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GENERALIZED SOLUTIONS IN OPTIMIZATION

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Summary. An extension of optimization problems on non-compact domains is constructed by means of methods of the uniform and proximity space theory, the topology and even the uniformity of the original domains being preserved. The solution of the extended problems is considered as a generalized solution of the original problem. The existence and stability (which generally do not take place for the classical solutions) are ensured for the generalized solutions of various abstract optimization problems. Moreover, the convergence (in the sense of the extended spaces) of approximate classical solutions of perturbed or penalized problems to the generalized solutions is shown.

1. Motivations

Let us consider a very simple minimization problem:

(P) minimize $f(x)$
subject to $x\in X$,

where $f:X\to\bar{R}$, X is a topological space, $\bar{R}=R\cup\{-\infty,+\infty\}$ is the usual two-point compactification of the real line R. We denote $\inf P=\inf f(X)$, $\mathrm{Arginf}\, P=\{x\in X,\ f(x)=\inf P\}$. It may happen that (P) has no solutions, i.e. $\mathrm{Arginf}\, P=\emptyset$. It is, however, only theoretical drawback because there is always a minimizing sequence containing elements which "solve" (P) with an arbitrary small tolerance. Such "solutions" are satisfactory from a "practical" point of view if f is e.g. a cost function. Another theoretical drawback, closely related with the previous one, consists in the fact that the set Arginf P may not be stable in the sense that, roughly speaking, small perturbations of f may considerably enlarge the set Arginf P. Such behaviour only indicates that there exists a minimizing sequence containing elements from $X\setminus B$, B being a neighbourhood of Arginf P. It again does not represent any "practical" difficulty.

For problems with certain special properties we avoid the above outlined theoretical drawbacks by means of extension of the domain X and of the problem (P). The extended problem thus obtained will be called generalized, and its solutions will be considered as generalized solutions of (P). We suppose X to be a uniform space – in practice, X will be mostly a metric space. If the stability of the generalized solutions is required, there is an "optimal" extension of X, for which even only a proximity structure of X is essential. Unfortunately, except very special cases, the extended spaces will not be metrizable, hence the notion of sequences is not a sufficiently powerful tool here, and the generalized solutions will be characterized in terms of minimizing filters on X.

This paper represents a brief survey of some author's results[4,5,6], the proofs being omitted. For another approaches to the topics we refer e.g. to the works of Polak, Wardi[3], and Warga[7].

2. Topological preliminaries

We briefly recall some definitions and assertions from general topology[1,2] needed in what follows. A filter on a set M is a nonempty collection of nonempty subsets of M with the properties $R,S\in\mathcal{F}\Rightarrow R\cap S\in\mathcal{F}$, and $R\in\mathcal{F}$, $S\supset R\Rightarrow S\in\mathcal{F}$. A uniformity $\mathcal{U}_X$ on X is a filter on $X\times X$ with the following properties:

a) $\forall U\in\mathcal{U}_X\ \forall x\in X$: $(x,x)\in U$,
b) $V^{-1}\in\mathcal{U}_X$ whenever $V\in\mathcal{U}_X$,
c) $\forall U\in\mathcal{U}_X\ \exists V\in\mathcal{U}_X$: $V\circ V\subset U$,

where $V^{-1}=\{(x,y)\in X\times X;\ (y,x)\in V\}$ and $V\circ V=\{(x,y)\in X\times X;\ \exists z\in X,\ (x,z)\in V,\ (z,y)\in V\}$. In what follows we confine ourselves to separated uniformities (also called Hausdorff), i.e. $x\neq y\Rightarrow\exists U\in\mathcal{U}_X$: $(x,y)\notin U$. A filter $\mathcal{F}$ on X is called $\mathcal{U}_X$-Cauchy if $\forall U\in\mathcal{U}_X\ \exists R\in\mathcal{F}$: $R\times R\subset U$. If every $\mathcal{U}_X$-Cauchy filter on X converges to some element of X, the uniform space $(X,\mathcal{U}_X)$ is called complete. For a uniform space $(X,\mathcal{U}_X)$ we define its completion $(\bar{X},\bar{\mathcal{U}}_X)$ as a complete uniform space such that X is dense in $\bar{X}$ and the trace of $\bar{\mathcal{U}}_X$ on $X\times X$ is just $\mathcal{U}_X$. There is a one-to-one correspondence between the points of $\bar{X}$ and the minimal (with respect to the ordering by inclusion) $\mathcal{U}_X$-Cauchy filters on X. The uniformity $\mathcal{U}_X$ is called precompact if $\forall V\in\mathcal{U}_X$ $\exists$ a finite set $S\subset X$: $U(S)=X$, where $U(S)=\{x\in X;\ \exists y\in S,\ (x,y)\in U\}$. The completion $(\bar{X},\bar{\mathcal{U}}_X)$ is compact iff $\mathcal{U}_X$ is precompact.

A proximity Π_X on X is a binary relation on the collection of all subsets of X with the following properties:

a) $(S_1,S_2)\in\pi_X \Rightarrow (S_2,S_1)\in\pi_X$,
b) $S_1\cap S_2\neq\emptyset \Rightarrow (S_1,S_2)\in\pi_X$,
c) $(S_1,S_2)\in\pi_X$, $S_1\subset R_1$, $S_2\subset R_2 \Rightarrow (R_1,R_2)\in\pi_X$,
d) $(\emptyset,X)\notin\pi_X$,
e) $(S_1,R)\notin\pi_X$, $(S_2,R)\notin\pi_X \Rightarrow (S_1\cup S_2,R)\notin\pi_X$
f) $(S_1,S_2)\notin\pi_X \Rightarrow \exists R_1,R_2$: $R_1\cap R_2=\emptyset$, $(S_1,X\setminus R_1)\notin\pi_X$, $(S_2,X\setminus R_2)\notin\pi_X$.

If $(S_1,S_2)\in\pi_X$, the sets S_1, S_2 are called near to each other (with respect to π_X), while in the opposite case they are called far to each other. A uniformity $\mathcal{U}_X$ induces on X a proximity $\pi(\mathcal{U}_X)=\{(R,S);\ S,R\subset X,\ \forall U\in\mathcal{U}_X\colon (S\times R)\cap U\neq\emptyset\}$. Different uniformities may induce the same proximity. Besides, for each proximity π_X there exists exactly one precompact uniformity, denoted by $\mathcal{U}(\pi_X)$, inducing π_X. The completion $(\overline{X},\overline{\mathcal{U}(\pi_X)})$ is called a <u>Smirnov compactification</u> of the proximity space (X,π_X). A filter $\mathcal{F}$ on X is called π_X-round if $\forall R\in\mathcal{F}\ \exists S\in\mathcal{F}$: $(X\setminus R,S)\notin\pi_X$. There is a one-to-one correspondence between the points of X and the maximal (with respect to the ordering by inclusion) π_X-round filters on X. For any uniformity $\mathcal{U}_X$ we define its <u>precompact modification</u> $\mathcal{U}_X^*$ by $\mathcal{U}_X^*=\mathcal{U}(\pi(\mathcal{U}_X))$. The completion $(\overline{X},\overline{\mathcal{U}}_X^*)$ is called a <u>Samuel compactification</u> of the uniform space $(X,\mathcal{U}_X)$, and it is nothing else than the Smirnov compactifiaction of the proximity space $(X,\pi(\mathcal{U}_X))$.

Any proximity π_X induces a <u>topology</u> on X by declaring $\{x\in X;\ (\{x\},S)\in\pi_X\}$ as the closure of S for any $S\subset X$. The topology induced by $\mathcal{U}_X$ is defined as the topology induced by $\pi(\mathcal{U}_X)$.

<u>An example</u>: Let (X,d) be a metric space. Its metric d induces the uniformity $\mathcal{U}_X=\{U\subset X\times X;\ \exists\varepsilon>0\colon\ d(x,y)\leq\varepsilon \Rightarrow (x,y)\in U\}$. Then $\pi(\mathcal{U}_X)=\{(R,S);\ R,S\subseteq X,\ \forall\varepsilon>0\ \exists x\in R\ y\in S\colon d(x,y)\leq\varepsilon\}$. Furthermore, $\mathcal{U}_X^*$ is projectively generated by the family of all $(\mathcal{U}_X,\mathcal{W})$-uniformly continuous functions from X to $[0,1]$, where denotes the unique uniformity on $[0,1]$ inducing the usual topology of $[0,1]$; i.e. $\varphi\colon X\to[0,1]$ is $(\mathcal{U}_X,\mathcal{W})$-uniformly continuous iff $\forall\varepsilon>0\ \exists\delta>0\ \forall x,y\in X$: $d(x,y)\leq\delta \Rightarrow |\varphi(x)-\varphi(y)|\leq\varepsilon$, which is obviously of the usual meaning. The Smirnov compactification of $(X,\pi(\mathcal{U}_X))$ is metrizable if and only if $\mathcal{U}_X$ is precompact.

<u>3. Generalized solutions in unconstrained optimization</u>

We consider the minimization problem (P) where $(X,\mathcal{U}_X)$ is a uniform space and the function $f\colon X\to\overline{R}$ is lower semicontinuous (briefly l.s.c.) with respect to the topology induced by $\mathcal{U}_X$. In practice, $\mathcal{U}_X$ will by induced by a metric as in the above example. Let $\mathcal{U}_X^+$ be some uniformity on X inducing the same topology as $\mathcal{U}_X$. We extend (P) to the completion $(\overline{X},\overline{\mathcal{U}}_X^+)$, defining $\overline{f}\colon\overline{X}\to\overline{R}$ by

$$\overline{f}(y)= = \liminf_{x\to y,x\in X} f(x) = \sup_{S\in\mathcal{N}(y)} \inf_{x\in S} f(x)$$

where $\mathcal{N}(y)=\{R\cap X;\ R$ is a neighbourhood of y in $\overline{X}\}$. Obviously $\mathcal{N}(y)$ is a minimial $\mathcal{U}_X^+$-Cauchy filter on X. Since f is l.s.c. and $\mathcal{U}_X$ is supposed to be separated, $\overline{f}(x)=f(x)$ for any $x\in X$, which justifies the term extension. We define the generalized problem:

(GP) minimize $\overline{f}(x)$
subject to $x\in\overline{X}$.

Clearly, $\inf\overline{f}(\overline{X})=\inf f(X)=\inf P$. The elements of Arginf GP$=\{x\in\overline{X};\ \overline{f}(x)=\inf P\}$ are to be considered as generalized solutions of (P). Of course, Arginf GP depends on $\mathcal{U}_X^+$.

We will require the set-valued mapping $f\mapsto$Arginf GP to be upper semicontinuous (u.s.c) with respect to the topology on the set $\{f\colon X\to\overline{R}\}$ defined as follows: We say that $f_n\to f$ iff epi $f_n\to$epi f (where epi $f=\{(x,a)\in X\times\overline{R};\ f(x)\leq a\}$) in the topology on the hyperspace of all subsets of $X\times\overline{R}$ induced by the so-called Hausdorff uniformity $\mathcal{H}(\mathcal{U}_X\times\mathcal{W})$, where $\mathcal{W}$ is the uniformity of $\overline{R}$. For a uniformity $\mathcal{V}$ on a set, the Hausdorff uniformity $\mathcal{H}(\mathcal{V})$ on its subsets is generated by the base $\{V^H;\ V\in\mathcal{V}\}$, where $V^H=\{(R,S);\ R\subset V(S),\ S\subset V(R)\}$. As for the topology on $\overline{X}$, we take naturally that induced by $\overline{\mathcal{U}}_X^+$. We come to a fundamental result:

<u>Theorem 3.1</u> The set-valued mapping $f\mapsto$ Arginf GP is u.s.c. (in the topologies mentioned above) if and only if $\mathcal{U}_X^+$ is coarser than the precompact modification of $\mathcal{U}_X$, i.e. $\mathcal{U}_X^+\subset\mathcal{U}_X^*$.

Thus $\mathcal{U}_X^*$ may be considered as an "optimal" uniformity which yields stable generalized solutions. Moreover, $\mathcal{U}_X^*$ induces not only the same topology as $\mathcal{U}_X$, but also the same proximity as $\mathcal{U}_X$.

In what follows, we confine ourselves to the case when $\mathcal{U}_X^+=\mathcal{U}_X^*$. Then X is the Samuel compactification of $(X,\mathcal{U}_X)$ for which, as explained in §2, only the proximity structure of $(X,\mathcal{U}_X)$ is essential.

The set Arginf GP can be characterized among all closed subsets of X by means of the filter $\mathcal{N}(\text{Arginf GP})=\bigcap\{\mathcal{N}(x);\ x\in\text{Arginf GP}\}$ on X. The following theorem shows a connection with the classical notion of level sets of f, defined by $\text{lev}(a)=\{x\in X;\ f(x)\leq a\}$, $a\in\overline{R}$.

<u>Theorem 3.2</u> $\mathcal{N}(\text{Arginf GP})=\{U(\text{lev}(a));\ U\in\mathcal{U}_X,\ a>\inf P\}$.

Note that we have got the effective characterization of Arginf GP using the original uniformity $\mathcal{U}_X$, whereas the particular elements of Arginf GP can be desribed only either by the uniformity $\mathcal{U}_X^*$ (as f-minimizing minimal $\mathcal{U}_X^*$-Cauchy filters on X) or by the axiom of choice (as f-minimizing maximal $\pi(\mathcal{U}_X)$-round filters on X). The filter $\mathcal{F}$ on X is called f-minimizing if $\inf f(S)=\inf f(X)$ for any $S\in\mathcal{F}$.

<u>An example</u>: Let $X=\ell_2$ (i.e. the Hilbert space of all square summable squences) and $\mathcal{U}_X$ induced by its norm. Consider f defined by $f(x)=\sum_{i=1}^{+\infty}2^{-i}x_i^2$ for $\|x\|\leq1$ and $f(x)=+\infty$ elsewhere (x_i denotes the i-th component of the sequence $x\in\ell_2$). Obviously, Arginf P$=\{0\}$ and f is even strictly convex and coercive, but

there is a minimizing sequence $\{x^j\}$ with $\|x^j\|\geq 1$, namely $(x^j)_i=\delta_{ij}$ (δ is the Kronecker symbol). Since $\bar{X}$ is compact, $\{x^j\}$ has a cluster point in $\bar{X}$ which must belong to Arginf GP and cannot be zero. Hence we see that Arginf GP may be actually larger than Arginf P.

4. Constrained optimization problems

Let us consider a functionally constrained minimization problem:

$$(P^c) \qquad \text{minimize } f(x) \quad \text{subject to } x\in X,\ F(x)\in D,$$

where $F:X\to Y$, $D\subset Y$, and $(X,\mathcal{U}_X)$, $(Y,\mathcal{U}_Y)$ are uniform spaces. Denote $\inf P^c = \inf f(\{x\in X;\ F(x)\in D\})$ and $\text{Arginf } P^c = \{x\in X;\ F(x)\in D,\ f(x)=\inf P^c\}$. We suppose, as in §3, that $f:X\to\mathbb{R}$ is l.s.c., which again allows us to extend f to $\bar{X}$. Now we meet the problem how to extend F to $\bar{X}$. Accordingly to the agreemant from §3, we consider only the "optimal" extensions, i.e. the Samuel compactifications $(\bar{X},\bar{\mathcal{U}}_X^*)$ and $(\bar{Y},\bar{\mathcal{U}}_Y^*)$. As the $(\mathcal{U}_X,\mathcal{U}_Y)$-uniform continuity of F guarantees that F is $(\mathcal{U}_X^*,\mathcal{U}_Y^*)$-uniformly continuous as well, there is a unique continuous mapping $\bar{F}:\bar{X}\to\bar{Y}$ such that $\bar{F}(x)=F(x)$ for any $x\in X$. We obtain immediately the generalized problem:

$$(GP^c) \qquad \text{minimize } \bar{f}(x) \quad \text{subject to } x\in\bar{X},\ \bar{F}(x)\in\bar{D},$$

where $\bar{D}$ is the closure of D in $\bar{Y}$. We denote $\inf GP^c = \inf \bar{f}(\{x\in\bar{X};\ \bar{F}(x)\in\bar{D}\})$ and $\text{Arginf } GP^c = \{x\in\bar{X};\ \bar{F}(x)\in\bar{D},\ \bar{f}(x)=\inf GP^c\}$. Like in Thm.3.2, we can characterize the set of the generalized solutions Arginf GP^c by means of the level sets of the original problem (P^c), which are defined now as $\text{lev}(a,V)=\{x\in X;\ f(x)\leq a,\ F(x)\in V(D)\}$, where $a\in\bar{\mathbb{R}}$ and $V\in\mathcal{U}_Y$. Note that we have relaxed also the constraint. There is, however, a strong difference between the problem investigated here and that in §3, because here $\inf GP^c$ may be less than $\inf P^c$ (then the generalized solutions are "better" than the classical ones and $\text{Arginf } GP^c\cap X=\emptyset$). Here we have to characterize also $\inf GP^c$.

Theorem 4.1 The collection $\{U(\text{lev}(a,V));\ U\in\mathcal{U}_X,\ V\in\mathcal{U}_Y,\ a>\alpha\}$ is a filter on X if and only if $\alpha\geq\inf GP^c$. Besides, if $\alpha=\inf GP^c$, then this collection is just the filter $\mathcal{N}(\text{Arginf } GP^c)$.

For $\varepsilon>0$ and $V\in\mathcal{U}_Y$, we investigate also the sets $\text{Arginf}_\varepsilon P_V^c=\{x\in X;\ f(x)\leq\inf P_V^c+\varepsilon,\ F(x)\in V(D)\}$, where $\inf P_V^c=\inf f(\{x\in X;\ F(x)\in V(D)\})$. The elements of $\text{Arginf}_\varepsilon P_V^c$ are obviously "almost" optimal (with a tolerance prescribed by ε) "solutions" of the original problem, the constraints being violated only a little (with atolerance prescribed by V). Such solutions are of the same "technical" or "engineering" applicability as the classical solutions, i.e. the elements of Arginf P^c. However, $\text{Arginf}_\varepsilon P_V^c$ generally does not converge to Arginf P^c when $\varepsilon\to 0$ and V ranges the filter $\mathcal{U}_Y$. Thus the following theorem indicates that the generalized problem (GP^c) may be considered as a realistic setting of the classical problem (P^c).

Theorem 4.2 If V ranges the filter $\mathcal{U}_Y$, then $\inf P_V^c\nearrow\inf GP^c$ (the convergence is monotone). Besides, if $\varepsilon>\inf GP^c-\inf P_V^c$ and $\varepsilon\to 0$, then $\text{Arginf}_\varepsilon P_V^c\to\text{Arginf } GP^c$ (with respect to the topology induced by the Hausdorff uniformity $\mathcal{H}(\bar{\mathcal{U}}_X^*)$ on the hyper-space of all subsets of $\bar{X}$).

Similar convergence results can be proved for approximate solutions obtained by using the well-known exterior penalty funtion method to the problem (P^c) provided f is bounded from below and the penalty function $\varphi:Y\to\mathbb{R}$ is uniformly continuous, $\varphi(y)=0$ whenever $y\in D$, and $\forall V\in\mathcal{U}_Y\ \exists\delta>0:\ \varphi(Y\setminus V(D))\geq\delta$. Of course, the penalized problem then consits in minimization of the function $f+K.\varphi\circ F$ over X, $K>0$. For some special cases it can be also shown that $\inf GP^c$ is nothing else than the supremum of the dual problem when the classical augmented Lagrangian method is employed.

An example: We construct an example for $\inf GP^c<\inf P^c$, considering, like in §3, $X=\ell_2$, $Y=\bar{\mathbb{R}}$, $D=\{a\geq 0\}$, $f(x)=\sum_{i=1}^{+\infty}-2^{-i}x_i$ and $F(x)=\sum_{i=1}^{+\infty}2^{-5i}x_i^2$. Obviously, $\inf P^c=0$ and $\text{Arginf } P^c=\{0\}$. Taking the sequence $\{x^j\}$ defined by $(x^j)_i=2^{2i}\delta_{ij}$ (δ is again the Kronecker symbol), we have $f(x^j)=-2^j$ and $F(x^j)=2^{-j}$. Since $\bar{X}$ is compact, $\{x^j\}$ has a cluster point $x\in\bar{X}$ with $\bar{f}(x)=-\infty$ and $\bar{F}(x)=0$. Thus $\inf GP^c=-\infty$.

5. Optimal control problems

Finally we consider a problem with more complicated structure, namely an abstract optimal-control problem:

$$(P^{oc}) \qquad \text{minimize } j(u,x) \quad \text{subject to } u\in U,\ x=A(u),\ f(u,x)\in D,$$

where $j:U\times X\to\mathbb{R}$ is a cost function, $A:U\to X$ is a state operator, and $f:U\times X\to Y$ is a mapping which forms, together with a set $D\subset Y$, a constraint imposed on the control u and the state x. The set of admissible controls U, as well as the sets X and Y, are supposed as uniform spaces endowed with uniformities $\mathcal{U}_U$, $\mathcal{U}_X$, $\mathcal{U}_Y$, respectively. We denote by $(\bar{U},\bar{\mathcal{U}}_U^*)$, $(\bar{X},\bar{\mathcal{U}}_X^*)$, $(\bar{Y},\bar{\mathcal{U}}_Y^*)$ the corresponding Samuel compactifications. We may transform (P^{oc}) to a mathematical-programming problem over U:

$$(P^{oct}) \qquad \text{minimize } J(u)=j(u,A(u)) \quad \text{subject to } u\in U,\ F(u)=f(u,A(u))\in D.$$

Note that (P^{oct}) has the form of (P^c) from §4, and thus we may use the previous results provided $J:U\to\bar{\mathbb{R}}$ is l.s.c. and $F:U\to Y$ is $(\mathcal{U}_U,\mathcal{U}_Y)$-uniformly continuous. Extending J to $\bar{J}:\bar{U}\to\bar{\mathbb{R}}$ and F to $\bar{F}:\bar{U}\to\bar{Y}$ like in §4, we get the generalized problem:

$$(GP^{oct}) \qquad \text{minimize } \bar{J}(u) \quad \text{subject to } u\in\bar{U},\ \bar{F}(u)\in\bar{D},$$

$\bar{D}$ is again the closure of D in $\bar{Y}$. We define the set of the generalized optimal controls Arginf $GP^{oct} = \{u\in\bar U;\ \bar F(u)\in\bar D,\ \bar J(u)=\inf GP^{oct}\}$, where $\inf GP^{oct} = \inf \bar J(\{u\in\bar U;\ \bar F(u)\in\bar D\})$. As in §4, we may see that the sets of approximate solutions of a perturbed original problem, i.e. $\mathrm{Arginf}_\varepsilon P_V^{oc} = \{u\in U;\ J(u)\leq\inf P_V^{oc}+\varepsilon,\ F(u)\in V(D)\}$ with $\inf P_V^{oc} = \inf J(\{u\in U;\ F(u)\in V(D)\})$, $V\in\mathcal{U}_Y$, $\varepsilon>0$, converge just to Arginf GP^{oct} with respect to the topology induced by $\mathcal{X}(\bar{\mathcal{U}}_U^*)$. Note that $\mathrm{Arginf}_\varepsilon P_V^{oc}$ contains elements which are "almost" optimal (with a tolerance ε), the constraint being fulfilled only with a certain accuracy (prescribed by V), while the state equation is satisfied exactly. This situation is in harmony with the very realistic approach of J.Warga[7] who distinguishes the state operator, which is supposed as governed by "absolute" physical laws (thus the state equation is to be satisfied exactly), and the constraints given by "engineering" requirements (which fulfilment is sufficient only with a certain acccuracy).

On the other hand, when A is $(\mathcal{U}_U,\mathcal{U}_X)$-uniformly continuous, F is $(\mathcal{U}_U^*\times\mathcal{U}_X^*,\mathcal{U}_Y^*)$-uniformly continuous, and f is l.s.c., we may extend directly the problem (P^{oc}) without any mathematical-programming transformation. By a straightforward way, we define $\bar A:\bar U\to\bar X$ and $\bar f:\bar U\times\bar X\to\bar Y$ (as continuous mappings) and $\bar j:\bar U\times\bar X\to\bar R$ (as a l.s.c. function). Thus we get the generalized problem having the form of an optimal-control problem like the original problem (P^{oc}):

$$(GP^{oc}) \qquad \text{minimize } \bar j(u,x) \quad \text{subject to } u\in\bar U,\ x=\bar A(u),\ \bar f(u,x)\in\bar D.$$

It should be remarked that $\bar U\times\bar X$ generally differs from $\overline{U\times X}$ which is the Samuel compactification of the uniform space $(U\times X,\mathcal{U}_U\times\mathcal{U}_X)$. The set of the generalized optimal controls is now defined naturally as Arginf $GP^{oc} = \{u\in\bar U;\ \bar f(u,\bar A(u))\in\bar D,\ \bar j(u,\bar A(u)) = \inf GP^{oc}\}$, where $\inf GP^{oc} = \inf \bar j(\{(u,x)\in\bar U\times\bar X;\ x=\bar A(u),\ \bar f(u,x)\in\bar D\})$.

Theorem 5.1 $\inf GP^{oc}\leq\inf GP^{oct}$. If j is, in addition, $(\mathcal{U}_U^*\times\mathcal{U}_X^*,\mathcal{W})$-uniformly continuous, then $\inf GP^{oc} = \inf GP^{oct}$ and thus also Arginf GP^{oc}= Arginf GP^{oct}.

The problem (GP^{oc}) corresponds to approximate problems when, in addition, perturbations of the state operator are allowed. More precisely, we are to deal with the sets $\mathrm{Arginf}_\varepsilon P_{VW}^{oc} = \{u\in U;\ \exists\, x\in W(A(u)):\ f(u,x)\in V(D),\ j(u,x)\leq\inf P_{VW}^{oc}+\varepsilon\}$ with $V\in\mathcal{U}_Y$, $W\in\mathcal{U}_X$, $\varepsilon>0$, and $\inf P_{VW}^{oc} = \inf j(\{(u,x)\in U\times X;\ \exists x\in W(A(u)),\ f(u,x)\in V(D)\})$.

Theorem 5.2 Let V and W range the filters $\mathcal{U}_Y$ and $\mathcal{U}_X$, respectively. Then $\inf P_{VW}^{oc}\nearrow\inf GP^{oc}$. Moreover, let $\varepsilon>\inf GP^{oc}-\inf P_{VW}^{oc}$ and $\varepsilon\to 0$. Then $\mathrm{Arginf}_\varepsilon P_{VW}^{oc}\to$ Arginf GP^{oc} with respect to the topology induced by $\mathcal{X}(\bar{\mathcal{U}}_U^*)$.

Analogous convergence results can be stated for the sets of approximate solutions obtained by the classical exterior penalty function method possibly with an approximation of the state operator, which is a typical situation in solving optimal control problems numerically.

If the state operator is compact in the sense that it maps bounded sets into precompact sets, we can even prove, under certain conditions, that the optimal generalized states belong not only to the extended space $\bar X$, but also to the original space X. This behaviour is very similar to that in the theory of J.Warga[7], where the so-called relaxed controls belongs to an extended space, while the corresponding states, i.e. solutions of a differential equation, are functions in the usual sense.

An example: We consider $U=X=[-1,1]\setminus\{0\}$ and $\mathcal{U}_U=\mathcal{U}_X$ as the restriction of the usual additive unifomity on the real line. Hence $\mathcal{U}_U$ and $\mathcal{U}_X$ are precompact and we have rather trivial situation $\bar U=\bar X=[-1,1]$. Taking $D=Y$ (i.e. no constraints), $A(u)=u$, $j(u,x)=u/x$, we observe that $\bar J(u)=1$ for all $u\in\bar U$, and $\bar j(u,\bar A(u))$ is either 1 (if $u\in U$) or $-\infty$ (if $u\in\bar U\setminus U=\{0\}$). Thus we have constructed a simple example for $\inf GP^{oc}=-\infty<\inf GP^{oct}=1$. Note that J is even continuous, but not uniformly continuous. We see that, although (P^{oc}) and (P^{oct}) are obviously equivalent to each other, (GP^{oc}) and (GP^{oct}) may not be so.

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