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HYBRID SOLUTION OF WEAKLY FORMULATED BOUNDARY-VALUE PROBLEMS *

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A weak formulation of linear two-point boundary-value problems is introduced. Then the factorization method, which is suitable for hybrid computation, is applied. So we can treat the problems with right-hand side containing Dirac distributions or some more general ones. In the general case, the coefficients of the differential operator are measurable bounded and the right-hand side belongs to the relevant Sobolev space of distributions. Practical examples are given.

1. Introduction

TPBVP's (two-point boundary-value problems) represent a fundamental class of the tasks which are effectively solvable on hybrid computing systems. We will consider the TPBVP for a linear differential equation of order of $2k$ ($k \geqslant 1$) in a divergent form

$$\sum_{i,j=0}^{k} \frac{\mathrm{d}^i}{\mathrm{d}x^i}\left(a_{ij}\frac{\mathrm{d}^j u}{\mathrm{d}x^j}\right) = f \tag{1}$$

with Dirichlet boundary conditions

$$\frac{\mathrm{d}^i u}{\mathrm{d}x^i}(a) = A_i, \quad \frac{\mathrm{d}^i u}{\mathrm{d}x^i}(b) = B_i, \qquad i = 0,\ldots,k-1, \tag{1'}$$

where $-\infty < a < b < +\infty$. Some generalizations will be outlined in Section 6. We will not engage in an existence or stability question, so that we need not make any ellipticity assumption. Only $1/a_{kk}$ is supposed to be bounded. In the opposite case, BVP (1)–(1′) degenerates.

The well-known classical formulation of the problem requires $f \in C^0$ and $a_{ij} \in C^i$ (for the definition of spaces C^i see Appendix). Then the classical solution u, if it exists, belongs to C^{2k}, fulfils (1′), and the equation (1) holds everywhere in the open interval $]a, b[$.

Let us introduce some typical practical examples which require certain generalized formulation of BVP. The steady-state temperature in a bar is described by

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(a_1\frac{\mathrm{d}u}{\mathrm{d}x}\right) + a_0 u = f \tag{2}$$

with $a_0 = 0$. Equation (2) is a special case of (1), the so-called self-adjoint case for $k = 1$. If the bar consists of several parts with different heat conductivities, then a_1 has discontinuities, while the classical formulation requires $a_1 = a_{11} \in C^1$.

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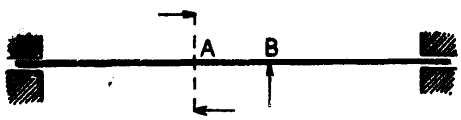

Fig. 1. Fixed-end beam, loaded by a force couple (acts in point A) and by a single force (acts in B).

Another example: A taut string deflection is described also by (2). If a transversal load of the string is concentrated in isolated points, then the right-hand side f contains Dirac distributions in these points. The deflection of a transversally loaded, taut and elastically supported beam is described by

$$\frac{d^2}{dx^2}\left(a_2\frac{d^2u}{dx^2}\right)+\frac{d}{dx}\left(a_1\frac{du}{dx}\right)+a_0u=f. \tag{3}$$

Equation (3) is the self-adjoint case of (1) for $k = 2$. Now f may even contain first-order derivatives of the Dirac distributions. This situation corresponds to loading by a force couple acting in a point – cf. Fig. 1.

A 'natural' mathematical tool which enables to describe above introduced practical situations is a weak formulation of BVP's in Sobolev spaces. This theory (called variational theory) is very simple in our one-dimensional case (for general case see e.g. Lions and Magenes [1, Chap. 2, §9]). For $a_{ij} \in L_\infty$, $f \in H^{-k}$ let u be a weak solution of (1)–(1′) iff $u \in H^k$, the boundary conditions (1′) are valid, and

$$\sum_{i,j=0}^{k}\int_a^b(-1)^i a_{ij}\frac{d^ju}{dx^j}\frac{d^iv}{dx^i}dx=\langle f,v\rangle \tag{4}$$

holds for every $v \in H_0^k$ (for the definition of spaces L_∞, H^k, H_0^k, H^{-k} see Appendix, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between H^{-k} and H_0^k).

The analog treatment of discontinuous coefficients is usual, while the treatment of distributions brings some problems. Note that if a regularization of f is used, then a function with an inconveniently large norm is obtained, which makes this way unsuitable from the analog viewpoint (the norm in L_∞ is considered).

The aim of this paper is to give a methodology for the hybrid solution of the weakly formulated BVP (1)–(1′) with f as general as possible. Computing on a digital computer by using numerical integration is, of course, also possible. Proofs are omitted for the sake of brevity and simple explanatory examples are given instead of them.

2. Hybrid treatment of the weak formulation of TPBVP's

The 'natural' method for the hybrid solution of linear TPBVP's is a factorization (also called decomposition or Riccati transformation), see Vichnevetsky [5]. By using the factorization method we obtain two initial-value problems which are directly programmable on the analog part of the hybrid computing system.

There are several ways to solve weakly formulated TPBVP's via factorization. We may use the following theorem which characterizes the distributions belonging to H^{-k} (for the proof see e.g. Lions and Magenes [1, Chap. 1, Theorem 12.1]): $f \in H^{-k}$ iff f may be represented (in non-unique fashion) by

$$f=\sum_{i=0}^{k}\frac{d^if_i}{dx^i},\quad f_i \in L_2. \tag{5}$$

Now we may apply the factorization method to problem (1)–(1′) with (5). This problem can be rewritten into the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}A_1 & A_2\\A_3 & A_4\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}+\begin{pmatrix}F_1\\F_2\end{pmatrix}$$

where u_1, u_2, F_1, F_2 are k-vectors and $A_1,\dots,A_4$ are $k\times k$-matrices. We can find the components

$$(A_1)_{ij}=\begin{cases}-a_{k,j-1}/a_{kk} & \text{for } i=k,\\ 1 & \text{for } i=j-1,\\ 0 & \text{elsewhere,}\end{cases}$$

$$(A_2)_{ij}=\begin{cases}1/a_{kk} & \text{for } i=k, j=1,\\ 0 & \text{elsewhere,}\end{cases}$$

$$(A_3)_{ij}=-a_{k-i,j-1}+a_{k-i,k}a_{k,j-1}/a_{kk},$$

$$(A_4)_{ij}=\begin{cases}-a_{k-i,k}/a_{kk} & \text{for } j=1,\\ 1 & \text{for } i=j-1,\\ 0 & \text{elsewhere,}\end{cases}\qquad (F_1)_i=\begin{cases}0 & \text{for } i<k,\\ f_k/a_{kk} & \text{for } i=k,\end{cases}$$

$$(F_2)_i=f_{k-i}-a_{k-i,k}f_k/a_{kk},\qquad (u_1)_i(a)=A_{i-1},\qquad (u_1)_i(b)=B_{i-1},$$

where $i, j=1,\dots,k$. Then we can construct the factorized equation (see e.g. Mufti et al. [2], Taufer [3]):

$$\frac{\mathrm{d}G}{\mathrm{d}x}=A_1G-GA_4+GA_3G-A_2,\qquad G(a)=0,\tag{6}$$

$$\frac{\mathrm{d}g}{\mathrm{d}x}=(A_1+GA_3)g+F_1+GF_2,\qquad g(a)=u_1(a),\tag{7}$$

where G is a $k\times k$-matrix and g a k-vector. This initial-value problem is solved 'forward' and afterwards we can solve 'backward' the initial-value problem

$$\frac{\mathrm{d}z}{\mathrm{d}x}=(A_4-A_3G)z+A_3g+F_2,\qquad z(b)=G(b)^{-1}(g(b)-u_1(b)),\tag{8}$$

$$u=g_1-\sum_{j=1}^{k}G_{1,j}z_j,\tag{9}$$

where z is a k-vector and u a scalar. The generality of f must be somewhat restricted in order that the factorized equation could be treated on the hybrid computer. We must assume $f_i\in L_\infty$. The factorized equation is considered in the well-known Caratheodory sense, therefore we may integrate it in usual way on the analog part of the hybrid computer. The functions g, z depend on the choice of f_i in the representation (5), while u is independent of this choice. If any solution of the factorized equation (6)–(9) exists (in the Caratheodory sense), then u from (9) is the (weak) solution of BVP (1)–(1′) and (5).

Certain rearrangements of the factorized equation are possible. We can use the factorization method with $F_1=0$ and $F_2=F_{2,0}+\mathrm{d}F_{2,1}/\mathrm{d}x$, e.g.

$$(F_{2,0})_i=f_{k-i},\quad (F_{2,1})_1=f_k,\quad (F_{2,1})_i=0\quad \text{for } i>1.$$

After usual analog-methodology arrangement we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(g-GF_{2,1})=(A_1+GA_3)g-\frac{\mathrm{d}G}{\mathrm{d}x}F_{2,1}+GF_{2,0},\tag{7′}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(z-F_{2,1})=(A_4-A_3G)z+A_3g+F_{2,0}.\tag{8′}$$

The initial conditions and other equations remain the same as in (6)–(9). Equations (7′), (8′) are considered again in the Carathedorys sense, but now g and z may be discontinuous, $g, z \in L_\infty$. On the other hand, g and z do not depend on the choice of f_k. Certain disadvantage consists in the fact, that the norm of $\mathrm{d}G/\mathrm{d}x$ must be estimated in advance. If we want to go on in this procedure, we may set $f_k = f_{k-1} = 0$, $F_2 = F_{2,0} + \mathrm{d}F_{2,1}/\mathrm{d}x + \mathrm{d}^2F_{2,2}/\mathrm{d}x^2$. But equations solvable by analog technique are of a very complicated structure (we must set $g = g_0 + \mathrm{d}g_1/\mathrm{d}x$, similarly for z), so that such an arrangement is unsuitable.

There exists a further possibility provided f has a more special form, namely

$$f = f_0 + \sum_{i=0}^{k-1} \frac{\mathrm{d}^i}{\mathrm{d}x^i}\Big(\sum_j c_{ij}\delta(x_j)\Big), \tag{10}$$

where $f_0 \in L_\infty$, c_{ij} are reals and $\delta(x_j)$ is the Dirac distribution in the point $x_j \in]a, b[$. Now (7) and (8) can be used with $f_i = 0$, $i > 0$, but some generalization must be made. Define the distributions g, z to be the distributive solution of (7), (8) iff the right-hand sides of (7), (8) exist in the sense of distributions, (7) and (8) represent equalities of distributions on $[a, b]$, g and z are regular and continuous in the points a, b and the prescribed initial conditions are fulfilled. Let us point out that the multiplication of two distributions is, generally, not well-defined, thus the distributive solution of (7), (8) need not exists.

However, the notion of the distributive solution is not sufficiently general in case $k > 1$. In addition, we can use a regularization of the coefficients a_{ij}. A special character of the investigated problem causes the regularization approach to be very simple. We will demonstrate it in Section 4. The notions of the distributive solution and the regularization enable to establish a behavior of g, z in the points x_j from (10). The irregular part of the distributions g, z is not essential from the analog point of view, hence only the magnitudes of the jumps are needed. Again we can show that if any generalized solution of the factorized equation exists, then u from (9) is the weak solution of the problem (1)–(1′) with (10).

Discontinuities of the solution of the factorized equation require special programming techniques. We can either interrupt an operating mode of the analog computer and reset new initial conditions on integrators in each point x_j, or add piecewise constant functions with jumps in the points x_j to outputs of the integrators. In case of a mere digital treatment, the realization of the discontinuous solution brings no difficulty. Obviously, because of technical realization, the number of the points x_j must be finite (this also guarantees that f in the form (10) belongs to H^{-k}).

In the following sections, our methodology will be briefly demonstrated.

3. Second-order problems

For simplicity we consider only the special problem (2), (1′) with (5) or (10).

Using the respresentation (5) before the factorization, we obtain the equations

$$\frac{\mathrm{d}G}{\mathrm{d}x} = -a_0G^2 - 1/a_1, \qquad G(a) = 0,$$

$$\frac{\mathrm{d}g}{\mathrm{d}x} = -a_0Gg + Gf_0 + f_1/a_1, \qquad g(a) = A_0,$$

$$\frac{\mathrm{d}z}{\mathrm{d}x} = a_0(Gz - g) + f_0, \qquad z(b) = (g(b) - B_0)/G(b),$$

$$u = g - Gz.$$

Now G, g, z are scalar functions, g and z are continuous and depend on the choice of f_0, f_1 from the representation (5).

Using the representation (5) after the factorization with $f_1 = 0$ via (7′), (8′), we obtain equations for g, z

$$\frac{\mathrm{d}}{\mathrm{d}x}(g - Gf_1) = -a_0Gg + Gf_0 - \frac{\mathrm{d}G}{\mathrm{d}x}f_1, \qquad \frac{\mathrm{d}}{\mathrm{d}x}(z - f_1) = a_0(Gz - g) + f_0.$$

In this case g, z may have discontinuities, but do not depend on the choice of f_0, f_1 (for fixed $f = f_0 + \mathrm{d}f_1/\mathrm{d}x$). The norm of $\mathrm{d}G/\mathrm{d}x$ must be known for the analog computation.

If $f = f_0 + \Sigma_j c_{0j}\delta(x_j)$, we can use the equations for g, z

$$\frac{\mathrm{d}g}{\mathrm{d}x} = -a_0 G g + G f, \qquad \frac{\mathrm{d}z}{\mathrm{d}x} = a_0(Gz - g) + f$$

considered in the distributive sense. The solution G of the Riccati equation, if it exists, is continuous, hence the multiplication $Gf = Gf_0 + \Sigma_j c_{0j} G(x_j)\delta(x_j)$ is well-defined. So we can easily determine the jumps of g (their magnitudes are $c_{0j}G(x_j)$ in the points x_j, respectively) and of z (their magnitudes are c_{0j}).

4. Higher-order problems

Consider the fourth-order problem (3), (1′) with (5) or (10). By using the representation (5) before the factorization, we obtain (after some simple arrangements) the following equations:

$$\left.\begin{aligned}
&\frac{\mathrm{d}G_1}{\mathrm{d}x} = G_3 - a_1 G_1 G_3 - a_0 G_1 G_2, && G_1(a) = 0,\\
&\frac{\mathrm{d}G_2}{\mathrm{d}x} = -2G_1 + a_1 G_1^2 - a_0 G_2^2, && G_2(a) = 0,\\
&\frac{\mathrm{d}G_3}{\mathrm{d}x} = -a_1 G_3^2 + a_0 G_1^2 - 1/a_2, && G_3(a) = 0,
\end{aligned}\right\} \tag{11}$$

$$\left.\begin{aligned}
&\frac{\mathrm{d}g_1}{\mathrm{d}x} = g_2 - a_1 G_1 g_2 - a_0 G_2 g_1 + G_2 f_0 + G_1 f_1, \qquad g_1(a) = A_0,\\
&\frac{\mathrm{d}g_2}{\mathrm{d}x} = -a_1 G_3 g_2 + a_0 G_1 g_1 - G_1 f_0 + G_3 f_1 + f_2/a_2, \qquad g_2(a) = A_1,\\
&\frac{\mathrm{d}z_1}{\mathrm{d}x} = z_2 - a_1 g_2 + a_1 G_3 z_1 - a_1 G_1 z_2 + f_1,\\
&\frac{\mathrm{d}z_2}{\mathrm{d}x} = -a_0 g_1 + a_0 G_1 z_1 + a_0 G_2 z_2 + f_0,\\
&z_1(b)G_1(b) + z_2(b)G_2(b) = g_1(b) - B_0,\\
&z_1(b)G_3(b) - z_2(b)G_1(b) = g_2(b) - B_1,
\end{aligned}\right\} \tag{12}$$

$$u = g_1 - G_1 z_1 - G_2 z_2. \tag{13}$$

All solutions of these equations, if they exist, are continuous.

Using (7′), (8′), we obtain equations for g_i, z_i with the same initial conditions

$$\left.\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}(g_1 - G_1 f_2) = g_2 - a_1 G_1 g_2 - a_0 G_2 g_1 + G_2 f_0 + G_1 f_1 - \frac{\mathrm{d}G_1}{\mathrm{d}x} f_2,\\
&\frac{\mathrm{d}}{\mathrm{d}x}(g_2 - G_3 f_2) = -a_1 G_3 g_2 + a_0 G_1 g_1 - G_1 f_0 + G_3 f_1 - \frac{\mathrm{d}G_3}{\mathrm{d}x} f_2,\\
&\frac{\mathrm{d}}{\mathrm{d}x}(z_1 - f_2) = z_2 - a_1 g_2 + a_1 G_3 z_1 - a_1 G_1 z_2 + f_1,\\
&\frac{\mathrm{d}}{\mathrm{d}x} z_2 = -a_0 g_1 + a_0 G_1 z_1 + a_0 G_2 z_2 + f_0.
\end{aligned}\right\} \tag{12'}$$

In this case the norms of $\mathrm{d}G_1/\mathrm{d}x$ and $\mathrm{d}G_3/\mathrm{d}x$ must be known.

Let $f=f_0+\Sigma_j(c_{0j}+c_{1j}\mathrm{d}/\mathrm{d}x)\delta(x_j)$. If we want to use the equation

$$\left.\begin{aligned}\frac{\mathrm{d}g_1}{\mathrm{d}x}&=g_2-a_1G_1g_2-a_0G_2g_1+G_2f,\\ \frac{\mathrm{d}g_2}{\mathrm{d}x}&=-a_1G_3g_2+a_0G_1g_1-G_1f,\\ \frac{\mathrm{d}z_1}{\mathrm{d}x}&=z_2-a_1g_2+a_1G_3z_1-a_1G_1z_2,\\ \frac{\mathrm{d}z_2}{\mathrm{d}x}&=-a_0g_1+a_0G_1z_1+a_0G_2z_2+f\end{aligned}\right\}\tag{12''}$$

in the distributive sense, we must suppose certain smoothness of the coefficients, namely $a_i\in C^0$. Then $G_i\in C^1$ and the multiplications G_if are well-defined. They are given by

$$G_if=G_if_0+\sum_j\left(c_{0j}G_i(x_j)-c_{1j}\frac{\mathrm{d}G_i}{\mathrm{d}x}(x_j)+c_{1j}G_i(x_j)\frac{\mathrm{d}}{\mathrm{d}x}\right)\delta(x_j).$$

Denote $J(x_j,g_1)$ and $D(x_j,g_1)$ the magnitude of the jump and the Dirac distribution of g_1 in the point x_j respectively, and similarly for other functions. Then

$$\begin{aligned}&J(x_j,g_1)=c_{0j}G_2(x_j)+c_{1j}G_1(x_j), && D(x_j,g_1)=c_{1j}G_2(x_j),\\ &J(x_j,g_2)=-c_{0j}G_1(x_j)+c_{1j}G_3(x_j), && D(x_j,g_2)=-c_{1j}G_1(x_j),\\ &J(x_j,z_1)=c_{1j}, && D(x_j,z_1)=0,\\ &J(x_j,z_2)=c_{0j}, && D(x_j,z_2)=c_{1j}\end{aligned}$$

clearly result from the definition of the distributive solution. Let us notice, that u from (13) is continuously differentiable (it is necessary for $u\in H^2$), in spite of g_i, z_i contain discontinuities and Dirac distributions.

Now we can bring the regularization to an end. For this purpose, it is important that the coefficients of BVP do not explicitly occur in the expressions for $J(x_j,g_i)$, $J(x_j,z_i)$. Thus the analog treatment may be extended onto the general coefficients from L_∞. Note that in this case the factorized equation need not be fulfilled in the distributive sense. There is an analogous situation for $k=3$ and probably also for higher k, but such BVP's do not occur in practice (we must require $a_{ij}\in C^{k-2}$ for the regularization).

We could also combine the above given approaches (via the representation (5) or via the generalized solution). Number of the feasibilities increases with increasing k.

5. Illustrative example

Consider the following simple problem (cf. Fig. 1)

$$\left.\begin{aligned}&\frac{\mathrm{d}^4u}{\mathrm{d}x^4}=f,\qquad u(0)=u(1)=\frac{\mathrm{d}u}{\mathrm{d}x}(0)=\frac{\mathrm{d}u}{\mathrm{d}x}(1)=0,\\ &\text{with } f=\delta(0.6)+\frac{\mathrm{d}}{\mathrm{d}x}\delta(0.4).\end{aligned}\right\}\tag{14}$$

First the problem (14) is solved via equations (11), (12), (13) with $a_2=1$, $a_1=a_0=0$ and $A_0=B_0=A_1=B_1=0$. The solution of the factorized equation (see Fig. 2) is continuous. Secondly, the same problem is

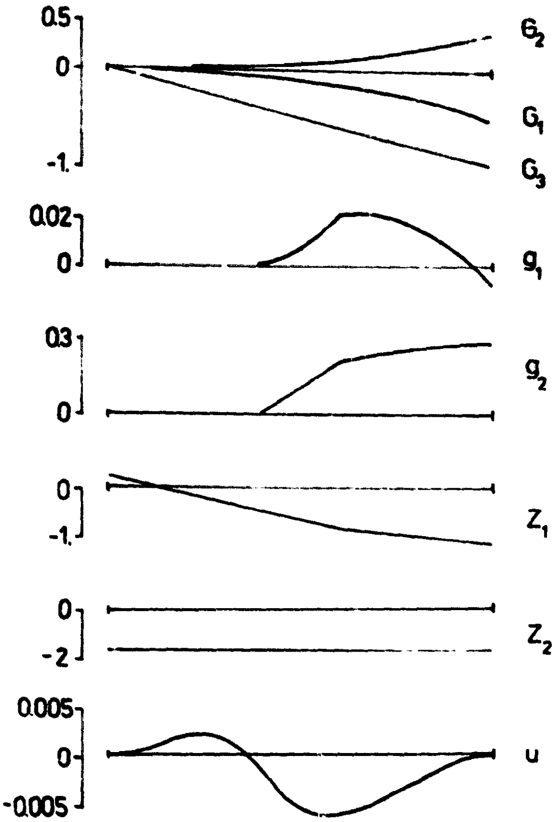

Fig. 2. Solution of the problem from Fig. 1 (deflection of beam) via equations (11), (12), (13).

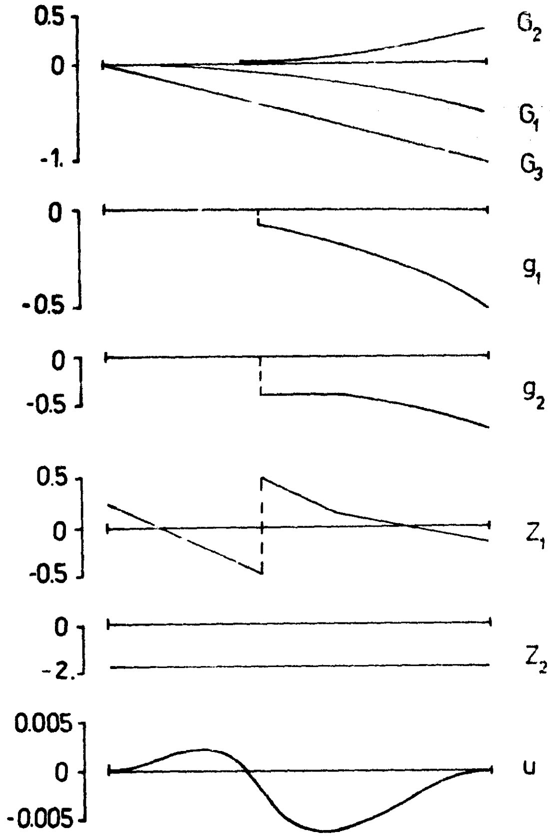

Fig. 3. Solution of the problem from Fig. 1 (deflection of beam) via equations (11), (12′), (13).

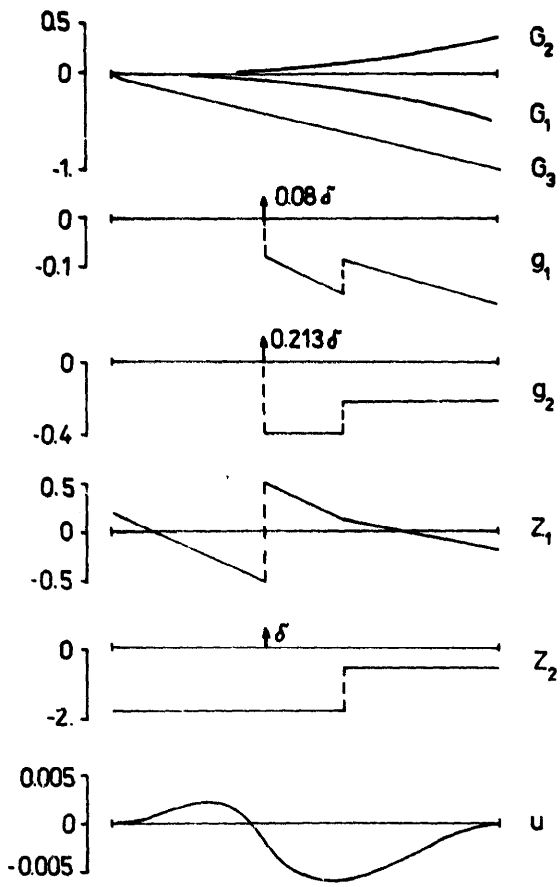

Fig. 4. Solution of the problem from Fig. 1 (deflection of beam) via equations (11), (12″), (13).

solved via equations (11), (12′), (13). Now the solution of the factorized equation (see Fig. 3) has discontinuities. In this two cases the representation (5) with

$$f_0 = 0, \qquad f_1 = Y(x - 0.6), \qquad f_2 = Y(x - 0.4),$$

where $Y(x)$ is the unit jump (Heaviside function), has been used. At the end, we solve the problem (14) via equations (11), (12″), (13). The generalized solution of these equations (see Fig. 4) has discontinuities and Dirac distributions (demonstrated schematically by darts, in fact they are not essential from the analog point of view).

Of course, the (weak) solution of the problem (14) is always the same, and also G_t are the same.

The example has been solved by a digital simulation.

6. Concluding remarks

6.1. Existence and uniqueness

If any solution of the factorized equation exists, as it has been supposed in the foregoing sections, then a weak solution of the corresponding BVP is obtained. Because of the Riccati equation (6), the solution of the factorized equation need not exist. Unfortunately, it does not imply the non-existence of the solution of BVP.

One problem may still occur. If the matrix $G(b)$ is singular, then the initial condition in (8) can be found only for such f, A_i, B_i that $g(b) - u_1(b) \in \operatorname{Im} G(b)$. Then we may choose the initial condition in a non-unique manner thus obtaining infinitely many weak solutions of BVP. If such situation appears, then the linear operator from H^k into $H^{-k} \times R^{2k}$ (R is the set of reals), defined by

$$u \mapsto \left(f, u(a), \ldots, \frac{\mathrm{d}^{k-1}u}{\mathrm{d}x^{k-1}}(a), u(b), \ldots, \frac{\mathrm{d}^{k-1}u}{\mathrm{d}x^{k-1}}(b) \right)$$

where f is uniquely determined by the integral identity (4), has no inverse.

6.2. Some generalizations

The preceeding considerations may be easily utilized in the case of TPBVP's for systems of linear differential equations in the divergent form.

Some corrections must be made, provided general boundary conditions (non-Dirichlet) are considered. Then the weak formulation of BVP is quite different (see e.g. Lions and Magenes [1, Chap. 2, §9]). Furthermore, the initial conditions in the factorized equation are different and also the factorized equation may be different (before the factorization we make a permutation of components between the vectors u_1, u_2).

6.3. Parabolic PDE's

A recent hybrid method for PDE's involving one space dimension and time is a semidiscretization of the type continuous-space-discrete-time (called CSDT method), see Vichnevetsky [4]. A 'natural' space of right-hand sides f is $L_2(0, T; H^{-k})$ when the linear initial-boundary-value problem

$$\frac{\partial u}{\partial t}+\sum_{i,j=0}^{k}\frac{\partial^i}{\partial x^i}\left(a_{ij}\frac{\partial^j u}{\partial x^j}\right)=f, \quad t\in[0, T], u|_{t=0}=u_0\in L_2$$

with the Dirichlet boundary conditions is considered. By the CSDT method and the above hybrid methodology for BVP's formulated on the spaces H^{-k}, we can obtain an approximate solution for the right-hand sides belonging to $L_2(0, T; H^{-k})$. It shows that the methodology for the hybrid treatment of weakly formulated BVP's and the CSDT method are well compatible.

Appendix. Spaces of functions and distributions

C^k = space of k-times continuously differentiable functions on $[a, b]$.
L_∞ = space of (classes of) measurable essentially bounded functions on $[a, b]$.
L_2 = space of (classes of) squared Lebesgue integrable functions on $[a, b]$.
H^k = $\{u;\ \mathrm{d}^i u/\mathrm{d}x^i$ is absolutely continuous on $[a, b]$ for $i=0,\dots,k-1$ and $\mathrm{d}^k u/\mathrm{d}x^k\in L_2\}$.
H_0^k = $\{u;\ u\in H^k$ and $\mathrm{d}^i u/\mathrm{d}x^i(a)=\mathrm{d}^i u/\mathrm{d}x^i(b)=0$ for $i=0,\dots,k-1\}$.
H^{-k} = $(H_0^k)^*$ = space of distributions, which can be obtained as a dual to H_0^k when identifying the Hilbert space L_2 with its dual (hence we have $H_0^k\subset L_2\subset H^{-k}$ with the continuous and dense embeddings).
$L_2(0,T;H^{-k})$ = space of squared integrable (in a Bochner sense) abstract functions on $[0, T]$ with values in H^{-k}.

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References

[1] J.L. Lions and E. Magenes, *Problèmes aux Limites Non Homogènes et Applications, Vol. 1* (Dunod, Paris, 1968); translation into English (Springer, Berlin, 1972).

[2] I.H. Mufti, C.K. Chow and F.T. Stock, Solution of Ill-Conditioned linear TPBVP by the Riccati transformation, *SIAM Rev. 11* (4) (1969) 616–619.

[3] J. Taufer, On factorization method, *Apl. Mat. 11* (6) (1966) 427–451.

[4] R. Vichnevetsky, State of the art in hybrid methods for partial-differential equations, *Proc. AICA-IFIP International Conference on Hybrid Computation*, Munich August 31–September 4 (1970) 33–43.

[5] R. Vichnevetsky, The method of decomposition for unstable two-point boundary value problems, *IEEE Trans. Comput.* (August, 1971) 910–914.