On σ-compact extensions of Banach spaces

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Abstract. A general method to construct σ-compact extensions of a Banach space X preserving greater or lesser part of the structure of X is developed by a modification and enrichment of the technique of compactification of semitopological semigroups, exploiting here continuous linear multiplicative nontrivial functionals on a suitable "Fréchet algebra" of functions on X. When being fine enough, these extensions are also locally compact. Such method of σ-compactification covers, in a unified manner, e.g. the bidual space X^{**} (endowed with the weak* topology) or the space of types $\mathcal{T}(X)$ by Krivine, Maurey [5].

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1. INTRODUCTION, NOTATION

The aim of the paper is to develop a general method for extension of a Banach space X to a σ-compact space together with a "certain part" of the original algebraico/topological structure of X. We will use systematically the compactification technique by means of linear multiplicative functionals (see e.g. Engelking [3; Sec.3.12.21]), which seems to be most suitable for our purposes and which also enables to employ rich results from the theory of compactifications of semitopological semigroups; for a comperhensive survey we refer to Berglund, Milnes, Junghen [1].

Let us recall briefly some of these results, introducing also some notation. As we will treat only Banach spaces, the general situation (i.e. 11 kinds of compactifications) reduces considerably (to 3 kinds only, cf. [1;p.130]) thanks to the fact that the addition in a Banach space makes it a commutative, complete metrizable group. We will refer to these kinds in accord with their main representants: *UC* (= uniformly continuous), *WAP* (= weakly almost periodic), *AP* (= almost periodic). By $C(X)$ we denote the C^*-algebra (i.e. the Banach algebra with an involution "$*$" and with the property $\|ff^*\|=\|f\|^2$) of all continuous, bounded, complex valued functions on X; the involution is the complex conjugation and the norm is defined by $\|f\|=\sup_{x\in X}f(x)$, X will be always considered with its norm topology. If $\mathcal{F}$ is a subspace of $C(X)$, the topological dual to $\mathcal{F}$ is denoted, as usual, by $\mathcal{F}^*$. We will always consider $\mathcal{F}^*$ endowed with the weak* topology, i.e. the relativized $\sigma(C(X)^*,C(X))$ topology. If $\mathcal{F}$ is a C^*-subalgebra of $C(X)$, $\mathcal{F}^\bullet$ will denote the subset of $\mathcal{F}^*$ containing all multiplicative functionals but the trivial one (=0). Clearly, $\mathcal{F}^\bullet$

is compact in $\mathcal{F}^*$. We define the so-called evaluation mapping $e: X \longrightarrow \mathcal{F}^*$ by

$$(1.1) \qquad e(x)(f) = f(x), \quad x \in X, \ f \in \mathcal{F}.$$

Supposing that $\mathcal{F}$ contains constant functions, $e(X)$ is a dense subset of $\mathcal{F}^\bullet$, and we thus get a compactification of X. In the literature, the elements of $\mathcal{F}^\bullet$ are sometimes also called multiplicative means ($\mu \in \mathcal{F}^*$ is a mean if $\mu(1)=1$ and $\mu(f) \geq 0$ whenever $f \geq 0$), and $\mathcal{F}^\bullet$ is then denoted by $MM(\mathcal{F})$.

For $\mu \in \mathcal{F}^\bullet$ the mapping $T_\mu : \mathcal{F} \longrightarrow B(X)$, $B(X)$ is the space of all bounded functions on X, is defined by $T_\mu(f)(x) = \mu(f(.+x))$, $f \in \mathcal{F}$, $x \in X$. Clearly, for $x \in X$, $T_{e(x)}$ represents just the shifts $f(.) \mapsto f(.+x)$. If $\mathcal{F}$ is m-introverted, i.e. $T_\mu \mathcal{F} \subset \mathcal{F}$ for all $\mu \in \mathcal{F}^\bullet$, we can define a binary operation "+" on $\mathcal{F}^\bullet$ by the formula

$$(1.2) \qquad \mu + \nu = \mu \circ T_\nu, \quad \mu, \nu \in \mathcal{F}^\bullet.$$

In the literature this operation is denoted rather by "*" and called "convolution", but we will denote it again as the addition to emphasize that it is an extension of the original addition of X (note that evidently $e(x_1)+e(x_2)=e(x_1+x_2)$ for any $x_1, x_2 \in X$, and in a very special case it can even coincide with the original addition on X; see Section 4.1 when X is reflexive). On the other hand, "+" defined on $\mathcal{F}^\bullet$ by (1.2) has nothing common with the addition in $\mathcal{F}^*$. In general, $\mathcal{F}^\bullet$ with the operation "+" is a right topological semigroup with zero; "right topological" means that the mapping $\mu \mapsto \mu + \nu$ from $\mathcal{F}^\bullet$ to itself is continuous for every $\nu \in \mathcal{F}^\bullet$. Moreover, also the mapping $(x, \nu) \mapsto e(x) + \nu$ from $X \times \mathcal{F}^\bullet$ to $\mathcal{F}^\bullet$ is continous; note that, by the results of Namioka [7] and Rao [8], see also [1; pp. 103, 105, 127], $\mathcal{F} \subset UC(X) = \{f\in C(X);\ f$ is uniformly continuous$\}$ provided $\mathcal{F}$ is an m-introverted C^*-subalgebra of $C(X)$, which just ensures the continuity of the mapping $(x,\nu)\mapsto e(x)+\nu$.

A function $f\in C(X)$ is called (weakly) almost periodic if the set $T_{e(X)}f$ is relatively (weakly) compact in $C(X)$; the set of all (weakly) almost periodic functions on X will by denoted by $AP(X)$ (or $WAP(X)$), respectively. It holds $UC(X)\subset WAP(X)\subset AP(X)$. If, in addition, $\mathcal{F}\subset WAP(X)$, $\mathcal{F}^*$ with "+" defined by (1.2) is a commutative semitopological semigroup ("semitopological" means that the addition is separately continuous), and if $\mathcal{F}\subset AP(X)$, $\mathcal{F}^*$ is even a topological group. Conversely, if $\mathcal{F}^*$ is a semitopological semigroup or a topological group, then $\mathcal{F}\subset WAP(X)$ or $\mathcal{F}\subset AP(X)$, respectively.

The aim of this paper is, by subjecting $\mathcal{F}$ to some further requirements, to extend on $\mathcal{F}^*$ also the multiplication by scalars and possibly the norm, and eventually also the scalar product if X is a Hilbert space. However, not to deteriorate the continuity of the multiplication (and also of the scalar product), we must exclude, roughly speaking, the points of the compactification at infinity. The remaining subset is then no longer compact, but it is still σ-compact (i.e. it is a countable union of compact sets), and sometimes even locally compact.

In Sec.4 we show that certain special choices of $\mathcal{F}$ lead to some extensions already known, namely $\mathcal{F}^*$ can be thus homeomorphic to the bidual space X^{**}, or to the space $\mathcal{T}(X)$ of types on X, introduced by Krivine, Maurey [5], or to the Leader local compactification [6]. The general method developed here can be thus viewed as a unified theory of these particular extensions. Besides, the σ-compact extensions can serve as a proper tool in optimization theory. The σ-compactness then ensures readily existence and stability of (generalized) minimizers or maximizers of coercive optimization problems (cf. [9,10]), while the remains of the linear structure still enable to treat optimality conditions because even in the purest cases we can still speak about cones in the extended space. However, in this paper we will not deal with the applications in optimization theory.

2. LOCALIZATION ON BOUNDED SETS, EXTENSION OF SCALAR MULTIPLICATION

Let $\mathcal{B}(X)$ denote the Fréchet space of all continuous (not neccessarily bounded) functions on X that are bounded on every ball $X_r = \{x\in X;\ \|x\|\le r\}$, $r>0$, endowed by the collection of seminorms $\{\|.\|_r\}_{r>0}$ defined by $\|f\|_r=\|R_r f\|$, where $R_r:\mathcal{B}(X)\longrightarrow C(X_r)$ is the operator of the restriction $f\mapsto f|_{X_r}$. Enlarging the classes $UC(X)$, $WAP(X)$, and $AP(X)$ by localization of the respective properties on bounded subsets only, we define:

$\mathcal{UB}(X) = \{f\in\mathcal{B}(X);\ \forall r>0:\ R_r f\in UC(X_r)\}$,

$\mathcal{WAP}(X) = \{f\in\mathcal{B}(X);\ \forall r>0:\ R_r(T_{e(X_r)}f)$ is relatively weakly compact in $C(X_r)\}$, and

$\mathcal{AP}(X) = \{f\in\mathcal{B}(X);\ \forall r>0:\ R_r(T_{e(X_r)}f)$ is relatively norm compact in $C(X_r)\}$.

Let $\mathbb{K}$ be the field of scalars of the Banach space X (we consider

$\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$). For $\alpha\in\mathbb{K}$, we define the dilatation operator $D_\alpha:\mathcal{C}(X)\longrightarrow\mathcal{C}(X)$ by $D_\alpha(f)(x)=f(\alpha x)$, $x\in X$. For a subspace $\mathcal{F}$ of $\mathcal{C}(X)$, endowed with the topology induced from $\mathcal{C}(X)$, we again define $\mathcal{F}^*$ and $\mathcal{F}^\bullet$ as in Section 1; $\mathcal{F}^\bullet$ has a sense only if $\mathcal{F}$ is an algebra. The following hypotheses about $\mathcal{F}$ will be employed:

(2.1) for all $r>0$, $R_r\mathcal{F}$ is a C^*-subalgebra of $C(X_r)$ containing constant functions,

(2.2) $\mathcal{F}$ is a Fréchet space,

(2.3) $\mathcal{F}$ is m-intorverted, i.e. $T_\mu\mathcal{F}\subset\mathcal{F}$ for every $\mu\in\mathcal{F}^\bullet$,

(2.4) $\mathcal{F}$ is dilatation invariant, i.e. $D_\alpha\mathcal{F}\subset\mathcal{F}$ for every $\alpha\in\mathbb{K}$, and

(2.5) $\mathcal{F}$ separates points, i.e. $\forall x_1,x_2\in X\ \exists f\in\mathcal{F}$: $f(x_1)\neq f(x_2)$.

It is clear that $[R_r\mathcal{F}]^\bullet$ are certain compactifications of the balls X_r provided (2.1) is valid. Note that (2.1) makes $\mathcal{F}$ a subalgebra of $\mathcal{C}(X)$ and, if $\mathcal{F}$ is closed in $\mathcal{C}(X)$ (that means (2.2) is valid), we may call $\mathcal{F}$ a "Fréchet algebra".

Lemma 2.1. *Let (2.1) be valid and R_r be considered as a mapping $\mathcal{F}\longrightarrow\mathcal{F}|_{X_r}$. Then the adjoint operator R_r^* clearly maps $[R_r\mathcal{F}]^*$ into $\mathcal{F}^*$. Moreover, $[R_r\mathcal{F}]^\bullet$ is homeomorphically imbedded via R_r^* into $\mathcal{F}^\bullet$.*

Proof. Suppose $\mu\in[R_r\mathcal{F}]^*$ is multiplicative. Then $R_r^*(\mu)(f_1f_2)$ = $\mu(R_r(f_1f_2))$ = $\mu(R_rf_1)\mu(R_rf_2)$ = $R_r^*(\mu)(f_1)R_r^*(\mu)(f_2)$, that means $R_r^*(\mu)$ is multiplicative, too. If $\mu\neq0$, i.e. $\mu(R_r(f))\neq0$ for some $f\in\mathcal{F}$, then $R_r^*(\mu)(f)\neq0$, which means $R_r^*(\mu)\neq0$. Thus we have proved that R_r^* maps $[R_r\mathcal{F}]^\bullet$ actually into $\mathcal{F}^\bullet$.

Now, let us consider a net $\{\mu_\alpha\}$ in $[R_r\mathcal{F}]^\bullet$ converging to some $\mu\in[R_r\mathcal{F}]^\bullet$ weakly*. In other words, $\mu_\alpha(R_rf)\longrightarrow\mu(R_rf)$ for every $f\in\mathcal{F}$.

It is equivalent to the convergence $R_r^*(\mu_\alpha)(f)\longrightarrow R_r^*(\mu)(f)$, which means precisely that $\{R_r^*(\mu_\alpha)\}$ converges to $R_r^*(\mu)$ weakly*, i.e. in the topology $\sigma(\mathcal{F}^*,\mathcal{F})$. We have thus proved that R_r^* is a homeomorphism between $[R_r\mathcal{F}]^\bullet$ and $R_r^*([R_r\mathcal{F}]^\bullet)$. ∎

Let us denote $\mathcal{F}$ endowed only with one seminorm $\|.\|_r$ by $\mathcal{F}_r$, and define again $\mathcal{F}_r^*$ and $\mathcal{F}_r^\bullet$ as above; clearly $\mathcal{F}_r^*\subset\mathcal{F}^*$ and $\mathcal{F}_r^\bullet=\mathcal{F}_r^*\cap\mathcal{F}^\bullet$.

Lemma 2.2. *Let (2.1) be valid. Then* $\mathcal{F}^\bullet=\bigcup_{r>0}\mathcal{F}_r^\bullet$, *and* $R_r^*([R_r\mathcal{F}]^\bullet)=\mathcal{F}_r^\bullet$.

Proof. We modify the arguments by Kolmogorov, Fomin [4; Sec. IV.1.4]: for every $\mu\in\mathcal{F}^*$ there is a neighbourhood of $0\in\mathcal{F}$ on which μ is bounded, that means $\exists r,\varepsilon>0\ \forall f\in\mathcal{F}$: $\|f\|_r\le\varepsilon \Rightarrow |\mu(f)|\le C$, which implies that μ is continuous with respect to some seminorm $\|.\|_r$, hence $\mu\in\mathcal{F}_r^*$. Thus we have proved $\mathcal{F}^*=\bigcup_{r>0}\mathcal{F}_r^*$. The modification for the multiplicative case is obvious.

Now take any $\mu'\in[R_r\mathcal{F}]^\bullet$ and put $\mu=R_r^*(\mu')$. By Lemma 2.1, $\mu\in\mathcal{F}^\bullet$. Moreover, for every $f\in\mathcal{F}$, $|\mu(f)| = |\mu'(R_rf)| \le \|f\|_r$, hence μ is continuous with respect to the seminorm $\|.\|_r$, i.e. $\mu\in\mathcal{F}_r^\bullet$. Conversely, take any $\mu\in\mathcal{F}_r^\bullet$. As μ is continuous with respect to $\|.\|_r$, we have $\mu(f_1)=\mu(f_2)$ whenever $\|f_1-f_2\|_r=0$, which means precisely $R_rf_1=R_rf_2$. Define $\mu':R_r\mathcal{F}\longrightarrow\mathbb{K}$ by $\mu'(f')=\mu(f)$ where $f\in\mathcal{F}$ is an arbitrary extension of $f'\in R_r\mathcal{F}$ (since $\mu\in\mathcal{F}_r^\bullet$, the particular choice of f' is not important, and thus μ' is well defined). The facts that $\mu'\in[R_r\mathcal{F}]^\bullet$ and $R_r^*(\mu')=\mu$ are then obvious. ∎

The following assertion, exploiting (2.2), is based on the well-known uniform boundedness principle.

Lemma 2.3. *Let* (2.1) *and* (2.2) *be fulfilled and* $\{\mu_\alpha\}$ *be a converging net in* $\mathcal{F}^*$. *Then* $\{\mu_\alpha\}\subset\mathcal{F}^*_r$ *for* r *sufficiently large.*

Proof. As the net $\{\mu_\alpha\}$ weakly* converges, $\{\mu_\alpha(f)\}$ is bounded for every $f\in\mathcal{F}$. Due to (2.2), $\mathcal{F}$ is complete metrizable space, and we can apply the uniform boundedness principle in the form [12; II.1, Theorem 1]. It gives some $\delta>0$ and $r>0$ such that, for all α, $\|f\|_r\le\delta$ implies $|\mu_\alpha(f)|\le1$. Hence $|\mu_\alpha(f)|\le1/\delta$ whenever $\|f\|_r\le1$. Particularly, $\mu_\alpha\in\mathcal{F}^*_r$. ∎

Let us note that the evaluation mapping e defined by (1.1) maps X into $\mathcal{F}^*$. Supposing (2.3), we can extend the addition to a binary operation on $\mathcal{F}^*$ again by (1.2). Besides, making use (2.4), we can extend the scalar multiplication $(\alpha,x)\mapsto\alpha x:\mathbb{K}\times X\longrightarrow X$ to a mapping $\mathbb{K}\times\mathcal{F}^*\longrightarrow\mathcal{F}^*$ by putting

$$\alpha\mu = \mu\circ D_\alpha \quad , \ \mu\in\mathcal{F}^*,\ \alpha\in\mathbb{K}. \tag{2.6}$$

It is evident that $\alpha e(x)=e(\alpha x)$ for every $x\in X$ and $\alpha\in\mathbb{K}$, hence the definition (2.6) actually extends the original scalar multiplication on X. The following theorem summarizes the general properties of $\mathcal{F}^*$.

Theorem 2.1. *Let* $\mathcal{F}$ *fulfils* (2.1)-(2.3). *Then* $\mathcal{F}^*$ *endowed with* "+" *defined by* (1.2) *is a Hausdorff,* σ*-compact, right topological semigroup with* 0, *the evaluation mapping* $e:X\longrightarrow\mathcal{F}^*$ *is continuous,* $e(X)$ *is dense in* $\mathcal{F}^*$, $e(x)+\mu=\mu+e(x)$ for every $x\in X,\mu\in\mathcal{F}^*$, *and the mapping* $(x,\mu)\mapsto e(x)+\mu:X\times\mathcal{F}^*\longrightarrow\mathcal{F}^*$ *is (jointly) continuous. Moreover,* e *is injective provided* (2.5) *is valid. Supposing* (2.4), *the* *scalar multiplication defined by (2.6) is jointly continuous. If, in addition, $\mathcal{F}\subset\mathcal{WAP}(X)$, then $\mathcal{F}^\bullet$ with "+" is a commutative semitopological semigroup. After all, if $\mathcal{F}\subset\mathcal{AP}(X)$, $\mathcal{F}^\bullet$ is also a topological group.*

Remark 2.1. By continuity, we can transfer some properties of the scalar multiplication from X on $\mathcal{F}^\bullet$. Thus we always have: $0\mu=0$, $1\mu=\mu$, $(\alpha_1\alpha_2)\mu=\alpha_1(\alpha_2\mu)$, $\alpha(\mu_1+\mu_2)=\alpha\mu_1+\alpha\mu_2$. On the other hand, the second distributive rule, i.e. $(\alpha_1+\alpha_2)\mu=\alpha_1\mu+\alpha_2\mu$, is not valid in general. Nevertheless, if $\mathcal{F}\subset\mathcal{AP}(X)$, it is valid, which makes $\mathcal{F}^\bullet$ a linear topological space.

Proof of Theorem 2.1. As the weak* topology of $\mathcal{F}^*$ is a Hausdorff one, $\mathcal{F}^\bullet\subset\mathcal{F}^*$ is a Hausdorff space, too. As $[R_r\mathcal{F}]^\bullet$ is compact, $\mathcal{F}^\bullet$ is σ-compact as a consequence of Lemmas 2.1 and 2.2. As $e(X_r)$ is dense in $[R_r\mathcal{F}]^\bullet$, $e(X)$ is dense in $\mathcal{F}^\bullet$, too. The continuity of $e:X\longrightarrow\mathcal{F}^\bullet$ is guaranteed by the fact that $\mathcal{F}\subset\mathcal{C}(X)$. The fact that $(\mathcal{F}^\bullet,+)$ is a right topological semigroup has been proved essentially in [1; p.21].

For every $f\in\mathcal{F}$ we have $(e(x)+\mu)(f) = (e(x)\circ T_\mu)(f) = (T_\mu f)(x) = \mu(T_{e(x)}f) = (\mu+e(x))(f)$, thus $e(x)+\mu=\mu+e(x)$ for every $x\in X,\mu\in\mathcal{F}^\bullet$. Now we are going to prove the joint continuity of the mapping $(x,\mu)\longmapsto e(x)+\mu$. We will modify the technique by [1;p.105]: Take $x\in X$ and $\mu\in\mathcal{F}^\bullet$ and nets $\{x_\alpha\}$ and $\{\mu_\alpha\}$ converging in X and in $\mathcal{F}^\bullet$ to x and μ, respectively. By Lemmas 2.1-2.3, for all α and r large enough we have $e(x_\alpha)$, $\mu_\alpha\in\mathcal{F}^\bullet_r$. Then also $e(x_\alpha)+\mu_\alpha\in\mathcal{F}^\bullet_{2r}$. Now we will only show that $\{h(e(x_\alpha)+\mu_\alpha)\}$ converges to $h(e(x)+\mu)$ for every $h\in C(\mathcal{F}^\bullet_{2r})$. As the mapping $\nu\longmapsto\nu+e(x)$ is continuous and $e(x)+\nu=\nu+e(x)$, also $\nu\longmapsto e(x)+\nu$ is continuous. By means of [1;

Lemma I.1.8], and by the compactness of $\mathcal{F}_r^*$, it suffices to show that $\{h(e(x_\alpha)+.)\}$ converges to $h(e(x)+.)$ uniformly on $\mathcal{F}_r^*$. It follows from the fact that h, being continuous on a compact set $\mathcal{F}_{2r}^*$, is uniformly continuous and $\{e(x_\alpha)+\nu\}$ converges to $e(x)+\nu$ uniformly with respect to $\nu\in\mathcal{F}_r^*$. The latter fact is a consequence of the estimate: $\forall f\in\mathcal{F}$, $\nu\in\mathcal{F}_r^*$: $|(e(x_\alpha)+\nu)(f) - (e(x)+\nu)(f)| = |\nu((T_{e(x_\alpha)} - T_{e(x)})f)| \leq \|\nu\|_r^* \|(T_{e(x_\alpha)} - T_{e(x)})f\|_r$ where $\|\nu\|_r^*=\sup_{f\in\mathcal{F}}|\nu(f)|/\|f\|_r$; realize that $\|\nu\|_r^*=1$ since $\nu\in\mathcal{F}_r^*$, and $\{T_{e(x_\alpha)}f\}$ converges to $T_{e(x)}f$ in the norm topology of $C(X_r)$ since $\{x_\alpha\}$ converges to x and f is uniformly continuous on X_{2r}; cf. Theorem 4.3 below.

Similarly we can prove the joint continuity of the extended scalar multiplication $(a,\mu)\mapsto a\mu$, exploiting the convergence of $\{D_{a_\alpha}f\}$ to $D_a f$ in the norm topology of $C(X_r)$ provided $\{a_\alpha\}$ converges to a in $\mathbb{K}$ and f is uniformly continuous on X_{r_1} with r_1 sufficiently large.

Now we go on to the $\mathcal{WAP}$-case. If $h\in\mathcal{WAP}(X)$, modifying [1; p.108] we can see that, for every bounded nets $\{x_\alpha\}$, $\{y_\beta\}$ in X, $\lim_\alpha \lim_\beta h(x_\alpha+y_\beta) = \lim_\beta \lim_\alpha h(x_\alpha+y_\beta)$ whenever all the limits do exist. By Lemmas 2.1 and 2.2, for every $\mu,\nu\in\mathcal{F}^*$ we can take bounded nets $\{x_\alpha\}$, $\{y_\beta\}$ in X converging to μ and ν in $\mathcal{F}^*$, respectively. By continuity arguments we thus obtain $h(\mu+\nu)=h(\nu+\mu)$ for every continuous function h on $\mathcal{F}^*$ provided $\mathcal{F}\subset\mathcal{WAP}(X)$. It yields that "+" is commutative on $\mathcal{F}^*$, from which the separate continuity of $(\mu,\nu)\mapsto\mu+\nu$ appearantly follows.

It remains to prove the $\mathcal{AP}$-case. For $f\in\mathcal{AP}(X)$, the set $\{R_r T_\nu f;\ \nu\in\mathcal{F}_r^*\}$ is relatively norm compact in $C(X_r)$ because it is contained in the closure of $\{R_r T_{e(x)} f;\ x\in X_r\}$ which is compact in

$C(X_r)$ by the very definition of the class $\mathscr{WAP}(X)$; note that, if a net $\{e(x_\alpha)\}$ converges to ν in $\mathscr{F}_r^*$, $\{T_{e(x_\alpha)}f\}$ converges to $T_\nu f$ pointwise on X_r, and, by sequential compactness, it contains a subnet converging in the norm of $C(X_r)$ to some g, but then $g=T_\nu f$ and the whole net $\{T_{e(x_\alpha)}f\}$ must converge to T_ν in the norm topology of $C(X_r)$. Now take $\mu,\nu\in\mathscr{F}^*$, a continuous function h on $\mathscr{F}^*$, and nets $\{\mu_\alpha\}$, $\{\nu_\alpha\}$ converging in $\mathscr{F}^*$ to μ and ν, respectively, and suppose $\mathscr{F}\subset\mathscr{AP}(X)$. We are to show $\lim\limits_\alpha h(\mu_\alpha+\nu_\alpha) = h(\mu+\nu)$, that means $\lim\limits_\alpha \mu_\alpha(T_{\nu_\alpha}\hat{h}) = \mu(T_\nu\hat{h})$ with $\hat{h}\in\mathscr{F}$ such that $\xi(\hat{h})=h(\xi)$ for every $\xi\in\mathscr{F}^*$; cf. also (3.3) below. It follows obviously from the compactness of the net $\{T_{\nu_\alpha}\hat{h}\}$ in $C(X_r)$ for every r (because $\hat{h}\in\mathscr{F}\subset\mathscr{AP}(X)$) and from the fact that $\mu_\alpha,\nu_\alpha\in\mathscr{F}_r^*$ for r large enough (thanks to Lemma 2.3). ∎

3. FURTHER STRUCTURE ON SUFFICIENTLY FINE σ-COMPACTIFICATIONS

In view of Theorem 2.1 we can observe that the structure on $\mathscr{F}^*$ follows the original structure of X more faitfully provided $\mathscr{F}$ is purer, or we may say provided the σ-compactification is coarser; we say that $\mathscr{F}_1^*$ is coarser σ-compactification than $\mathscr{F}_2^*$ (or $\mathscr{F}_2^*$ is finer than $\mathscr{F}_1^*$) if, for every r, $(\mathscr{F}_1)_r^*$ is coarser compactification of X_r than $(\mathscr{F}_2)_r^*$ in the usual sense, i.e. there is a continuous surjection $(\mathscr{F}_2)_r^*\to(\mathscr{F}_1)_r^*$ fixing $e(X_r)$. On the other hand, if $\mathscr{F}$ is rich enough (i.e. the σ-compactification is sufficiently fine), some more structure can be transferred from X onto $\mathscr{F}^*$, which is just to be shown in this Section.

First we mention a general construction of a continuous extension of mappings. Let X_1, X_2 be two Banach spaces, $\mathcal{F}_1$, $\mathcal{F}_2$ Fréchet algebras of functions on X_1 and X_2, respectively, and $F: X_1 \longrightarrow X_2$ be a mapping such that

$$\mathcal{F}_2 \circ F \subset \mathcal{F}_1 \ . \tag{3.1}$$

Then we can define a continuous mapping $\mathcal{F}_1^{\bullet} \longrightarrow \mathcal{F}_2^{\bullet}$, denoted again by F without causing any misplacing, by the formula

$$F(\mu)(f) = \mu(f \circ F) \ , \quad \mu \in \mathcal{F}_1^{\bullet}, \ f \in \mathcal{F}_2. \tag{3.2}$$

The facts that $F(\mu) \in \mathcal{F}_2^{\bullet}$ and F is $(\sigma(\mathcal{F}_1^{*}, \mathcal{F}_1), \sigma(\mathcal{F}_2^{*}, \mathcal{F}_2))$-continuous are obvious. Besides, for all $x \in X_1$, $F(e_1(x)) = e_2(F(x))$, where $e_i: X_i \longrightarrow \mathcal{F}_i^{\bullet}$, $i=1,2$, are the respective evaluation mappings. Thus $F: \mathcal{F}_1^{\bullet} \longrightarrow \mathcal{F}_2^{\bullet}$ can be actually considered as the contiuous extension of the mapping $F: X_1 \longrightarrow X_2$.

In the special case $X_1 = X$, $X_2 = \mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) and $\mathcal{F}_2^{\bullet} \cong \mathbb{K}$ (then $\mathcal{F}_2$ contains the identity on $\mathbb{K}$) we can extend continuously every $f \in \mathcal{F}_1 \equiv \mathcal{F}$, the extended mapping being denoted again by $f: \mathcal{F}^{\bullet} \longrightarrow \mathbb{K}$ for simplicity, and (3.2) in this special case looks as follows:

$$f(\mu) = \mu(f) . \tag{3.3}$$

Let us now investigate the case when the algebra $\mathcal{F}$ contains the norm of X, i.e.

$$\|.\| \in \mathcal{F}. \tag{3.4}$$

Remark 3.1. By (3.3) we can then extend the norm continuously on $\mathcal{F}^*$, and write $\|\mu\|$ for $\mu\in\mathcal{F}^*$. Supposing (2.1)-(2.4), by continuity we obtain all the usual properties of the norm: $0\le\|\mu\|<+\infty$, $\|\mu+\nu\|\le\|\mu\|+\|\nu\|$, $\|\alpha\mu\|=|\alpha|\,\|\mu\|$, and $\|\mu\|>0$ provided $\mu\ne e(0)$.

Theorem 3.1. *Let* (2.1),(2.3) and (3.4) *be fulfilled. Then* $\mathcal{F}^*$ *is locally compact and* $e:X\longrightarrow\mathcal{F}^*$ *defined by* (1.1) *realizes a homeomorphical imbedding of* X *into* $\mathcal{F}^*$.

Proof. Take any $\mu\in\mathcal{F}^*$ and put $B=\{\nu\in\mathcal{F}^*;\ \|\nu\|\le\|\mu\|+1\}$. As $\|.\|$ is continuous on $\mathcal{F}^*$, B is obviously a closed neighbourhood of μ. Now we want to show $B\subset\mathcal{F}^*_r$ with $r=\|\mu\|+1$, which will yield compactness of B as a consequence of the compactness of $[R_r\mathcal{F}]^*$ and of Lemmas 2.1 and 2.2. Let $\nu\in B$. In view of density of $e(X)$ in $\mathcal{F}^*$ there is a net $\{e(x_\alpha)\}\subset X$ converging to ν in $\mathcal{F}^*$. Since the case $\nu=e(0)$ is trivial, we may suppose $\nu\ne e(0)$ and also $\|x_\alpha\|\ge\varepsilon>0$ for all α because $\{\|x_\alpha\|\}$ converges to $\|\nu\|>0$. Put $\hat{x}_\alpha=x_\alpha\|\nu\|/\|x_\alpha\|$. Obviously, $\hat{x}_\alpha\in X_r$ and $\|\hat{x}_\alpha-x_\alpha\| = |\,\|\hat{x}_\alpha\|-\|\nu\|\,|$, thus $\{\|\hat{x}_\alpha-x_\alpha\|\}$ converges to zero thanks to the continuity of the extended norm. As $e(\hat{x}_\alpha) = e(\hat{x}_\alpha-x_\alpha)+e(x_\alpha)$, the net $\{e(\hat{x}_\alpha)\}$ converges to $0+\nu$ because of the joint continuity of $+:X\times\mathcal{F}^*\longrightarrow\mathcal{F}^*$. Since $0+\nu=\nu$, we see that ν belongs to the closure of $e(X_r)$ in $\mathcal{F}^*$, that means to $\mathcal{F}^*_r$.

Realizing that e is injective because (3.4) with (2.3) implies (2.5), we are only to show that the inverse mapping $e^{-1}:e(X)\longrightarrow X$ is continuous. Indeed, the convergence of a net $\{e(x_\alpha)\}$ to $e(x)$ in $\mathcal{F}^*$ means precisely that $\{f(x_\alpha)\}$ converges to $f(x)$ for every $f\in\mathcal{F}$. Thanks to (2.3) and (3.4) we can choose for $f\in\mathcal{F}$ the function $f(y)=\|y-x\|$. Then $f(x_\alpha)=\|x_\alpha-x\|$ and $f(x)=0$, which offers immediately the convergence of $\{x_\alpha\}$ to x in X. ▪

Let us consider now the case when X is a Hilbert space, $\langle .,. \rangle$ denoting its scalar product, and $\mathcal{F}$ contains particularly all linear continuous functionals on X, i.e.

$$(3.5) \qquad X^* \subset \mathcal{F}.$$

Then we can extend the scalar product to a mapping $\langle .,. \rangle : \mathcal{F}^\bullet \times \mathcal{F}^\bullet \longrightarrow \mathbb{K}$ by means of the formula

$$(3.6) \qquad \langle \mu, \nu \rangle = \mu(L_\nu), \ \mu,\nu \in \mathcal{F}^\bullet, \quad \text{with} \quad L_\nu(x) = \nu(\langle x,. \rangle), \ x \in X.$$

Note that $L_\nu \in X^*$. Indeed, L_ν is linear since $\nu \in \mathcal{F}^*$. By Lemma 2.2, $\nu \in \mathcal{F}^\bullet_r$ for r large enough, and a net $\{\langle x_\alpha,. \rangle\}$ converges to $\langle x,. \rangle$ in $C(X_r)$ provided $\{x_\alpha\}$ converges to x in X, hence $\{\nu(\langle x_\alpha,. \rangle)\}$ converges to $\nu(\langle x,. \rangle)$. In other words, L_ν is continuous, and thus actually $L_\nu \in X^*$. Besides, it is obvious that $\langle e(x), e(y) \rangle = \langle x,y \rangle$ for every $x,y \in X$, hence $\langle .,. \rangle : \mathcal{F}^\bullet \times \mathcal{F}^\bullet \longrightarrow \mathbb{K}$ defined by (3.6) actually represents an extension of the original scalar product.

Theorem 3.2. *Let (2.1)-(2.2) and (3.5) be valid. Then the scalar product extended on $\mathcal{F}^\bullet$ by (3.6) is separately continuous and the mapping $(x,\mu) \mapsto \langle e(x), \mu \rangle : X \times \mathcal{F}^\bullet \longrightarrow \mathbb{K}$ is (jointly) continuous.*

Remark 3.2. By the separate continuity quoted above all the usual properties of the scalar product can be transferred on $\mathcal{F}^\bullet$. Thus we get: $\langle a\mu, \nu \rangle = a\langle \mu, \nu \rangle$, $\langle \mu_1 + \mu_2, \nu \rangle = \langle \mu_1, \nu \rangle + \langle \mu_2, \nu \rangle$, and $\langle \mu, \nu \rangle = \overline{\langle \nu, \mu \rangle}$ provided (2.3) and (2.4) are valid; the bar denotes the complex conjugation for the case $\mathbb{K} = \mathbb{C}$. Moreover, $\langle \mu, \nu \rangle \leq \|\mu\| \, \|\nu\|$ provided also (3.4) is valid.

Proof of Theorem 3.2. The continuity of the mapping $\mu\mapsto\langle\mu,\nu\rangle$ follows readily from the definition (3.6). Let μ be fixed and $\{\nu_\alpha\}$ be a net converging to ν in $\mathcal{F}^\bullet$. Then $\{\nu_\alpha(f)\}$ converges to $\nu(f)$ for every $f\in\mathcal{F}$, and, due to (3.5), particularly for every $f\in X^*$, which gives the weak* (and by reflexivity also weak) convergence of $\{L_{\nu_\alpha}\}$ to L_ν in X^*. Since μ is continuous and linear on $\mathcal{F}$ and thus also on X^*, we see that $\{\mu(L_{\nu_\alpha})\}$ converges to $\mu(L_\nu)$, which means precisely that $\{\langle\mu,\nu_\alpha\rangle\}$ converges to $\langle\mu,\nu\rangle$.

The convergence of $\{\langle e(x_\alpha),\nu_\alpha\rangle\}$ to $\langle e(x),\nu\rangle$ for $\{x_\alpha\}$ converging to x in X and $\{\nu_\alpha\}$ converging to ν in $\mathcal{F}^\bullet$ can be proved completely as in the case of the addition (cf. the proof of Theorem 2.1), using the facts that $\nu_\alpha\in\mathcal{F}^\bullet_r$ for r large enough (due to Lemma 2.3) and $\{\langle x_\alpha,.\rangle\}$ converges to $\langle x,.\rangle$ in $C(X_r)$, which implies the convergence of $\{\nu_\alpha(\langle x_\alpha,.\rangle)\} = \{\langle e(x_\alpha),\nu_\alpha\rangle\}$ to $\nu(\langle x,.\rangle) = \langle e(x),\nu\rangle$. ■

4. EXAMPLES

In this section we want to show how the σ-compactifications of X by means of suitable Fréchet algebras $\mathcal{F}$ can cover various standard extensions that can be thus seen from a unified point of view.

4.1. AN EXAMPLE OF THE $\mathcal{AP}$-KIND: THE BIDUAL SPACE X^{**}.

Let us take the class $\mathcal{B}_{\rm aff}$ of all continuous affine functions on X, i.e. $\mathcal{B}_{\rm aff}=\{y(.)+a;\ y\in X^*,\ a\in\mathbb{K}\}$. Denote by $\mathcal{R}(\mathcal{B}_{\rm aff})$

the minimal ring containing $\mathcal{B}_{\mathrm{aff}}$, i.e.

$$\mathcal{R}(\mathcal{B}_{\mathrm{aff}}) = \left\{ \sum_{i=1}^{m} \prod_{j=1}^{n} f_{ij}(.);\ f_{ij}\in\mathcal{B}_{\mathrm{aff}},\ m,n\in\mathbb{N} \right\} .$$

Finally, let $\mathcal{F}(\mathcal{B}_{\mathrm{aff}})$ denote the closure of $\mathcal{R}(\mathcal{B}_{\mathrm{aff}})$ in $\mathcal{C}(X)$. Thus $\mathcal{F}(\mathcal{B}_{\mathrm{aff}})$ satisfies (2.1) and (2.2). Since $\mathcal{B}_{\mathrm{aff}}$ is dilatation invariant, $\mathcal{F}(\mathcal{B}_{\mathrm{aff}})$ satisfies also (2.4). Since $\mathcal{B}_{\mathrm{aff}}$ is appearantly translation invariant (i.e. $T_{e(x)}\mathcal{B}_{\mathrm{aff}}\subset\mathcal{B}_{\mathrm{aff}}$ for every $x\in X$), $\mathcal{R}(\mathcal{B}_{\mathrm{aff}})$ has this property, too, and modifying [1; p.113] we verify also (2.3) with $\mathcal{F}=\mathcal{F}(\mathcal{B}_{\mathrm{aff}})$. Moreover, $\mathcal{F}(\mathcal{B}_{\mathrm{aff}})$ fulfils evidently (2.5) and (3.5). It is evident that $\mathcal{B}_{\mathrm{aff}}\subset\mathcal{AP}(X)$. Modifying the arguments of [1; pp. 115 and 110] we can show that $f_1,f_2\in\mathcal{AP}(X)$ implies $f_1f_2\in\mathcal{AP}(X)$, thus $\mathcal{R}(\mathcal{B}_{\mathrm{aff}})\subset\mathcal{AP}(X)$. By the technique [1; pp.25-27] we can eventually show that also $\mathcal{F}(\mathcal{B}_{\mathrm{aff}})\subset\mathcal{AP}(X)$. In view of Remark 2.1 we thus can see that $\mathcal{F}(\mathcal{B}_{\mathrm{aff}})^{\bullet}$ is a linear topological space. The following assertion even identifies it.

Theorem 4.1. $[\mathcal{F}(\mathcal{B}_{\mathrm{aff}})]^{\bullet}$ *is homeomorphic with the bidual space* X^{**} *of* X *endowed with the weak** *topology, and the following diagram commutes* (J *denotes the cannonical imbedding of* X *into* X^{**}, *and* $\Phi: X^{**}\longrightarrow[\mathcal{F}(\mathcal{B}_{\mathrm{aff}})]^{\bullet}$ *the homeomorphism*):

$$\begin{tikzcd} & X \arrow[dl, "J"'] \arrow[dr, "e"] & \\ X^{**} \arrow[rr, leftrightarrow, "\Phi"] & & {[\mathcal{F}(\mathcal{B}_{\mathrm{aff}})]^{\bullet}} \end{tikzcd}$$

Proof. Let us recall that $J: X\longrightarrow X^{**}$ is defined by $J(x)(y)=y(x)$, $y\in X^{*}$, $x\in X$. Let us investigate the mapping $\varphi: J(X)\longrightarrow[\mathcal{F}(\mathcal{B}_{\mathrm{aff}})]^{\bullet}$ defined by $\varphi(J(x))(f)=f(x)$, $x\in X$, $f\in\mathcal{F}(\mathcal{B}_{\mathrm{aff}})$. Note that $\varphi\circ J=e$. We will show that φ is uniformly continuous on bounded subsets of X^{**} from the (relativized) $\sigma(X^{**},X^{*})$-uniformity to the $\sigma([\mathcal{F}(\mathcal{B}_{\rm aff})]^{*},\mathcal{F}(\mathcal{B}_{\rm aff}))$-uniformity, it means $\forall r>0$ $\forall\varepsilon>0$ $\forall f\in\mathcal{F}(\mathcal{B}_{\rm aff})$ $\exists$ a finite set M in X^{*} $\exists\delta>0$ $\forall x_1,x_2\in X_r$: ($\forall y\in M$: $|y(x_1-x_2)|\le\delta$) => $|f(x_1)-f(x_2)|\le\varepsilon$. Indeed, for every r, ε, and f, we can take $f_{ij}\in\mathcal{B}_{\rm aff}$, i.e. $f_{ij}(.) = y_{ij}(.)+a_{ij}$ with $y_{ij}\in X^{*}$ such that $|\sum_{i=1}^{m}\prod_{j=1}^{n}f_{ij}(x) - f(x)| \le \frac{\varepsilon}{2}$ for all $x\in X_r$. Now it is clear that we get the desired estimate $|f(x_1)-f(x_2)| \le \varepsilon$ when take $M = \{y_{ij};\ i=1,n,\ j=1,m\}$ and $\delta>0$ small enough.

Moreover, φ is injective due to (2.5), and $\varphi^{-1}:e(X)\longrightarrow J(X)$ is uniformly continuous in the mentioned uniformities. Indeed, $\forall\varepsilon>0$ $\forall y\in X^{*}$ $\exists f\in\mathcal{F}(\mathcal{B}_{\rm aff})$ $\exists\delta>0$ $\forall x_1,x_2\in X$: $|f(x_1)-f(x_2)|\le\delta$ => $|y(x_1-x_2)|\le\varepsilon$ (it suffices to put simply $f=y$ and $\delta=\varepsilon$).

Now the assertion to be proved follows from the facts that the completion of $J(X_r)$ with respect to the relativized $\sigma(X^{**},X^{*})$-uniformity (which coincides with the $\sigma(X,X^{*})$-one) is just X_r^{**} (see e.g. [12; Sec. IV.8]), the completion of $e(X_r)$ with respect to the relativized $\sigma([\mathcal{F}(\mathcal{B}_{\rm aff})]^{*},\mathcal{F}(\mathcal{B}_{\rm aff}))$-uniformity is just $\mathcal{F}(\mathcal{B}_{\rm aff})^{\bullet}_{r}$, and a uniformly continuous mapping can be extended continuously on the respective completion. Thus, extending φ on the $\sigma(X^{**},X^{*})$-completion of $J(X_r)$, we obtain a homeomorphism between X_r^{**} and $\mathcal{F}(\mathcal{B}_{\rm aff})^{\bullet}_{r}$, and by continuation with r passing to $+\infty$ we get eventually the homeomorphism $\Phi:X^{**}\longrightarrow\mathcal{F}(\mathcal{B}_{\rm aff})^{\bullet}$. ∎

Remark 4.1. By continuity, Φ is simultaneously an isomorphism of the respective algebraic structures (here linear spaces). Φ^{-1} can be defined alternatively by assigning to every $\mu\in\mathcal{F}(\mathcal{B}_{\rm aff})^{*}$ its restriction on $X^{*}\subset\mathcal{F}(\mathcal{B}_{\rm aff})$ which is appeharantly a linear continuous functional on X^{*}, hence an element of X^{**}.

4.2. AN EXAMPLE OF THE $\mathcal{WAP}$-KIND: THE SPACE OF TYPES.

Let us confine ourselves to the case when X is a stable Banach space, which means $\lim_n \lim_m \|x_n+y_m\| = \lim_m \lim_n \|x_n+y_m\|$ whenever all the limits exist. This class of Banach spaces has been introduced by Krivine and Maurey [5] (see also [2; Chap.VII], e.g.); it is known that, e.g., the spaces L^p of p-integrable functions are stable if $1 \le p < +\infty$ (while L^∞ is not stable). The space of types $\mathcal{T}(X)$ is defined as the closure of $\{\tau_x;\ x \in X\}$ in $\mathbb{R}_+^X$ when $\mathbb{R}_+^X$ is endowed with the product topology; $\mathbb{R}_+$ is the set of non-negative reals and $\tau_x : X \to \mathbb{R}_+$ is defined by $\tau_x(y) = \|x+y\|$. The mapping $x \mapsto \tau_x$ realizes a homeomorphical imbedding of X into $\mathcal{T}(X)$, let us denote it by $\mathcal{J}$. The addition, scalar multiplication, and the norm can be extended onto $\mathcal{T}(X)$, the (separately continuous) extended addition being called "convolution of types"; cf. [2,5].

To obtain the space of types by our method of σ-compactifications, we consider the set

$$\mathcal{B}_{\text{type}} = \{a\tau_x + b;\ a,b \in \mathbb{K},\ x \in X\}$$

and define $\mathcal{F}(\mathcal{B}_{\text{type}})$ analogously as $\mathcal{F}(\mathcal{B}_{\text{aff}})$ in the previous section. We can again verify (2.1)-(2.4), (3.4), and (3.5) with $\mathcal{F} = \mathcal{F}(\mathcal{B}_{\text{type}})$, using also the stability assumption, which implies by [1; p.108] that $\mathcal{B}_{\text{type}} \subset \mathcal{WAP}(X)$, and thus $\mathcal{F}(\mathcal{B}_{\text{type}}) \subset \mathcal{WAP}(X)$ by similar arguments as those used for $\mathcal{F}(\mathcal{B}_{\text{aff}}) \subset \mathcal{AP}(X)$ in Section 4.1.

Theorem 4.2. $[\mathcal{F}(\mathcal{B}_{\text{type}})]^*$ *is homeomorphic with the space of types* $\mathcal{T}(X)$ *and the following diagram commutes* $(\Phi : \mathcal{T}(X) \to [\mathcal{F}(\mathcal{B}_{\text{type}})]^*$

denotes the mentioned homeomorphism):

$$
\begin{tikzcd}
 & X \arrow[dl, "\mathcal{J}"'] \arrow[dr, "e"] & \\
\mathcal{T}(X) \arrow[rr, leftrightarrow, "\Phi"] & & {[\mathcal{F}(\mathcal{B}_{\mathrm{type}})]^{\bullet}}
\end{tikzcd}
$$

Proof. Defining $\varphi\colon \mathcal{J}(X)\longrightarrow[\mathcal{F}(\mathcal{B}_{\mathrm{type}})]^{\bullet}$ as $\varphi(\tau_x)=e(x)$ for all $x\in X$, the proof merely pharaphrazes that one of Theorem 4.1. ▨

Remark 4.2. Using general considerations, our theory together with Theorem 4.2 enables to derive readily most of algebraico/topological properties of $\mathcal{T}(X)$, usually stated in the space-of-types theory; see [2,5].

4.3. THE FINEST σ-COMPACTIFICATION

Like the Stone-Čech compactification plays an important role, being the finest compactification of a completely regular topological space, here it is natural to look for the finest σ-compactification preserving still the algebraico/topological structure transferred partly from the original Banach space. It is clear that if $\mathcal{F}$ would not be a subalgebra of $\mathcal{C}(X)$, the continuity both of $e\colon X\longrightarrow\mathcal{F}^{\bullet}$ and of $(x,\mu)\longmapsto e(x)+\mu\colon X\times\mathcal{F}^{\bullet}\longrightarrow\mathcal{F}^{\bullet}$ would be lost. Hence we require $\mathcal{F}\subset\mathcal{C}(X)$. Moreover, to extend the addition by (1.2) we need neccessarily the m-introversion of $\mathcal{F}$. Using and modifing the results and arguments of Namioka [7] and Rao [8] (see also [1; pp.103, 105, 127]) we obtain the class we are looking for.

Theorem 4.3. *The largest m-introverted subalgebra of $\mathcal{E}(X)$ coincides with $\mathcal{UE}(X)$, the set of all functions that are uniformly continuous on bounded subsets (defined in Section 2).*

Proof. Put $\mathcal{ME}(X)=\{f\in\mathcal{E}(X);\ \forall\mu\in\mathcal{E}(X)^*:\ \mu(T_{e(.)}f)\in\mathcal{E}(X)\}$; "$\mathcal{ME}$" stands for "multiplicatively continuous". By the very definitions, $\mathcal{ME}(X)$ is appearantly the (only) maximal m-introverted subalgebra of $\mathcal{E}(X)$; cf. also Rao [8] or [1; pp.102-103]. By the arguments of the proof of Lemma 2.2, every $\mu\in\mathcal{E}(X)^*$ belongs to some $\mathcal{E}(X)^*_r$ for r sufficiently large, and $\mathcal{E}(X)^*_r \cong C(X_r)^*$ $(=\beta X_r$, the Stone-Čech compactification of $X_r)$. For any $f\in\mathcal{UE}(X)$, $T_{e(.)}f$ is continuous as a mapping $X_r\to C(X_r)$, which implies $R_r(\mu(T_{e(.)}f))\in C(X_r)$. Thus $\mu(T_{e(.)}f))\in\mathcal{E}(X)$. In other words, $f\in\mathcal{ME}(X)$, and thus $\mathcal{UE}(X)\subset\mathcal{ME}(X)$.

It remains to prove that, conversely, $\mathcal{ME}(X)\subset\mathcal{UE}(X)$. As $\mathcal{ME}(X)$ is m-introverted, the addition $\mu+\nu$ is well defined and the mapping $\mu\mapsto\mu+\nu$ is continuous. Since $e(x)+\mu=\mu+e(x)$ (cf. the proof of Theorem 2.1) the mapping $(x,\mu)\mapsto e(x)+\mu: X\times\mathcal{ME}(X)^*\to\mathcal{ME}(X)^*$ is separately continuous. Since $\mathcal{UE}(X)$ obviously satisfies (3.4) and $\mathcal{ME}(X)\supset\mathcal{UE}(X)$, $\mathcal{ME}(X)$ satisfies it, as well. Thus $\mathcal{ME}(X)^*$ is locally compact; cf. the proof of Theorem 3.1. Then we can use readily the result by Rao [8] that yields the joint continuity of the mapping $(x,\mu)\mapsto e(x)+\mu$, employing also the facts that X is a group acting on $\mathcal{ME}(X)^*$ and a complete metric space, hence a Čech complete, and therefore a strongly countably complete space; for details see [8]. Following [1; p.105], take any $f\in\mathcal{ME}(X)$. Then, for $x,y\in X$, $\|T_{e(x)}f-T_{e(y)}f\|_r = \sup\{|f(e(x)+\mu)-f(e(y)+\mu)|;\ \mu\in\mathcal{ME}(X)^*_r\}$, where we have employed also the continuous extension of f on $\mathcal{ME}(X)^*$ by (3.3) and the density of $e(X_r)$ in $\mathcal{ME}(X)^*_r$. By the compactness of $\mathcal{ME}(X)^*_r$ and the joint continuity of

$(x,\mu)\mapsto e(x)+\mu$, the mapping $x\mapsto f(e(x)+.): X\to C(\mathcal{ME}(X)^*_r)$ is continuous, which gives eventually $\|T_{e(x_\alpha)}f-f\|_r$ converging to zero whenever a net $\{x_\alpha\}$ converges to 0. However, it means $R_r f\in UC(X_r)$, and passing with r to $+\infty$, we get $f\in\mathcal{UE}(X)$. The inclusion $\mathcal{ME}(X)\subset\mathcal{UE}(X)$ has been thus proved. ▪

Remark 4.3. Note that the largest m-introverted subalgebra of $\mathcal{E}(X)$ satisfies not only (2.3), but also the other conditions (2.1)-(2.5), (3.4), and (3.5). The compactification $\mathcal{F}^*_r$, $\mathcal{F}=\mathcal{UE}(X)$, of the ball X_r is appearantly homeomorphic with the Smirnov compactification of X_r endowed naturally with the norm proximity; for details we refer to [11]. Then it is clear that the finest (according to Theorem 4.3) σ-compactification $\mathcal{F}^*$ is, disregarding the algebraic structure, homeomorphic with the local compactification by S.Leader [6] of the local proximity space $(X,\beta,\mathcal{B})$ when the local proximity relation β as well as the boundedness $\mathcal{B}$ is induced naturally by the norm; for details see [6].

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