Písemka 8.6.2009

1

$$\frac{\sqrt[3]{n^{111}+n^{110}}-\sqrt[3]{n^{111}-n^{110}}}{\sqrt{n^{10}+n^9}-\sqrt{n^{10}-n^9}} =$$

$$= \frac{2n^{110}}{2n^9}\cdot\frac{\sqrt{n^{10}+n^9}+\sqrt{n^{10}-n^9}}{\sqrt[3]{(n^{111}+n^{110})^2}+\sqrt[3]{(n^{111}+n^{110})(n^{111}-n^{110})}+\sqrt[3]{(n^{111}-n^{110})^2}}$$

$$= n^{101}\cdot\frac{n^5}{n^{222/3}}\cdot\underbrace{\frac{\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}}{\sqrt[3]{\left(1+\frac{1}{n}\right)^2}+\sqrt[3]{\left(1+\frac{1}{n}\right)\left(1-\frac{1}{n}\right)}+\sqrt[3]{\left(1-\frac{1}{n}\right)^2}}}_{=:A_n}$$

$$= A_n\cdot n^{32}, \text{ kde } \lim A_n = \frac{2}{3}.$$

$$\frac{(n^2+19)^3-(n^2-19)^3}{(n^2+3)^{19}-(n^2-3)^{19}} = \frac{(n^6+3.19n^4)-(n^6-3.19n^4)+\sum_{j=0}^{3}\alpha_j n^j}{(n^{38}+19.3.n^{36})-(n^{38}-19.3n^{36})+\sum_{j=0}^{35}\beta_j n^j}$$

$$= \frac{n^4}{n^{36}}\cdot\underbrace{\frac{\left(2.3.19+\sum_{j=0}^{3}\alpha_j n^{j-4}\right)}{\left(2.3.19+\sum_{j=0}^{35}\beta_j n^{j-36}\right)}}_{=:B_n} \qquad (\alpha_j, \beta_j \in \mathbb{Z})$$

$$= \frac{1}{n^{32}}\cdot B_n, \text{ kde } \lim B_n = 1$$

Celkově je

$$\lim \frac{\sqrt[3]{n^{111}+n^{110}}-\sqrt[3]{n^{111}-n^{110}}}{\sqrt{n^{10}+n^9}-\sqrt{n^{10}-n^9}}\cdot\frac{(n^2+19)^3-(n^2-19)^3}{(n^2+3)^{19}-(n^2-3)^{19}} = \frac{2}{3}$$

2

$$\lim_{x\to 0}\left(\frac{1^{x+1}+2^{x+1}+3^{x+1}}{7}\right)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{1}{x}\ln\frac{1^{x+1}+2^{x+1}+3^{x+1}}{7}}$$

Přitom

$$\frac{1}{x}\cdot\frac{\ln\frac{1+2^{x+1}+4^{x+1}}{7}}{\underbrace{\frac{1+2^{x+1}+4^{x+1}}{7}-1}_{\to 1 \text{ pro } x\to 0}}\cdot\frac{1+2^{x+1}+4^{x+1}-7}{7}$$

a

$$\frac{1}{7}\cdot\frac{1+2^{x+1}+4^{x+1}-7}{x} = \frac{1}{7}\left(\underbrace{\frac{2(2^x-1)}{x}}_{\downarrow\; 2\ln 2}+\underbrace{\frac{4(4^x-1)}{x}}_{\downarrow\; 4\ln 4}\right) \quad \text{pro } x\to 0$$

Celkem

$$\lim_{x\to 0}\left(\frac{1^{x+1}+2^{x+1}+3^{x+1}}{7}\right)^{\frac{1}{x}} = \exp\left(\frac{1}{7}(2\ln 2+4\ln 4)\right) =$$

$$= \sqrt[7]{2^2\cdot 4^4} = \sqrt[7]{2^{10}} = 2^{\frac{10}{7}} = \underline{\underline{2\cdot 2^{\frac{3}{7}}}}$$

3

$$a_n = \frac{1}{4^n}\frac{(2n)!}{n!\,n!} > 0 \quad \forall n$$

$$\frac{a_{n+1}}{a_n} = \frac{1}{4^{n+1}}\frac{(2n+2)!}{(n+1)!\,(n+1)!}\cdot\frac{4^n\, n!\, n!}{(2n)!} = \frac{1}{4}\frac{(\overset{2}{\cancel{2n}}+2)(2n+1)}{(n+1)^{\cancel{2}}} =$$

$$= \frac{2n+1}{2n+2} \longrightarrow 1 \quad \text{pro } n\to\infty$$

Podílové kritérium nerozhodlo.

Raabe:

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = n\left(\frac{2n+2}{2n+1} - 1\right) = n\,\frac{1}{2n+1} \longrightarrow \frac{1}{2}$$

pro $n \to \infty$

Řada diverguje dle Raabeho kritéria.

(4)

- $f(x) = \ln\left(\frac{x+2}{x^2+2x+1}\right) = \ln(x+2) - \ln(\overbrace{x^2+2x+1}^{(x+1)^2}) =$
$= \ln(x+2) - \ln((x+1)^2)$
- $\mathcal{D}(f) = (-2,-1) \cup (-1,+\infty)$, $f \in C(\mathcal{D}(f))$
- $\lim_{x\to -2} f(x) = -\infty$, $\lim_{x\to -1-} f(x) = \lim_{x\to -1+} f(x) = +\infty$

$\lim_{x\to +\infty} f(x) = -\infty$

- asymptota v $+\infty$: $\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{\ln(x+2) - 2\ln(x+1)}{x} \overset{\text{L'H}}{=}$

$= \lim_{x\to+\infty}\left(\frac{1}{x+2} - \frac{2}{x+1}\right) = 0$, ale $\lim_{x\to+\infty}(f(x) - 0\cdot x) = -\infty$

asymptota neexistuje

- $f'(x) = \frac{1}{x+2} - \frac{2}{x+1} = \frac{-(x+3)}{(x+2)(x+1)}$ $\quad x \in (-2,-1) \cup (-1,+\infty)$

f roste na $(-2,-1)$ $\quad$ f nemá lok. extr.
f klesá na $(-1,\infty)$

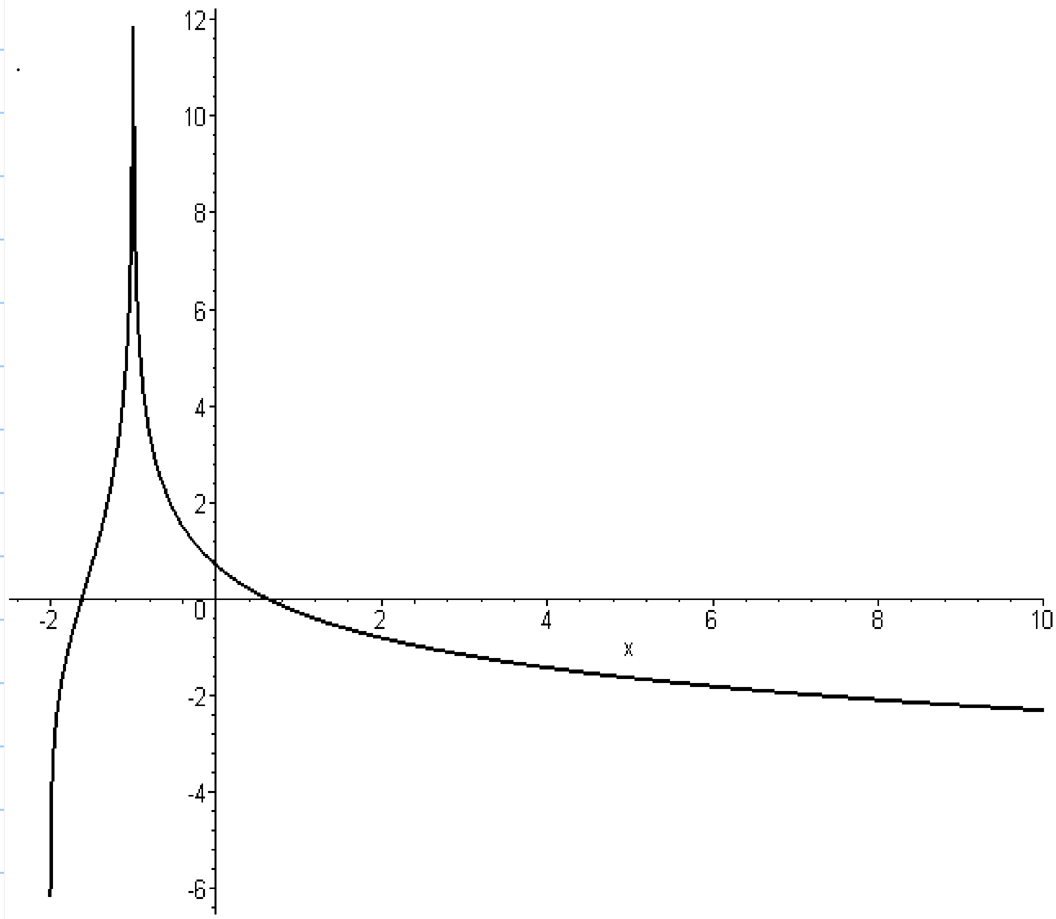
$\mathcal{H}(f) = \mathbb{R}$

- $f''(x) = -\frac{(x^2+3x+2) - (x+3)(2x+3)}{(x+1)^2(x+2)^2} = \frac{x^2+6x+7}{(x+1)^2(x+2)^2}$, $x \in (-2,-1) \cup (-1,\infty)$

$x^2+6x+7 = 0$ pro $x = \frac{-6 \pm \sqrt{36-28}}{2} = -3 \pm \sqrt{2}$ $\begin{cases} \approx -1.57 \\ \approx -4,\ldots \end{cases}$

$\Rightarrow$ f konkávní na $(-2, \sqrt{2}-3)$
f konvexní na $(\sqrt{2}-3, -1)$ a na $(-1,\infty)$

Inflexní bod: $\sqrt{2}-3$.



12
10
8
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