ISOMORPHISMS OF SPACES OF CONTINUOUS AFFINE FUNCTIONS ON COMPACT CONVEX SETS WITH LINDELÖF BOUNDARIES

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Abstract. Let X, Y be compact convex sets such that every extreme point of X and Y is a weak peak point and both $\operatorname{ext} X$ and $\operatorname{ext} Y$ are Lindelöf spaces. We prove that, if there exists an isomorphism $T : \mathfrak{A}^c(X) \to \mathfrak{A}^c(Y)$ with $\|T\| \cdot \|T^{-1}\| < 2$, then $\operatorname{ext} X$ is homeomorphic to $\operatorname{ext} Y$. This generalizes results of H. B. Cohen and C. H. Chu.

1. Introduction

If X is a compact convex set in a real locally convex space, let $\mathfrak{A}^c(X)$ stand for the space of all continuous affine functions, $\mathfrak{A}^b(X)$ for the space of all bounded affine functions on X, and $\operatorname{ext} X$ for the set of extreme points. The following results are proved in [5, Theorems 7 and 12] by H. B. Cohen and C. H. Chu:

Let X and Y be compact convex sets and let $T\colon \mathfrak{A}^c(X) \to \mathfrak{A}^c(Y)$ be an isomorphism satisfying $\|T\| \cdot \|T^{-1}\| < 2$. If

- *X and Y are metrizable and each point of $\operatorname{ext} X$ and $\operatorname{ext} Y$ is a weak peak point, or*
- *the sets $\operatorname{ext} X$ and $\operatorname{ext} Y$ are closed and each extreme point of X and Y is a split face,*

then the sets $\operatorname{ext} X$ and $\operatorname{ext} Y$ are homeomorphic.

We refer the reader to [5, pp. 72, 73, 75] for notions of the theory of compact convex sets. We just mention that X can be embedded to $(\mathfrak{A}^c(X))^*$ via the evaluation mapping $\phi\colon X \to (\mathfrak{A}^c(X))^*$ defined as $\phi(x)(f) = f(x)$, $f \in \mathfrak{A}^c(X)$, $x \in X$. The dual unit ball $B_{(\mathfrak{A}^c(X))^*}$ equals the convex hull $\operatorname{co}(X \cup -X)$ and $(\mathfrak{A}^c(X))^*$ coincides with $\operatorname{span} X$, the linear span of X. Further, any function $f \in \mathfrak{A}^b(X)$ has a unique extension to $\operatorname{span} X$, and this provides an identification of $(\mathfrak{A}^c(X))^{**}$ with $\mathfrak{A}^b(X)$.

We also recall that any weak peak point of a compact convex set X is a split face and the converse holds if $\operatorname{ext} X$ is closed; see [5, Proposition 1].

The aim of our paper is to show that the method of the proof of [5, Theorem 7] is applicable in a more general setting that covers both results mentioned above.

Theorem 1.1. *Let X, Y be compact convex sets such that every extreme point of X and Y is a weak peak point and both $\operatorname{ext} X$ and $\operatorname{ext} Y$ are Lindelöf spaces. Let*

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$T : \mathfrak{A}^c(X) \to \mathfrak{A}^c(Y)$ *be an isomorphism with* $\|T\| \cdot \|T^{-1}\| < 2$. *Then* $\operatorname{ext} X$ *is homeomorphic to* $\operatorname{ext} Y$.

As in [5, Corollaries 13 and 14], this yields a corollary for function algebras: *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be function algebras with Lindelöf Choquet boundaries, and let* $T\colon \operatorname{re}\mathcal{A} \to \operatorname{re}\mathcal{B}$ *be an isomorphism satisfying* $\|T\| \cdot \|T^{-1}\| < 2$. *Then the Choquet boundaries of* $\mathcal{A}$ *and* $\mathcal{B}$ *are homeomorphic.*

We recall that the construction from [3, Section VII] (see also [1, Proposition I.4.15] or [2, Theorem 3.2.4]) yields an example of a non-metrizable simplex X such that $\operatorname{ext} X$ is a Lindelöf non-closed subset of X and every extreme point of X is a weak peak point. To see this, let $B \subset [0,1]$ be a Bernstein set (see [9, Theorem 5.3]) and let

$$K = ([0,1] \times \{0\}) \cup \bigcup_{x \in B} (\{x\} \times [0,1])$$

be endowed with the "porcupine" topology (see [3, Section VII]). Precisely, if $x \in B$ and $t \in (0,1]$, then a basis of neighborhoods of (x,t) consists of sets of the form $\{x\} \times U$, where $U \subset [0,1]$ is a neighborhood of t. If $x \in [0,1]$, then a basis of neighborhoods of $(x,0)$ consists of sets of the form

$$(U \times \{0\}) \cup \big((U \times [0,1]) \setminus \bigcup_{i=1}^{n} (\{x_i\} \times F_i)\big),$$

where $n \in \mathbb{N}$, $U \subset [0,1]$ is a neighborhood of x, $x_1, \dots, x_n$ are points in $B \cap U$ and $F_1, \dots, F_n$ are compact subsets of $(0,1]$.

If λ stands for Lebesgue measure on $[0,1]$, let

$$H = \{f \in \mathcal{C}(K)\colon f(x,0) = \int_{[0,1]} f(x,t)\, d\lambda(t), x \in B\}$$

and

$$X = \{s \in H^*\colon s \geq 0, s(1) = 1\}.$$

Then X endowed with the weak* topology is a simplex and $\operatorname{ext} X$ is homeomorphic to $(([0,1] \setminus B) \times \{0\}) \cup (K \setminus ([0,1] \times \{0\}))$. It is easy to see that $\operatorname{ext} X$ is a Lindelöf non-closed set and every extreme point of X is its weak peak point.

Example 1 on [5, p. 83] shows that Theorem 1.1 need not hold even for compact convex sets in finite dimensional spaces if we omit the assumption that extreme points are weak peak points. An example due to H. U. Hess (see [7]) shows that *for every* $\varepsilon > 0$ *there exist metrizable simplices* X, Y *and an isomorphism* $T\colon \mathfrak{A}^c(X) \to \mathfrak{A}^c(Y)$ *such that* $\|T\| \cdot \|T^{-1}\| < 1 + \varepsilon$ *and* $\operatorname{ext} X$ *is not homeomorphic to* $\operatorname{ext} Y$. Nevertheless, it is not clear whether Theorem 1.1 remains valid if we omit the topological assumption on the sets of extreme points.

Question 1.2. *Let* X, Y *be compact convex sets such that every extreme point of* X *and* Y *is a weak peak point and let* $T : \mathfrak{A}^c(X) \to \mathfrak{A}^c(Y)$ *be an isomorphism with* $\|T\| \cdot \|T^{-1}\| < 2$. *Is it true that* $\operatorname{ext} X$ *is homeomorphic to* $\operatorname{ext} Y$*?*

We need to recall several notions not explained in [5]. If X is a compact (Hausdorff) space, we write $\mathcal{C}(X)$ for the space of all continuous functions on X and $\mathcal{M}^1(X)$ for the space of all probability Radon measures on X. (By a Radon measure we mean a complete measure that is inner regular with respect to compact sets and is defined on a σ-algebra including all Borel subsets of X. We refer the reader to [6, Section 416] for more information on Radon measures).) We always consider $\mathcal{M}^1(X)$ endowed with weak* topology. We say that a function $f : X \to \mathbb{R}$ is *universally measurable* if f is μ-measurable for every $\mu \in \mathcal{M}^1(X)$.

If X is a compact convex subset of a real locally convex space, any $\mu \in \mathcal{M}^1(X)$ has its unique barycenter $r(\mu) \in X$, i.e., the point $x \in X$ satisfying $f(x) = \mu(f)$ for any $f \in \mathfrak{A}^c(X)$. We sometimes say that μ *represents* x. A function $f : X \to \mathbb{R}$ is *strongly affine* (or *satisfies the barycentric formula*), if f is universally measurable, $\mu(f)$ exists and $f(r(\mu)) = \mu(f)$ for any $\mu \in \mathcal{M}^1(X)$. We write $\mathfrak{A}_{\mathrm{bf}}(X)$ for the space of all strongly affine functions on X and recall that it is easy to see that any strongly affine function is bounded (see the proof of [8, Satz 2.1(c)]). We also recall that any semicontinuous affine function on X is strongly affine; see [2, Theorem 1.6.1(ix)].

2. Proof of Theorem 1.1

The proof of the main theorem follows the idea of the proof of [5, Theorems 7 and 12]. Hence we recall the main steps of their proof and point out our modifications. We start the proof with a minimum principle which is crucial for us because then [10, Lemma 2.4] is applicable for functions $T^{**}f$, $f \in \mathfrak{A}_{\mathrm{bf}}(X)$.

Lemma 2.1. *Let X be a compact convex set such that $\operatorname{ext} X$ is Lindelöf. If $f \in \mathfrak{A}_{\mathrm{bf}}(X)$ satisfies $|f(x)| \le c$ for all $x \in \operatorname{ext} X$, then $|f(x)| \le c$ for all $x \in X$.*

Proof. Let $x \in X$ be given. We find a maximal measure $\mu \in \mathcal{M}^1(X)$ representing the point x (see [2, Theorem 1.6.4]) and define

$$A = \{y \in X : |f(y)| \le c\}.$$

Then A is a μ-measurable set and we claim that $\mu(A) = 1$.

Indeed, let $K \subset X$ be an arbitrary compact set disjoint from A. Since $A \supset \operatorname{ext} X$, for any $y \in \operatorname{ext} X$ we can find its closed neighborhood not intersecting K. The set $\operatorname{ext} X$ is Lindelöf, and thus we can select countably many closed sets $F_n \subset X$, $n \in \mathbb{N}$, such that

$$\operatorname{ext} X \subset \bigcup_{n=1}^{\infty} F_n \subset X \setminus K.$$

By [4, Theorem 27.11], $\mu(\bigcup_{n=1}^{\infty} F_n) = 1$, and hence $\mu(K) = 0$. By the regularity of μ, $\mu(X \setminus A) = 0$, and hence

$$|f(x)| = \left| \int_X f \, d\mu \right| \le \int_A |f| \, d\mu \le c.$$

This concludes the proof. □

Proof of Theorem 1.1. Let $T : \mathfrak{A}^c(X) \to \mathfrak{A}^c(Y)$ be an isomorphism satisfying $\|T\| \cdot \|T^{-1}\| < 2$. We assume that there exists $c, c' \in \mathbb{R}$ such that $1 < c < c' < 2$ and $\|T\| < 2$ and $\|Tf\| \ge c'\|f\|$ for all $f \in \mathfrak{A}^c(X)$ (otherwise we would find $1 < c < c' < 2$ such that $\|T\| \cdot \|T^{-1}\| < \frac{2}{c'} < 2$ and consider the mapping $c'\|T^{-1}\|T$; see [5, p. 76]).

Claim 1. *For any $f \in \mathfrak{A}^b(X)$ and $g \in \mathfrak{A}^b(Y)$ non-zero, $\|T^{**}f\| > c\|f\|$ and $\|(T^{-1})^{**}g\| > \frac{1}{2}\|g\|$.*

Proof of Claim 1. The first inequality follows from

$$\|f\| = \|(T^{-1})^{**}T^{**}f\| \le (c')^{-1}\|T^{**}f\| < c^{-1}\|T^{**}f\|,$$

the second one is analogous. □

If $x \in \operatorname{ext} X$, we recall that $(\mathfrak{A}^c(X))^* = \operatorname{span}\{x\} \oplus_{\ell^1} \operatorname{span}\{x\}'$ because $\{x\}$ is a split face; see [5, p. 72]. Hence, given $y \in Y$, following [5, p. 76] we can write

$$T^*y = \lambda x + \mu \quad \text{for some } \lambda \in \mathbb{R} \text{ and } \mu \in \operatorname{span}\{x\}'. \tag{1}$$

It is proved in [5, p. 77] that $\|\mu\| < 2 - c$ whenever $y \in Y$ satisfies $|\lambda| > c$.

We recall the construction of mappings $\rho\colon Y \to \operatorname{ext} X$ and $\tau\colon X \to \operatorname{ext} Y$. Given $x \in \operatorname{ext} X$, the function $h_x = \chi^*_{\{x\}}$ is upper semicontinuous and affine (see [5, p. 73]), and thus strongly affine (see [2, Theorem 1.6.1(ix)]). By [5, p. 77], for each $y \in \operatorname{ext} Y$ there is at most one point $x \in \operatorname{ext} X$ such that $|T^{**}h_x(y)| > c$. Let

$$Y' = \{y \in \operatorname{ext} Y : \text{there exists } x \in \operatorname{ext} X \text{ with } |T^{**}h_x(y)| > c\},$$

and let $\rho\colon Y' \to X$ be defined by the property that $\rho(y)$ equals the unique point $x \in \operatorname{ext} X$ satisfying $|T^{**}h_x(y)| > c$.

Analogously, if

$$X' = \{x \in \operatorname{ext} X : \text{there exists } y \in \operatorname{ext} Y \text{ with } |(T^{-1})^{**}h_y(x)| > \frac{1}{2}\},$$

then $\tau\colon X' \to \operatorname{ext} Y$ can be defined by the requirement that $\tau(x)$ is the unique $y \in \operatorname{ext} Y$ satisfying $|(T^{-1})^{**}h_y(x)| > \frac{1}{2}$.

Claim 2. *For any $x \in \operatorname{ext} X$, $T^{**}h_x \in \mathfrak{A}_{\mathrm{bf}}(Y)$.*

Proof of Claim 2. Since $T\colon \mathfrak{A}^c(X) \to \mathfrak{A}^c(Y)$, we have $T^*\colon \operatorname{span} Y \to \operatorname{span} X$. If $f \in \mathfrak{A}^b(X)$ and $\widetilde{f}$ is the linear extension of f to $\operatorname{span} X$, then $T^{**}f = \widetilde{f} \circ T^*$. Since $\|T\| < 2$,

$$T^*Y \subset 2B_{(\mathfrak{A}^c(X))^*} = \operatorname{co}(2X \cup -2X).$$

The sets $2X$ and $-2X$ are affinely homeomorphic to X, and hence $\widetilde{f}$ is strongly affine on both of them. By [10, Lemma 2.4(b)],

$$\widetilde{f} \in \mathfrak{A}_{\mathrm{bf}}(2B_{(\mathfrak{A}^c(X))^*}) = \mathfrak{A}_{\mathrm{bf}}(\operatorname{co}(2X \cup -2X)).$$

Since Y is affinely homeomorphic to T^*Y and $T^{**}f = \widetilde{f} \circ T^*$, we obtain that $T^{**}f \in \mathfrak{A}_{\mathrm{bf}}(Y)$. □

Claim 3. *The mappings $\rho\colon Y' \to \operatorname{ext} X$ and $\tau\colon X' \to \operatorname{ext} Y$ are surjective.*

Proof of Claim 3. Let $x \in \operatorname{ext} X$ be given and assume that $|T^{**}h_x(y)| \le c$ for all $y \in \operatorname{ext} Y$. By Claims 1, 2 and Lemma 2.1, $|T^{**}h_x| \le c$ on Y. Then

$$c \ge \|T^{**}h_x\| > c\|h_x\| = c$$

gives a contradiction. Hence ρ is surjective.

Analogously, using the second part of Claim 1 we obtain that τ is surjective. □

The following claim is essentially Lemma 6 of [5]. However, we recall its proof since it uses Lemma 2.1.

Claim 4. *We have $X' = \operatorname{ext} X$ and $Y' = \operatorname{ext} Y$ and, for any $x \in \operatorname{ext} X$ and $y \in \operatorname{ext} Y$, $\rho(\tau(x)) = x$ and $\tau(\rho(y)) = y$.*

Proof of Claim 4. We show that, for any $y \in Y'$,

$$|(T^{-1})^{**}h_y(\rho(y))| > \frac{1}{2}. \tag{2}$$

Assuming $|(T^{-1})^{**}h_y(\rho(y))| \leq \frac{1}{2}$, Claim 2 and Lemma 2.1 yield

$$d = \sup_{x\in \operatorname{ext} X} |(T^{-1})^{**}h_y(x)| = \sup_{x\in X} |(T^{-1})^{**}h_y(x)| = \|(T^{-1})^{**}h_y\|.$$

By Lemma 2.1 and Claim 1, $\frac{1}{2} < d$. Since $c > 1$, we have $d > \frac{d}{c}$. Let $x' \in \operatorname{ext} X$ be chosen such that

$$|(T^{-1})^{**}h_y(x')| > \max\{\frac{d}{c}, \frac{1}{2}\}.$$

By the assumption, $|(T^{-1})^{**}h_y(\rho(y))| \leq \frac{1}{2}$, and thus $\rho(y) \neq x'$.

By Claim 3 we can select $y' \in Y'$ with $\rho(y') = x'$. Then $y' \in \{y\}'$, and thus $h_y(y') = 0$. If $T^*y' = \lambda' x' + \mu'$, $\lambda' \in \mathbb{R}$ and $\mu' \in \operatorname{span}\{x'\}'$ (see (1)), then

$$0 = h_y(y') = (T^{-1})^{**}h_y(T^*y') = (T^{-1})^{**}h_y(\lambda' x') + (T^{-1})^{**}h_y(\mu'). \tag{3}$$

Since $\lambda' = T^{**}h_{x'}(y')$, it follows from the definition of ρ that $|\lambda'| > c$.

Using this, (3) and (1) along with the subsequent remark, we obtain

$$\begin{aligned} d < |\lambda'|\frac{d}{c} &< |\lambda'||(T^{-1})^{**}h_y(x')| \\ &= |(T^{-1})^{**}h_y(\lambda' x')| \\ &= |(T^{-1})^{**}h_y(\mu')| \\ &\leq d\|\mu'\| < d(2-c) < d. \end{aligned}$$

This contradiction yields the validity of (2).

Now, let $x \in \operatorname{ext} X$ be given. We find $y \in Y'$ with $\rho(y) = x$. It follows from (2) that $x \in X'$ and $\tau(x) = y$. Hence $X' = \operatorname{ext} X$ and $\tau(\rho(y)) = y$ for all $y \in Y'$.

If $y \in \operatorname{ext} Y$ is given, let $x \in \operatorname{ext} X$ be such that $\tau(x) = y$. If $y' \in Y'$ satisfies $\rho(y') = x$, from the previous argument we obtain

$$y = \tau(x) = \tau(\rho(y')) = y'.$$

Hence $Y' = \operatorname{ext} Y$ and it easily follows that $\rho(\tau(x)) = x$ for any $x \in \operatorname{ext} X$. □

By the proof of Theorem 7 on p. 78 in [5], the mappings ρ and τ are continuous (we point out that this part of the argument is valid for arbitrary compact convex sets as mentioned in [5, p. 83]). This concludes the proof. □

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