P3, 16.3.2021 

$$y'' + y = f(x)\, y(x) \qquad x \in (0,a) \quad a>0$$

$$y(0) = 1,$$

$$y'(0) = 0$$

$$\Updownarrow$$

$$y(x) = \cos x + \int_0^x f(t) \sin(x-t)\, y(t)\, dt$$

$$\boxed{y = u + Ty} \qquad \text{v } \mathcal{C}(\langle 0,a \rangle)$$

$$Ty(x) := \int_0^x f(t) \sin(x-t)\, y(t)\, dt$$

$$(Id - T)\, y = u$$

$$\Downarrow \text{(2)}$$

$$y = (Id - T)^{-1} u$$

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$$T: \mathcal{C}(\langle 0,a \rangle) \longrightarrow \mathcal{C}(\langle 0,a \rangle)$$

$T$ lineární

? $T$ omezený ?

$$\|y\|_{C(\langle 0,a\rangle)} = \sup_{x\in\langle 0,a\rangle} |y(x)| = \|y\|_\infty$$

$$\|Ty\|_\infty = \sup_{x\in\langle 0,a\rangle} \left| \int_0^x f(t)\sin(x-t)\, y(t)\,dt \right|$$

$$\leq \sup_{x\in\langle 0,a\rangle} \int_0^x \underbrace{|f(t)|\cdot 1\cdot |y(t)|}_{\geq 0}\,dt$$

$$\leq \underbrace{a\cdot\|f\|_\infty}\ \underbrace{\|y\|_\infty}$$

$$\|T\| = \sup_{\|y\|_\infty\leq 1} \|Ty\|_\infty \leq \sup_{\|y\|_\infty\leq 1} a\|f\|_\infty \|y\|_\infty$$

$$\leq \underbrace{a\|f\|_\infty}_{} = c < \infty$$

$$\Rightarrow \boxed{T\in\mathcal{L}(C(\langle 0,a\rangle))}$$

$\boxed{\text{Věta}}$ $X$ Banachův, $T\in\mathcal{L}(X)$. Definujme

$$T^0 := Id;\quad T^{j+1}y = T(T^j y)$$

$$(T^3 y = T(T(Ty))$$

Dále nechť je splněna alespoň 1 z násl.

Mějme podmínky:

(a) $\|T\|_{\mathcal{L}(X)} < 1$

(b) $\sum_{j=0}^{\infty} \|T^j\|_{\mathcal{L}(X)} < \infty$

(c) $\sum_{j=0}^{\infty} \|T^j y\|_X < \infty \quad \forall y \in X$.

Potom:

$y = (Id - T)^{-1} u$

(1) $\forall u \in X \ \exists!\, y \in X, \quad \underbrace{(Id - T) y = u}$

(2) Definujeme-li zobrazení "$u \longmapsto y$" a označíme-li ho $(Id - T)^{-1} : u \longmapsto y$

potom

$$(Id - T)^{-1} \circ (Id - T) = Id$$

$$(Id - T) \circ (Id - T)^{-1} = Id$$

(3) $\underbrace{(Id - T)^{-1} = \sum_{j=0}^{\infty} T^j}$ von Neumannova řada $T$

v $\mathcal{L}(X)$

$$\lim_{n \to \infty} \left\| (Id - T)^{-1} - \sum_{j=0}^{n} T^j \right\|_{\mathcal{L}(X)} = 0$$

Pozn.: analogie $|q|<1$ $\quad \sum_{j=0}^{\infty} q^j = \frac{1}{1-q} = (1-q)^{-1}$

$$\|T\|<1 \qquad \sum_{j=0}^{\infty} T^j = (Id - T)^{-1}$$

- Před důkazem ukážeme $(a) \Rightarrow (b) \Rightarrow (c)$

D. $(a) \Rightarrow (b)$: $\quad (a) \quad \|T\|_{\mathcal{L}(X)} < 1$ VÍME

$$(b) \quad \sum_{j=0}^{\infty} \|T^j\|_{\mathcal{L}(X)} < \infty \qquad \text{CHCEME}$$

$$\|T^2 y\|_X = \|T(\underbrace{Ty}_{z})\|_X \le \|T\| \cdot \|Ty\| \le \|T\| \cdot \|T\| \|y\|$$
$$= \|T\|^2 \cdot \|y\|$$

$$\underbrace{\|T^2\|_{\mathcal{L}(X)}}_{} = \sup_{\|y\|\le 1} \|T^2 y\| \le \sup_{\|y\|\le 1} \|T\|^2 \|y\| = \underbrace{\|T\|^2}_{}$$

Indukcí: $\underbrace{\|T^j\| \le \|T\|^j}_{} \quad \forall j$

$$\sum_{0}^{\infty} \|T^j\| \le \sum_{j=0}^{\infty} \|T\|^j < \infty \qquad \|T\| =: q < 1$$

$(b) \Rightarrow (c)$ $\quad \sum \|T^j y\| \le \sum \|T^j\| \cdot \|y\| = \|y\| \underbrace{\sum \|T^j\|}_{<\infty}$

$\vdots < \infty \ \forall y$

Tj. Bude stačit $\underbrace{(c) \Rightarrow (1),(2),(3)}$

Pozn.: „Náš" operátor $Ty(x) = \int_0^x f(t) \dots$

splňuje (a) nebo (b) nebo (c) ?

(a) $\|T\| < 1$ ? VÍME: $\|T\| \le \underbrace{a \|f\|_\infty < 1}$

LOKÁLNÍ EXISTENCE:

$\forall f(x) \in C(\langle 0, a \rangle) \ \exists \tilde{a} \in \langle 0, a \rangle$

$\tilde{a} \|f\|_\infty < 1$

na $\langle 0, \tilde{a} \rangle$ $\|T\| < 1$

$\exists!$ řeš. probl.

Ukážu: T splňuje (b) i (c) globálně

$$|Ty(x)| \le \int_0^x |f(t)|\,|y(t)|\,dt \qquad \forall x \ \forall y$$

$$\le x \|f\|_\infty \|y\|_\infty$$

$$|T^2 y(x)| \le \int_0^x |f(t)| \cdot |Ty(t)|\,dt$$

$$\le \int_0^x |f(t)| \cdot t \cdot \|f\|_\infty \|y\|_\infty$$

$$\le \|f\|^2 \|y\| \int_0^x t = \frac{x^2}{2}\|f\|_\infty^2 \|y\|_\infty$$

$\Rightarrow$

$$|T^j_y(x)| \le \frac{x^j}{j!}\|f\|^j \|y\| \quad \Big/ \sup_{x\in\langle 0,a\rangle}$$

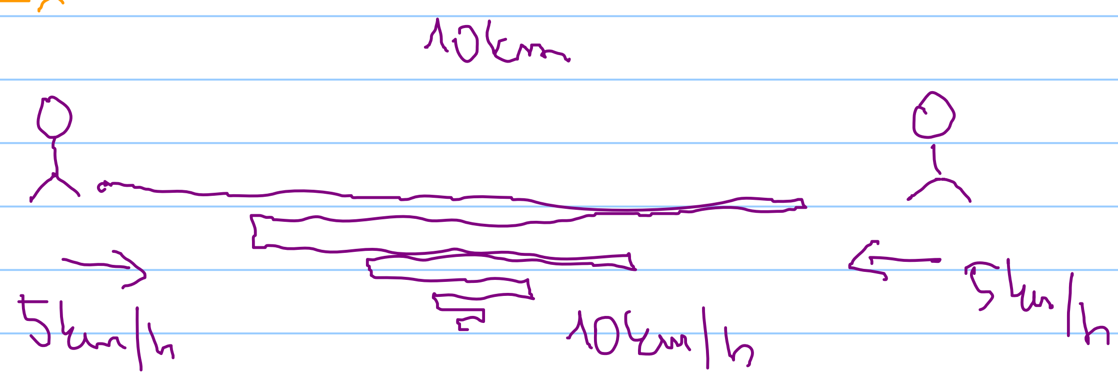
$$\|T^j y\| \le \frac{a^j}{j!}\|f\|^j \|y\|$$

$$\sum_0^\infty \|T^j y\| \le \|y\|_\infty \cdot \underbrace{\sum_0^\infty \frac{(a\|f\|)^j}{j!}}_{\exp(a\|f\|)} < \infty \quad \forall y$$

Podm. (c)

$\Downarrow$ konečná délka

$\Rightarrow \underbrace{\text{GLOB } \exists. \ \forall a>0 \ \exists! \text{ na } \langle 0,a\rangle}$



Důkaz věty:

$\sum_{j=0}^{\infty} \|T^j y\|_X < \infty \quad \forall y \in X$. VÍME

CHCEME $\forall u \in X \ \exists!\, y \in X, \ (Id - T)y = u$

Konstrukce řešení: $y_0 \in X$ lib.

(srov $y = u + Ty$) $\quad y_{n+1} = u + Ty_n$

$$y_1 = u + Ty_0$$

$$y_2 = u + Ty_1 = u + Tu + T^2 y_0$$

$$y_3 = u + Tu + T^2 u + T^3 y_0$$

$$y_n = \sum_{j=0}^{n-1} T^j u + T^n y_0$$

BÚNO $n > m$

$$y_n - y_m = \sum_{j=m}^{n-1} T^j u + T^n y_0 - T^m y_0$$

$$\|y_n - y_m\| \le \underbrace{\sum_{m}^{n-1} \|T^j u\|}_{< \varepsilon} + \underbrace{\|T^n y_0\|}_{< \varepsilon} + \underbrace{\|T^m y_0\|}_{< \varepsilon}$$

$\Rightarrow \{y_n\}$ v $X$ cauchyovská

$\Downarrow$ $X$ úplný

$\exists\, y \in X$ $\quad \underbrace{y_n \to y \text{ v } X}$

$$\underbrace{y_{n+1} = u + T y_n}$$

$$\downarrow \qquad \downarrow \qquad (y_n \to y \Rightarrow T y_n \to T y)$$

$$\underbrace{y = u + T y}$$

$$\underbrace{(Id - T) y = u}$$

jednoznačnost: nechť $\exists\, y \neq z$ $\quad \left.\begin{array}{l} y = u + Ty \\ z = u + Tz \end{array}\right\} -$

$$\underbrace{y - z} = Ty - Tz = T(y - z)$$

$$\underbrace{w = Tw} = T^2 w = T^3 w = \underbrace{T^n} w$$

víme $\sum^{\infty} \|T^i w\| < \infty$

$$\Rightarrow \lim_{n\to\infty} \|T^n w\| = 0$$

$$\|w\| = \underline{\|T^n w\| \longrightarrow 0} \quad \Rightarrow \|w\| = 0$$

$w = 0$

$y - z = 0$

$\underbrace{y = z}$

$\Rightarrow \underbrace{\forall u \in X \ \exists!\, y \in X \quad y = u + Ty}$

$\underbrace{(Id - T)^{-1} := \text{"} u \longmapsto y \text{"} \text{ je O.K.}}$

$$y_n = \sum_{j=0}^{n-1} T^j u + T^n y_0 \quad \Big/ \lim_{n \to \infty}$$

$$\underbrace{y = \sum_{j=0}^{\infty} T^j u}_{\substack{\| \\ (Id - T)^{-1} u}} \qquad 0 \quad \Rightarrow \text{"} u \mapsto y \text{"} \quad \Bigg\} \ \forall u$$

$$\left[ \underbrace{(Id - T)^{-1}} = \sum_{j=0}^{\infty} T^j \right]$$

$$S_N := \sum_{j=0}^{N} T^j$$

$$S_N \circ (Id - T) = \sum_{j=0}^{N} T^j - \sum_{j=1}^{N+1} T^j = T^0 - T^{N+1}$$

$N \to \infty$
$$= Id - T^{N+1}$$
$\to 0$

$$\underbrace{\sum_{j=0}^{\infty} T^j}_{(Id - T)^{-1}} \circ (Id - T) = Id$$

Podobně $(Id - T) \circ S_N = \ldots\ldots$

D.cv.

$\boxed{X}$

— X

Ok. $y'' + y = x^2 y$

$y(0) = 1$ $\quad \rightsquigarrow \quad y = e^{-\frac{x^2}{2}}$

$y'(0) = 0$

$\rightsquigarrow \; T \ldots\ldots \int_0^x f \ldots\ldots$

$\sum^{N} T^j \ldots$