In: Advances in Math. Sci. Appl. 10 (2000), 57-65.

DIRECT METHOD FOR PARABOLIC PROBLEMS

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Abstract. The variational principle by Brézis, Ekeland [10] and Nayroles [16] can characterize solutions to Cauchy or periodic problems for nonlinear parabolic equations or inequalities having a convex potential. Here, existence and uniqueness of solutions to such problems is shown by a direct method using this principle.

1. Introduction.

The so-called direct method for equations or inclusions means an investigation of a certain functional whose minimizers necessarily represent solutions to the original problem. The value of the minimum itself is not under question in this method. Classically, this method is used for elliptic equations, see e.g. Nečas [17]. Here we develop the direct method to parabolic problems, using the variational principle invented by Brézis, Ekeland [10] and Nayroles [16].

For $T > 0$ and V a reflexive separable Banach space, we will first consider the Cauchy problem for the abstract first-order non-autonomous evolution inclusion, i.e.

$$\frac{\mathrm{d}u}{\mathrm{d}t} + A(t, u(t)) \ni f(t) \ , \quad u(0) = u_0 \ , \tag{1}$$

where $A(t, u) = \partial_u \varphi(t, u)$ for some potential $\varphi : (0, T) \times V \to \mathbb{R}$ such that $\varphi(t, \cdot) : V \to \mathbb{R}$ is convex ("∂_u" stands for the subdifferential) for a.a. t. Alternatively, we will also consider a periodic problem for such inclusion, i.e.

$$\frac{\mathrm{d}u}{\mathrm{d}t} + A(t, u(t)) \ni f(t) \ , \quad u(0) = u(T) \ . \tag{2}$$

To avoid a lot of technicalities (cf. also Remark 4 below), we suppose the potential φ of A to be a Carathéodory function (i.e. measurable in t and continuous in u) satisfying the following growth and coercivity condition

$$c\|u\|_V^p \le \varphi(t, u) \le C(1 + \|u\|_V^p) \tag{3}$$

for some $c > 0$ and $1 < p < +\infty$. Let us define the Banach space

$$W^{p,q}(0,T;V,V^*) := \left\{ u \in L^p(0,T;V);\ \frac{\mathrm{d}u}{\mathrm{d}t} \in L^q(0,T;V^*) \right\} \tag{4}$$

equipped with the norm $\|u\| := \|u\|_{L^p(0,T;V)} + \|\frac{\mathrm{d}u}{\mathrm{d}t}\|_{L^q(0,T;V^*)}$, where V^* denotes the dual space to V and $q = p/(p-1)$ the conjugate exponent. As usual, we suppose V imbedded densely and continuously into some Hilbert space identified with its own dual so that $V \subset H \subset V^*$. Then it is well known that $W^{p,q}(0,T;V,V^*)$ is imbedded continuously into $C(0,T;H)$. Assuming $u_0 \in H$, we define two closed affine manifolds $\mathcal{D}_1, \mathcal{D}_2 \subset W^{p,q}(0,T;V,V^*)$ by

$$\mathcal{D}_1 := \{ u \in W^{p,q}(0,T;V,V^*);\quad u(0) = u_0 \}, \tag{5a}$$

$$\mathcal{D}_2 := \{ u \in W^{p,q}(0,T;V,V^*);\quad u(0) = u(T) \}. \tag{5b}$$

For $\ell = 1,2$ and $f \in L^q(0,T;V^*)$, we call $u \in \mathcal{D}_\ell$ a solution to the problem (ℓ) if $\frac{\mathrm{d}}{\mathrm{d}t}u + \partial\varphi(t,u(t)) \ni f(t)$ for a.a. $t \in [0,T]$. Furthermore, let us define $\Phi : W^{p,q}(0,T;V,V^*) \to \mathbb{R}$ by

$$\Phi(u) := \int_0^T \varphi(t,u(t)) + \varphi^*(t, f(t) - \frac{\mathrm{d}u}{\mathrm{d}t}) + \left\langle \frac{\mathrm{d}u}{\mathrm{d}t} - f(t), u(t) \right\rangle \mathrm{d}t, \tag{6}$$

where $\varphi^*(t,\cdot) : V^* \to \mathbb{R}$ is conjugate to $\varphi(t,\cdot) : V \to \mathbb{R}$, i.e. $\varphi^*(t,\xi) = \sup_{v \in V} \langle \xi, v \rangle - \varphi(t,v)$ and $\langle \cdot,\cdot \rangle$ denotes the canonical duality pairing $V^* \times V \to \mathbb{R}$.

The following assertion was proved by Brézis and Ekeland [10] and Nayroles [16] even under weaker assumptions than (3); cf. also Aubin and Cellina [3, Section 3.4] or Aubin [2]. For various generalizations see Auchmuty [7, Section 6] or Rios [18, 19]. For a usage to almost periodic solutions see Cieutat [11].

Theorem. (BRÉZIS-EKELAND-NAYROLES VARIATIONAL PRINCIPLE [10, 16].) *Let φ be a Carathéodory function satisfying (3) and $\varphi(t,\cdot)$ be convex. Then, for $\ell = 1,2$, the following two implications hold:*

(i) *If $u \in W^{p,q}(0,T;V,V^*)$ solves the problem (ℓ), then u minimizes Φ over $\mathcal{D}_\ell$ and, moreover, $\Phi(u) = 0$.*

(ii) *Conversely, if $\Phi(u) = 0$ for some $u \in \mathcal{D}_\ell$, then u minimizes Φ over $\mathcal{D}_\ell$ and solves the problem (ℓ).*

Note that Theorem (i) states the existence of a minimizer only by means of an apriori knowledge that a solution to the problem (ℓ) does exist. Likewise, Theorem (ii) does not imply any existence result for (ℓ) because $\min_{u \in \mathcal{D}_\ell} \Phi(u) = 0$ is not apriori obvious.

Thus the direct method to prove existence of solution to (ℓ) seemed difficult, which is why various convex/concave variational principles have been proposed and investigated by minimax theorems by Aubin [2] and Auchmuty [8]. Yet, the minimax theorems are less constructive than the direct method.

In an abstract setting, an attempt to the direct method has been done by Rios [18, 19], outlined already by Nayroles [16], who proved existence of a minimizer of Φ over $\mathcal{D}_\ell$ and the fact that the minimum is zero, which already yields the existence of a solution to (ℓ) by Theorem (ii).

The aim of this paper to show the existence of a solution to (ℓ) by calculation of the first-order necessary optimality condition for the minimizer of Φ (whose existence is ensured by compactness) without investigating the value of the minimum of Φ. This is the classical direct method. Note that, since Φ contains $\mathrm{d}u/\mathrm{d}t$, the standard Euler-Lagrange equation (resulting as a necessary condition) would involve $\mathrm{d}^2u/\mathrm{d}t^2$ (cf. also (11) below) which however does not appear in the original problem (ℓ). Beside a systematic exploitation of convex analysis, the fundamental trick relies on usage of a certain special symmetry in the necessary conditions, cf. (15) below. A similar symmetry has been also observed by Auchmuty [8, Theorem 4] in analyzing saddle points of the functional $L(u,v) = \int_0^T \varphi(t,u) - \varphi(t,v) + \langle f - \frac{\mathrm{d}}{\mathrm{d}t}u, v - u\rangle \mathrm{d}t$ (proposed incidentally already by Rios [19, Remark 1]) for the Cauchy problem (1). The development of the direct method may be promising for other problems admitting a variational-principle formulation, including some hyperbolic problems, cf. Aubin [2] or Aubin and Ekeland [4].

2. Direct method.

Let us first briefly recall some facts from convex analysis. Considering $\phi : V \to \mathbb{R}$ convex and continuous, its (so-called Fenchel) conjugate $\phi^* : V^* \to \mathbb{R} \cup \{+\infty\}$, defined as $\phi^*(\xi) := \sup_{v\in V} \langle \xi, v\rangle - \phi(v)$, is convex, lower semicontinuous, and

$$\xi \in \partial\phi(v) \qquad \Leftrightarrow \qquad v \in \partial\phi^*(\xi)\ , \tag{7}$$

and moreover any relation in (7) is equivalent to the extremality relation $\phi(v)+\phi^*(\xi) = \langle \xi, v\rangle$; cf. e.g. [5]. For any constant c, one has $[\phi + c]^* = \phi^* - c$ and, for c positive, $[c\phi]^* = c\phi^*(\cdot/c)$. Furthermore, $\phi_1^* \le \phi_2^*$ provided $\phi_1 \ge \phi_2$. If $\phi(v) = \frac{1}{p}\|v\|_V^p$, then $\phi^*(\xi) = \frac{1}{q}\|\xi\|_{V^*}^q$, which follows by the Hölder inequality, see, e.g., [5, p.205]. This also implies $(c\|\cdot\|_V^p)^* = (cp)^{1-q}\|\cdot\|_{V^*}^q/q$. From (3), one then gets analogous estimates for the conjugate φ^*, namely

$$\frac{(cp)^{1-q}}{q}\|\xi\|_{V^*}^q \ge \varphi^*(t,\xi) \ge \frac{(Cp)^{1-q}}{q}\|\xi\|_{V^*}^q - C. \tag{8}$$

Moreover, φ^* is itself a Carathéodory function: $\varphi^*(t,\cdot)$ is convex and bounded due to (8), and therefore it must be continuous, and $\varphi^*(\cdot,\xi)$, being the supremum of a countable collection of the measurable functions $\{\langle\xi, v\rangle - \varphi(\cdot, v)\}_{v\in V_0}$ with V_0 dense in V, is a measurable function.

Note that (3) ensures that the Nemytskiĭ mapping $\mathcal{N}_\varphi : u \mapsto (t \mapsto \varphi(t, u(t))) : L^p(0,T;V) \to L^1(0,T)$ is bounded and continuous, while (8) ensures that $\varphi^*(t,\cdot)$ has at most q-growth so that also $\mathcal{N}_{\varphi^*} : L^q(0,T;V^*) \to L^1(0,T)$ is bounded and continuous; cf. e.g. Krasnoselskiĭ [13] for V finite-dimensional or Lucchetti and Patrone [15] for V general. Altogether, Φ from (6) is well defined, continuous and bounded on $W^{p,q}(0,T;V,V^*)$ provided still $f \in L^q(0,T;V^*) \cong L^p(0,T;V)^*$.

The existence of a solution to (1) or (2) will be implied directly by the following two assertions. The first one is, in fact, a special case of [8, Theorem 1] as far as (1) concerns.

Proposition 1. *Let φ be a Carathéodory mapping satisfying (3), let $\varphi(t,\cdot)$ be convex, $u_0 \in H$, and $f \in L^q(0,T;V^*)$. Then, for $\ell = 1, 2$, Φ attains its minimum on $\mathcal{D}_\ell$.*

Proof. We already mentioned that Φ is convex and continuous, $W^{p,q}(0,T;V,V^*)$ is reflexive, and $\mathcal{D}_\ell$ is closed in $W^{p,q}(0,T;V,V^*)$ because obviously $D_1 = \delta_0^{-1}(u_0)$ and $D_2 = (\delta_0 \times \delta_T)^{-1}(\mathrm{diag}(H \times H))$ where the mappings $\delta_0 : u \mapsto u(0)$ and $\delta_T : u \mapsto u(T)$ are continuous from $W^{p,q}(0,T;V,V^*)$ to H. Moreover, by (3) and (8), Φ is coercive on $\mathcal{D}_\ell$ in the sense $\lim_{\|u\|\to\infty} \Phi(u) \to +\infty$, which follows from the estimate

$$\Phi(u) \geq \int_0^T \left[c\|u\|_V^p + \frac{(Cp)^{1-q}}{q} \left\| f - \frac{\mathrm{d}u}{\mathrm{d}t} \right\|_{V^*}^q - C + \langle \frac{\mathrm{d}u}{\mathrm{d}t} - f, u \rangle \right] \mathrm{d}t$$
$$\geq c\|u\|^p_{L^p(0,T;V)} + \frac{(Cp)^{1-q}}{q} \left\| f - \frac{\mathrm{d}u}{\mathrm{d}t} \right\|^q_{L^q(0,T;V^*)} - \ \|f\|_{L^q(0,T;V^*)} \|u\|_{L^p(0,T;V)} - C_0$$

with

$$C_0 = \begin{cases} CT + \frac{1}{2}\|u_0\|_H^2 & \text{if } \ell = 1, \\ CT & \text{if } \ell = 2, \end{cases}$$

we used also that $\int_0^T \langle \frac{\mathrm{d}}{\mathrm{d}t} u, u \rangle \mathrm{d}t$ is bounded from below by $-\frac{1}{2}\|u_0\|_H^2$ (if $u \in \mathcal{D}_1$) or by 0 (if $u \in \mathcal{D}_2$). Then the existence of a minimizer of Φ on $\mathcal{D}_\ell$ follows by the standard weak-compactness argument. □

Proposition 2. *Let the assumptions of Proposition 1 be fulfilled. Then:*
(i) *Any minimizer of Φ on $\mathcal{D}_1$ represents the solution to the Cauchy problem (1).*
(ii) *If also $\varphi(t, \cdot)$ is smooth for a.a. $t \in [0,T]$, then any minimizer of Φ on $\mathcal{D}_2$ represents the solution to the periodic problem (2).*

Proof. Having a minimizer $u \in \mathcal{D}_\ell$ of Φ, we must calculate the subdifferential $\partial\Phi(u)$ and then use simply the standard first-order optimality condition, i.e.

$$\exists \xi \in \partial\Phi(u) \quad \forall v \in \mathcal{T}_\ell : \qquad \langle\!\langle \xi, v \rangle\!\rangle \geq 0 \ , \tag{9}$$

where $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ denotes the canonical duality pairing $L^q(0,T;V^*) \times L^p(0,T;V) \to \mathbb{R}$ and $\mathcal{T}_\ell$ denotes the tangent cone to $\mathcal{D}_\ell$ at the point u. Note that (9) means equivalently $\partial\Phi(u) + \mathcal{N}_\ell \ni 0$ which is equivalent with the standard condition $\partial[\Phi + \delta_{\mathcal{D}_\ell}](u) \ni 0$, where $\delta_{\mathcal{D}_\ell}$ denotes the indicator function of $\mathcal{D}_\ell$ and $\mathcal{N}_\ell$ denotes the normal cone to $\mathcal{D}_\ell$ at u, i.e. $\mathcal{N}_\ell$ is the polar cone to $\mathcal{T}_\ell$. Here $\mathcal{T}_\ell$ is even a linear space independent of u, namely $\mathcal{T}_1 = \{v \in W^{p,q}(0,T;V,V^*) : v(0) = 0\}$ and $\mathcal{T}_2 = \mathcal{D}_2$, and therefore the inequality in (9) turns, in fact, into an equality.

Let us abbreviate $\psi(u) := \int_0^T \varphi(t, u(t))\mathrm{d}t$. Then $\psi^*(\xi) = \int_0^T \varphi^*(t, \xi(t))\mathrm{d}t$, which was proved for V finite-dimensional by Rockafellar [20, Theorem 2] but it holds also for V a separable Banach space because the needed measurable-selection results hold in this case, too, see [6]; for $p = 2$ see also Auchmuty [7, Lemma 6.1]. Using $L : u \mapsto \frac{\mathrm{d}u}{\mathrm{d}t} : W^{p,q}(0,T;V,V^*) \to L^q(0,T;V^*)$ and $\langle\!\langle Lu, u \rangle\!\rangle = \frac{1}{2}\|u(T)\|_H^2 - \frac{1}{2}\|u(0)\|_H^2$, we can write (6) as

$$\Phi(u) = \psi(u) + \psi^*(f - Lu) - \langle\!\langle f, u \rangle\!\rangle + \frac{1}{2}\|u(T)\|_H^2 - \frac{1}{2}\|u(0)\|_H^2. \tag{10}$$

Using the fact that $\psi_1 : \xi \mapsto \psi^*(f - \xi)$ is defined everywhere on $L^q(0,T;V^*)$ so that certainly $0 \in \mathrm{int}(\mathrm{Range}(L) - \mathrm{Dom}(\psi_1))$, we have $\partial[\psi + \psi_1 \circ L](u) = \partial\psi(u) + L^*\partial\psi_1(Lu)$; see Aubin and Ekeland [5, Sect. 4.3, Theorem 5]. As $\partial\psi_1(\xi) = -\partial\psi^*(f - \xi)$, we obtain

$$\partial\Phi(u) = \partial\psi(u) - L^*\partial\psi^*(f - Lu) - f + u(T) \cdot \delta_T - u(0) \cdot \delta_0 \tag{11}$$

where $\delta_t : u \mapsto u(t)$. Therefore, combining (9) as an equality with (11), we have

$$\left.\begin{array}{ll}\exists z\in L^q(0,T;V^*), \quad w\in L^p(0,T;V), \quad z\in\partial\psi(u), \quad w\in\partial\psi^*(f-Lu), \\ \forall v\in\mathcal{T}_\ell: \qquad \langle\!\langle z-f-L^*w\ ,\ v\rangle\!\rangle + \langle u(T),v(T)\rangle - \langle u(0),v(0)\rangle = 0. \end{array}\right\} \tag{12}$$

By (7) used for $\phi=\psi$, the inclusion $w\in\partial\psi^*(f-Lu)$ means precisely

$$\partial\psi(w)\ni f-Lu. \tag{13}$$

Using v with a compact support in $(0,T)$, we get from (12) that $\langle\!\langle z-f-L^*w,v\rangle\!\rangle = 0$, i.e. $\langle\!\langle z-f,v\rangle\!\rangle = \langle\!\langle w,Lv\rangle\!\rangle$. By integration by parts, it means precisely that $Lw=f-z$ in the sense of distributions on $(0,T)$. In particular, $Lw=f-z\in L^q(0,T;V^*)$ so that $w\in W^{p,q}(0,T;V,V^*)$. Moreover, as $z\in\partial\psi(u)$, one thus gets

$$\partial\psi(u)\ni z=f-Lw. \tag{14}$$

Equally, (13) and (14) mean

$$\frac{\mathrm{d}u}{\mathrm{d}t}+\partial_w\varphi(t,w(t))\ni f(t) \qquad \text{and} \tag{15a}$$

$$\frac{\mathrm{d}w}{\mathrm{d}t}+\partial_u\varphi(t,u(t))\ni f(t) \qquad \text{for a.a. } t\in[0,T], \tag{15b}$$

cf. Auchmuty [8, Proposition 3.1]. From (15) by the monotonicity of $\partial_u\varphi(t,\cdot)$, for a.a. $t\in[0,T]$ we get

$$0\le\left\langle\frac{\mathrm{d}u}{\mathrm{d}t}-\frac{\mathrm{d}w}{\mathrm{d}t},u(t)-w(t)\right\rangle=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u-w\|_H^2. \tag{16}$$

From (12) one can also obtain

$$\forall v\in\mathcal{T}_\ell: \quad \langle\!\langle z-f+Lw+(u(T)-w(T))\cdot\delta_T-(u(0)-w(0))\cdot\delta_0\ ,\ v\rangle\!\rangle=0, \tag{17}$$

where we used also the identity $L^*w=-Lw+w(T)\cdot\delta_T-w(0)\cdot\delta_0$ which follows by integration by parts for $w\in W^{p,q}(0,T;V,V^*)\subset L^p(0,T;V)\cong L^q(0,T;V^*)^*$:

$$\begin{aligned}\langle\!\langle Lw,v\rangle\!\rangle=\langle\!\langle\frac{\mathrm{d}w}{\mathrm{d}t},v\rangle\!\rangle&=-\langle\!\langle w,\frac{\mathrm{d}v}{\mathrm{d}t}\rangle\!\rangle+\langle w(T),v(T)\rangle-\langle w(0),v(0)\rangle\\&=-\langle\!\langle w,Lv\rangle\!\rangle+\langle w(T),v(T)\rangle-\langle w(0),v(0)\rangle\\&=-\langle\!\langle L^*w,v\rangle\!\rangle+\langle w(T),v(T)\rangle-\langle w(0),v(0)\rangle.\end{aligned}$$

Now we must distinguish two cases.

Case $\ell=1$ (The Cauchy problem): Note that δ_0 vanishes on $\mathcal{T}_1$. Taking $v\in\mathcal{T}_1$ vanishing on $[0,T-\varepsilon]$ and such that $v(T)=u(T)-w(T)$, and pushing ε to zero, we obtain from (17) that $0=v(T)\cdot(u(T)-w(T))=(u(T)-w(T))\cdot(u(T)-w(T))=\|u(T)-w(T)\|_H^2$, i.e.

$$u(T)=w(T). \tag{18}$$

Using (16) with (18) and the Gronwall inequality backward, we get $u=w$ on $(0,T)$. Putting this into (15a) (or equally into (15b)), we get $\mathrm{d}u/\mathrm{d}t+\partial_u\varphi(t,u(t))=f(t)$. As $u\in\mathcal{D}_1$, the initial condition $u(0)=u_0$ in (1) is satisfied, too.

Case $\ell = 2$ (The periodic problem): Now we cannot require $w = u$, cf. Remark 2 below. Taking $v \in \mathcal{T}_2$ vanishing on $[\varepsilon, T-\varepsilon]$ and such that $v(T) = v(0)$, and pushing ε to zero, we obtain from (17) that $(u(T)-w(T))\cdot v(T) - (u(0)-w(0))\cdot v(0) = 0$. As $v(0) \in \mathcal{T}_2 = \mathcal{D}_2$ can be arbitrary but $v(T) = v(0)$ and as $u(T) = u(0)$ because $u \in \mathcal{D}_2$, we obtain $w(T) = w(0)$, i.e. $w \in \mathcal{D}_2$. From (16) we then can deduce that $\|[u-w](\cdot)\|_H$ must be constant on $(0,T)$, hence

$$\langle Lu - Lw, u - w\rangle = 0 \qquad \text{a.e. on } [0,T]. \tag{19}$$

Subtracting (15a) and (15b) and testing it by $w(t) - u(t)$, one gets $\langle \nabla\varphi(t,w) - \nabla\varphi(t,u) + L(u-w), w-u\rangle = 0$ a.e. on $[0,T]$, where $\nabla\varphi(t,\cdot)$ denotes the Gâteaux differential. In view of (19), it means

$$\langle \nabla\varphi(t,w) - \nabla\varphi(t,u), w-u\rangle = 0 \qquad \text{a.e. on } [0,T]. \tag{20}$$

As $\varphi(t,\cdot)$ is assumed smooth, $\varphi^*(t,\cdot)$ is strictly convex because otherwise the graph of $\varphi^*(t,\cdot)$ would contain a segment $[(\xi_1,a_1),(\xi_2,a_2)]$, i.e. $\partial\varphi(t,u)$ would contain the nontrivial segment $[\xi_1,\xi_2]$ for a certain $u \in V$ hence $\varphi(t,\cdot)$, having a non-singleton subdifferential, would not be smooth at u. Therefore $\partial\varphi^*(t,\cdot)$ is strictly monotone, and so is also $[\nabla\varphi(t,\cdot)]^{-1} = \partial\varphi^*(t,\cdot)$, cf. (7). From (20), one can then deduce $\nabla\varphi(t,w) = \nabla\varphi(t,u)$. Putting it into (15a) proves that u solves (2). $\square$

The direct method can also be used for uniqueness of a solution to (ℓ) based on the argument of a possible strict convexity of Φ on the linear manifold $\mathcal{D}_\ell$.

Proposition 3. *Let the assumptions of Proposition 1 be fulfilled and let, for a.a.* $t \in [0,T]$, $\varphi(t,\cdot)$ *be smooth or strictly convex. Then:*

(i) *The functional* Φ *admits at most one minimizer on* $\mathcal{D}_1$.

(ii) *The functional* Φ *admits at most one minimizer on* $\mathcal{D}_2$ *provided* $\mathrm{meas}(S) > 0$, *where* $S := \{t\in(0,T) :\ \varphi(t,\cdot)$ *is strictly convex*$\}$.

Proof. Take $u_1, u_2 \in \mathcal{D}_1$, $u_1 \neq u_2$, and put $M = \{t\in(0,T);\ u_1(t) \neq u_2(t)\}$. Then M is a nonempty open set, hence $\mathrm{meas}(M) > 0$. Moreover, put $N = \{t\in M;\ Lu_1(t) \neq Lu_2(t)\}$. Also N has a positive Lebesgue measure otherwise $u_1 = u_2$ on $[0,T]$ because $u_1, u_2 \in C([0,T];H)$ and $[0,T)\setminus M$ is nonempty, containing certainly $t = 0$.

If $\varphi(t,\cdot)$ is smooth for some $t\in[0,T]$, then $\varphi^*(t,\cdot)$ is strictly convex, see the proof of Proposition 2. Hence, we have $\varphi(t,\cdot)$ or $\varphi^*(t,\cdot)$ strictly convex for a.a. $t \in [0,T]$ and, in particular, for a.a. $t \in N$ for which we have both $u_1(t) \neq u_2(t)$ and $Lu_1(t) \neq Lu_2(t)$. As $\mathrm{meas}(N) > 0$, it holds

$$\begin{aligned}\frac12\Phi(u_1) + \frac12\Phi(u_2) - \Phi(\frac{u_1+u_2}{2}) \geq \int_N \Big[&\frac12\varphi(t,u_1) + \frac12\varphi(t,u_2) - \varphi(t,\frac{u_1+u_2}{2})\\ &\frac12\varphi^*(t,f-\frac{\mathrm{d}u_1}{\mathrm{d}t}) + \frac12\varphi^*(t,f-\frac{\mathrm{d}u_2}{\mathrm{d}t}) - \varphi^*(t,f-\frac{\mathrm{d}}{\mathrm{d}t}\frac{u_1+u_2}{2})\Big]\,\mathrm{d}t > 0\end{aligned} \tag{21}$$

so that Φ is strictly convex on $\mathcal{D}_1$. This proves (i).

For $u_1, u_2 \in \mathcal{D}_2$, it may occur that $Lu_1 = Lu_2$ a.e. on $[0,T]$ but $u_1 \neq u_2$. Then $N = \emptyset$ but $u_1(t) \neq u_2(t)$ for all $t \in [0,T]$ because $u_1 - u_2$ is constant. In this case, the strict convexity of Φ on $\mathcal{D}_2$ follows like in (21) but using the strict convexity of the contribution $\int_S \varphi(t,u)\mathrm{d}t$. Otherwise, $Lu_1 \neq Lu_2$ so that $N \neq \emptyset$ and we are in the same situation as in (i). The point (ii) has thus been proved, too. $\square$

Remark 1. In fact, $\Phi(u) = 0$ just means that, for a.a. $t \in [0,T]$, the extremality relation $\varphi(t,u(t)) + \varphi^*(t, f(t) - \frac{\mathrm{d}u}{\mathrm{d}t}) = \langle f(t) - \frac{\mathrm{d}u}{\mathrm{d}t}, u(t)\rangle$ holds, which is then equivalent to the inclusion $f(t) - \frac{\mathrm{d}u}{\mathrm{d}t} \in \partial\varphi_u(t,u(t)) = A(t,u(t))$, cf. also (7).

Remark 2. In case $\ell = 2$, we cannot assume the auxiliary variable $w \in \mathcal{D}_2$ to be identical with the solution u like in case $\ell = 1$. Indeed, for u solving (2), $w := u$+konst. will satisfy (15)–(17), (19) and (20). The converse implication (which could abandon the smoothness assumption about $\varphi(t,\cdot)$ in Proposition 2(ii)) remains open. Yet, in the proof of Proposition 2(ii), we did not use fully the information from (15). For example, taking $V = \mathbb{R}^2$, $\varphi(t,u) = \max(0,|u|-1)^2$, and $f = 0$, then $u = 0$ is a solution of (2), and $w(t) = (\sin(2\pi t/T), \cos(2\pi t/T))$ satisfies (16), (19), (20), and hence also (15a), but not (15b), neither (17).

Remark 3. Let us note that Φ actually need not be strictly convex on $\mathcal{D}_2$ if $\varphi(t,\cdot)$ is not strictly convex, i.e. if $\mathrm{meas}(S) = 0$. Indeed, the last two terms in (6), i.e. $\psi^*(f - Lu)$ and $\langle\!\langle Lu - f, u\rangle\!\rangle$, are affine (or even constant if $\int_0^T f(t)\mathrm{d}t = 0$) over any affine manifold $v+\mathrm{Ker}\, L \subset \mathcal{D}_2$ and it is then easy to construct φ so that ψ (and thus Φ) is not strictly convex on such manifold.

Remark 4. Brézis and Ekeland [10, Remark 1] mentioned that the existence and uniqueness of a minimizer of Φ over $\mathcal{D}_\ell$ do not seem easy to prove directly, i.e. without passing through (ℓ) as in Theorem (i), as we did in Propositions 1 and 3 under the assumption (3). Indeed, without (3) this task requires a more complicated technique based on theory of normal convex integrands, see Rockafellar [20]. Then, the existence of a minimizer on $\mathcal{D}_1$ has been shown by Rios [18, 19] assuming there is some $\bar{u} \in \mathcal{D}_1$ such that $\psi(\bar{u}) < +\infty$ and $\psi^*(f - \cdot)$ is continuous at $\mathrm{d}\bar{u}/\mathrm{d}t$. Such technique fits with variational inequalities with unilateral constraints. As for the uniqueness of a minimizer of Φ on $\mathcal{D}_1$, it has been proved already by Auchmuty [8, Theorem 1] but only on the condition that $\psi : L^p(0,T;V) \to \mathbb{R}$ is strictly convex, which means here $\varphi(t,\cdot)$ strictly convex for a.a. $t \in [0,T]$.

Remark 5. Contrary to the elliptic problems, the presented direct method does not seem to be an optimal tool for parabolic problems. Indeed, already the assumption about monotonicity of $A(t,\cdot)$ (i.e. about convexity of the potential $\varphi(t,\cdot)$), which is essential for the used Brézis-Ekeland-Nayroles principle, is not needed in other methods; let us recall that standardly $A(t,\cdot) = \partial_u\varphi(t,\cdot)$ is required to be only pseudomonotone (see Brézis [9] or also Lions [14, Section II.9.3]) for existence results and to satisfy only $\langle A(t,u) - A(t,v), u - v\rangle \geq -c(t)\|u - v\|_H^2$ with some $c \in L^1(0,T)$ for uniqueness results. In case of the periodic problem (2), the smoothness of $\varphi(t,\cdot)$ has been additionally required in Proposition 2(ii) but it seems that a more sophisticated proof (cf. also Remark 2) or an additional limit passage might avoid this requirement.

3. Example: parabolic equations.

The above results can be directly applied to initial or periodic boundary value problems for parabolic equations or inequalities governed by an elliptic part having a convex potential. Then Φ can always be formally defined by the formula (6), though its concrete form can be mostly quite difficult to determine explicitly. Let us briefly mention an example consisting in an initial/Dirichlet-boundary value problem for a nonlinear parabolic equation on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, i.e.

$$\left.\begin{array}{ll} \dfrac{\partial u}{\partial t} - \mathrm{div}(|\mathrm{grad}\, u|^{p-2}\mathrm{grad}\, u) = g & \text{on } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0,\cdot) = u_0 & \text{on } \Omega, \end{array}\right\} \tag{22}$$

where $g \in L^q(0,T;L^r(\Omega))$ and $u_0 \in L^2(\Omega)$, and $r = 1$ for $p > n$, or $r > 1$ for $p = n$, or $r \geq np/(np-n+p)$ for $p < n$. Considering a standard weak formulation, the natural functional setting of (22) turns it into the abstract Cauchy problem (1) with $V = W_0^{1,p}(\Omega)$ the Sobolev space of functions vanishing on the boundary $\partial\Omega$, $\varphi(t,u) \equiv \phi(u) = \frac{1}{p}\int_\Omega |\mathrm{grad}\, u|^p \mathrm{d}x$ and $f \in L^q(0,T;W_0^{1,p}(\Omega)^*) \cong L^q(0,T;W^{-1,q}(\Omega))$ is determined by $\langle\!\langle f,u\rangle\!\rangle = \int_0^T\int_\Omega g(t,x)u(t,x)\mathrm{d}x\mathrm{d}t$.

Let us abbreviate $\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$ the p-Laplacean, this means $\Delta_p u := \mathrm{div}(|\mathrm{grad}\, u|^{p-2}\mathrm{grad}\, u)$. One can notice that $\varphi(u) = \frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)}$ provided $\|u\|_{W_0^{1,p}(\Omega)} := \|\mathrm{grad}\, u\|_{L^p(\Omega;\mathbb{R}^n)}$. Then $\varphi^*(\xi) = \frac{1}{q}\|\xi\|^q_{W^{-1,q}(\Omega)}$, cf. e.g. Aubin and Ekeland [5, p.205]. Moreover, it is easy to see (cf. Lions [14, Section II.2.2]) that $\Delta_p u = -J(u)$ where $J : V \to V^*$ is the duality mapping (with respect to the $(p-1)$-power) defined by the formulae $\langle J(u),u\rangle = \|J(u)\|_{V^*}\|u\|_V$ and $\|J(u)\|_{V^*} = \|u\|_V^{p-1}$. Hence, $\|\xi\|_{V*} = \|J^{-1}(\xi)\|_V^{p-1}$ implies here $\|\xi\|^q_{W^{-1,q}(\Omega)} = \|\Delta_p^{-1}\xi\|^p_{W_0^{1,p}(\Omega)}$ so that

$$\varphi^*(\xi) = \frac{1}{q}\|\xi\|^q_{W^{-1,q}(\Omega)} = \frac{1}{q}\|\Delta_p^{-1}\xi\|^p_{W_0^{1,p}(\Omega)} = \frac{1}{q}\|\mathrm{grad}\, \Delta_p^{-1}\xi\|^p_{L^p(\Omega;\mathbb{R}^n)} \,. \tag{24}$$

It yields the following explicit form of the functional Φ:

$$\Phi(u) = \int_0^T\int_\Omega \left[\frac{1}{p}|\mathrm{grad}\, u|^p \ + \ \frac{1}{q}\left|\mathrm{grad}\, \Delta_p^{-1}(g - \frac{\partial u}{\partial t})\right|^p \ + \ (\frac{\partial u}{\partial t} - g)u\right]\mathrm{d}x\mathrm{d}t \ , \tag{25}$$

where the product $\frac{\partial u}{\partial t}u$ should be understood in the weak sense (so that $\int_0^T\int_\Omega \frac{\partial u}{\partial t}u\,\mathrm{d}x\mathrm{d}t = \frac{1}{2}\int_\Omega u(T,x)^2 - u(0,x)^2\mathrm{d}x$) while the other terms have usual meaning. For the linear case (i.e. $p = 2$) see Brézis and Ekeland [10] or also [2].

We can observe that the integrand in (25) is nonlocal in space, which is actually inevitable as shown by Adler [1] who proved that there is no local variational principle yielding (22) as its Euler-Lagrange equation. Let us still remark that there exist various nonlocal-in-time variational principles for (22); cf. Hlaváček [12] for a survey. For minimax-type principles see Aubin [2] or Auchmuty [8], the latter one being even local.

Acknowledgements. The author is indebted to dr. Jan Malý for inspiring discussion and to a referee for valuable comments. This research was partly covered by the grants 201/96/0228 (GA ČR) and A 107 5707 (GA AV ČR).

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