kakutani's fixed point can now be proved by testing the Nernst-Planck equation by c_ℓ^- which is now indeed in duality with $\frac{\partial}{\partial t}c_\ell$. The argument for $K_\ell(c)v\cdot\nabla c_\ell^- = 0$ is now the same as used in the proof of Proposition 3.8 for $K_\ell(c)\nabla c_\ell^- = 0$.

Acknowledgement: Thanks are to Jiří Glaser who detected the above mentioned discrepancy in the proof of Proposition 3.12 concerning the non-negativity of c_ℓ.