Mechanical oscillators described by a system of differential-algebraic equations

Dalibor Pražák[*]
Department of Mathematical Analysis
Charles University, Prague
Sokolovská 83, 186 75 Czech Republic
E-mail: prazak@karlin.mff.cuni.cz

Kumbakonam R. Rajagopal
Department of Mechanical Engineering
Texas A&M University
College Station, TX 77843 USA
E-mail: krajagopal@tamu.edu

Abstract

The classical framework for studying the equations governing the motion of lumped parameter systems presumes one can provide expressions for the forces in terms of kinematical quantities for the individual constituents. This is not possible for a very large class of problems where one can only provide implicit relations between the forces and the kinematical quantities. In certain special cases, one can provide non-invertible expressions for a kinematical quantity in terms of the force, which then reduces the problem to a system of differential-algebraic equations.

We study such a system of differential-algebraic equations, describing motions of mass-spring-dashpot oscillator. Assuming a monotone relationship between the displacement, velocity and the respective forces, we prove global existence and uniqueness of solutions. We also analyze the behavior of some simple particular models.

1 Introduction

The equations governing the vibratory motion of a spring, dashpot and mass, represented as a lumped parameter system (see Figure 1) take the form

$$m\ddot{x} = F - F_s - F_d, \tag{1.1}$$

where x is the displacement, m the mass, F the externally applied force on the mass and F_s and F_d denote the forces in the spring and dashpot, respectively. The dot denotes differentiation with respect to time. It is customarily assumed that one can provide explicit

[*]Corresponding author.

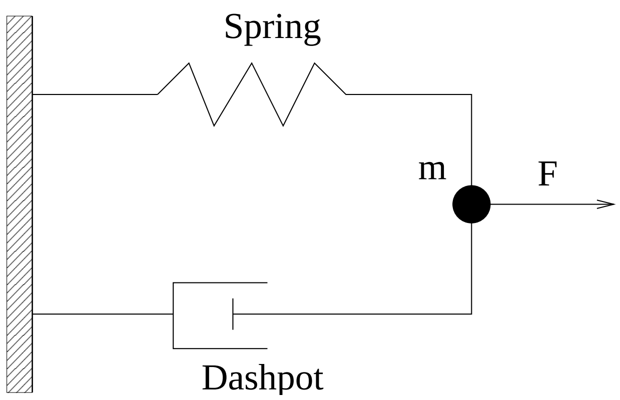

Figure 1: Mass-Spring-Dashpot lumped parameter system.

expression for the force in the spring, in terms of the displacement, and the force in the dashpot, in terms of the velocity, i.e.,

$$F_s = \hat{f}(x), \tag{1.2}$$
$$F_d = \hat{g}(\dot{x}). \tag{1.3}$$

Substituting (1.2) and (1.3) into (1.1) yields the equations governing the vibratory motion of the system. The equations are completed by providing the initial conditions

$$x(0) = x_0, \tag{1.4}$$
$$\dot{x}(0) = x_1. \tag{1.5}$$

Recently, Rajagopal [3] has studied the vibratory motion of a class of lumped parameter systems wherein the forces in the individual constituents that store and dissipate energy (springs and dashpots) cannot be expressed in terms of kinematical quantities such as the displacement, velocity, etc. Either one has the expression for the kinematics in terms of the forces in the constituents, or worse still one has an implicit relationship between the forces and the kinematics. In such lumped parameter systems, instead of dealing with a differential equation we are forced to deal with a system of differential-algebraic equations. Constituents wherein one cannot provide expressions for the forces explicitly in terms of the kinematical quantities arise naturally in physical systems, for example the frictional force in a dashpot consisting in a Bingham fluid cannot be expressed in terms of the velocity, rather the velocity can be expressed in terms of the frictional force (see Figure 2). Similarly, one cannot express the force in a system which consists in a spring and an inextensible spring in parallel in terms of the displacement, on the contrary one can express the displacement in terms of the force (see Figure 3). In the above mentioned examples, we note that the constitutive expression for the spring-inextensible spring is given by

$$x = f(F_s), \tag{1.6}$$

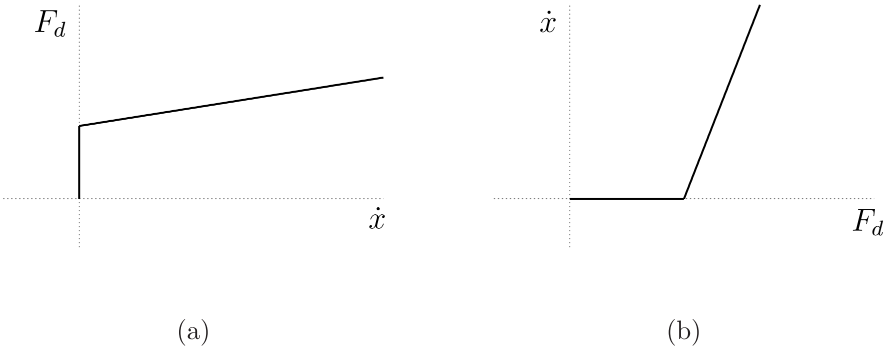

Figure 2: Frictional forces velocity relationship for a linear Bingham dashpot.

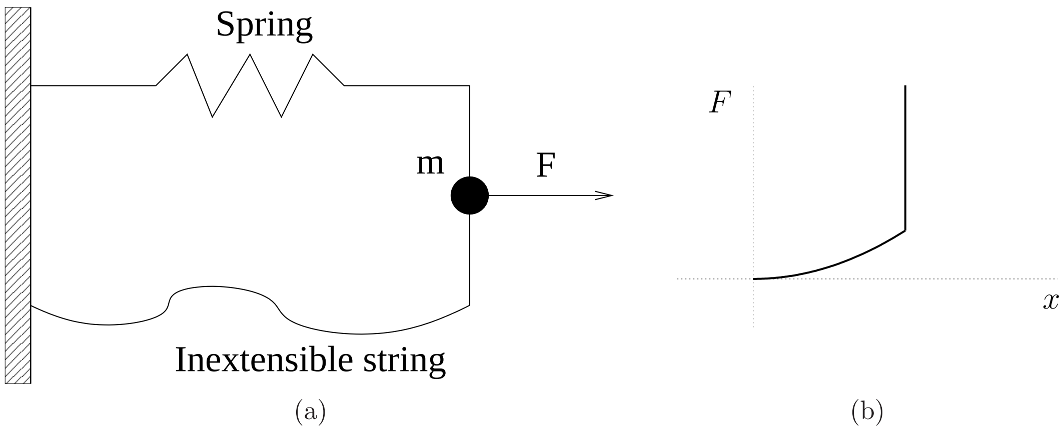

Figure 3: A spring and an inextensible string in parallel; the relation between the applied force and the displacement.

while the constitutive expression for the dashpot takes the form

$$\dot{x} = g(F_d). \tag{1.7}$$

Thus, we cannot substitute (1.6) and (1.7) into (1.1) to obtain a single differential equation for the displacement, and thus we need to solve the system of equations (1.1), (1.6) and (1.7) simultaneously.

The constituents of the lumped parameter system can be much more complex in that neither the force nor the kinematics can be expressed in terms of the other, but requiring implicit constitutive relations between these variables. This would indeed be the case if the individual constituents were viscoelastic or viscoplastic bodies. Thus, one might have to deal with the vibratory motion of a lumped parameter system which comprises of several constituents, the constitutive response of these components being given by

$$h_i(f_i, \dot{f}_i, x, \dot{x}) = 0, \tag{1.8}$$

where f_i is the force acting in the i^{th} constituent. In fact the problem can be even more daunting in that one might have higher time derivatives of the forces and the displacement. Equation (1.8) could be such that it is truly implicit, not allowing the displacement or any of the other kinematical variables in terms of the force and its time derivatives. However, there are components, such as a Maxwell element which will correspond to a relation of the form

$$\dot{x} = \alpha f + \beta \dot{f},$$

where x is the displacement and f the force.

A frictional force that arises often in vibratory motions is Coulomb friction. This also is an interesting situation wherein one cannot express the frictional force in terms of the velocity. When the velocity is zero, the frictional force can take any value $-\mu_s N \leq F_d \leq \mu_s N$ where μ_s is the coefficient of static friction and N is the normal traction acting on the mass. Classical solutions are not possible to the system of equations governing the problem. Even more importantly, the frictional force F_d that appears in (1.1) cannot be specified independent of the extension of the spring whenever the force in the spring $F_s \leq \mu_s N$. The problem can be cast as a differential inclusion and solutions can be established in the sense of Filippov, see [1].

However, one finds that there is a large body of work, especially in engineering, where the problem is incorrectly identified as possessing a classical solution. In fact, in such studies (see for example Meirovitch [2]) one persists in using the governing equation (1) to describe the vibrating system while obtaining the solution to the differential equation by patching up the solution for negative and positive velocity, not recognizing that the equation (1) is not meaningfully defined in that one cannot have an expression for the force due to Coulomb friction when the velocity is zero and the initial conditions are such that the spring force is less than the static frictional force. In this paper, we seek solutions for F_s, F_d that are integrable functions rather than solutions such as those sought by Filippov. We also require that the displacement and the velocity are absolutely continuous functions.

In this short paper, we will not be considering constituent equations of the form (1.8). We shall consider the simpler system (1.1), (1.6) and (1.7). We show there exists at least one solution to this system satisfying (1.4), (1.5). For the autonomous problem, i.e., when the external force F is constant, the solution is unique. We then specialize the constitutive expression (1.6) and (1.7) to correspond to that for a linear spring and a Bingham dashpot, respectively.

Throughout the paper, we will assume that

$$f,\ g : \mathbb{R} \to \mathbb{R} \quad \text{are continuous, nondecreasing} \tag{1.9}$$

$$f(0) = g(0) = 0 \tag{1.10}$$

$$c_1|u| - c_2 \leq |f(u)| \leq c_3(|u| + 1) \tag{1.11}$$

$$c_4|u| - c_5 \leq |g(u)| \leq c_6(|u| + 1) \tag{1.12}$$

Here (1.11), (1.12) are convenient technical assumptions, which together with (1.6), (1.7) imply that

$$\begin{aligned} x \text{ is bounded } &\iff F_s \text{ is bounded} \\ \dot{x} \text{ is bounded } &\iff F_d \text{ is bounded} \end{aligned} \tag{1.13}$$

Clearly, (1.11), (1.12) are satisfied if f and g grow linearly close to infinity; they also imply that these functions are onto, which seems necessary in order for us to prove global existence of solutions for arbitrary initial data.

The underlying analysis does not correspond to that used in standard ODE theory; therefore it is important to specify what we mean by a solution.

Definition 1. *By a solution we mean a triple of functions* (x, F_s, F_d) *defined on some time interval* I*, where* x*,* $\dot{x}$ *are absolutely continuous functions,* F_s*,* F_d *are integrable functions, and* (1.1), (1.6), (1.7) *hold almost everywhere (a.e.) in* I*.*

2 A general existence result

In this section, we are going to prove the following theorem.

Theorem 1. *For arbitrary* x_0*,* $x_1 \in \mathbb{R}$*,* $T > 0$ *and* $F \in L^2(0,T)$*, the system* (1.1), (1.6), (1.7) *has at least one solution, satisfying* (1.4), (1.5)*, and defined on the whole interval* $[0,T]$*.*

Proof. Replacing f, g by

$$\begin{aligned} f_k(u) &= f(u) + k^{-1}u, \\ g_k(u) &= g(u) + k^{-1}u, \end{aligned}$$

we can define the following approximating problem

$$\ddot{x}^k = F - F_s^k - F_d^k, \tag{2.1}$$

$$x^k = f_k(F_s^k), \tag{2.2}$$

$$\dot{x}^k = g_k(F_d^k). \tag{2.3}$$

We observe that f_k, g_k are invertible, hence this is equivalent to

$$\ddot{x}^k = F - (f_k)_{-1}(x^k) - (g_k)_{-1}(\dot{x}^k). \tag{2.4}$$

In fact, the functions $(f_k)_{-1}$, $(g_k)_{-1}$ are globally k-Lipschitz. Thus, for any $k \geq 1$, we have a global solution by virtue of the standard ODE theory; see for example Vrabie [5, Theorem 2.4.5].

To obtain estimates that are independent of k, we multiply (2.1) by $2\dot{x}^k$. Note that

$$\begin{aligned} F_d^k \dot{x}^k &= F_d^k g_k(F_d^k) \geq 0, \\ |F_s^k| &\leq c(|x^k| + 1); \end{aligned}$$

the second inequality follows from (1.11), with c being independent of k provided it is sufficiently large. Hence we deduce that

$$\frac{d}{dt}(\dot{x}^k)^2 \leq 2(|F| + c_1|x^k| + c_1)|\dot{x}^k| \leq |F|^2 + c_2(1 + (x^k)^2 + (\dot{x}^k)^2);$$

integrating over $(0, t)$, and noting that

$$\int_0^t (x^k)^2 \, \mathrm{d}s = \int_0^t \big(x_0 + \int_0^s \dot{x}^k \, \mathrm{d}\tau\big)^2 \, \mathrm{d}s \leq c_3\big(x_0^2 + \int_0^t (\dot{x}^k)^2 \, \mathrm{d}s\big), \tag{2.5}$$

we finally deduce

$$(\dot{x}^k)^2(t) \leq x_1^2 + c_4\big(x_0^2 + \int_0^t |F|^2 + (\dot{x}^k)^2 \, \mathrm{d}s\big), \qquad t \in [0, T],$$

where the constants c_3, c_4 possibly depend on T. We deduce from Gronwall's lemma (Vrabie [5, Lemma 1.5.2]), that

$$\sup_{t\in[0,T]} |\dot{x}^k(t)| + |x^k(t)| \leq K_1, \tag{2.6}$$

where K_1 only depends on x_0, x_1, F and $T > 0$, but is independent of k that is sufficiently large. Similarly, we have from (2.1)–(2.3) (invoking again (1.11), (1.12))

$$\sup_{t\in[0,T]} |F_s^k(t)| + |F_d^k(t)| \leq K_2, \tag{2.7}$$

$$\int_0^T |\ddot{x}^k|^2 \, \mathrm{d}t \leq K_3. \tag{2.8}$$

Now, we need to take the limit $k \to \infty$. From the above a priori estimates, there exists a triple (x, F_s, F_d) such that, taking a subsequence (not relabelled)

$$\left.\begin{aligned} x^k &\to x \\ \dot{x}^k &\to \dot{x} \end{aligned}\right\} \quad \text{uniformly on } [0, T],$$

and

$$\left.\begin{aligned} \ddot{x}^k &\to \ddot{x} \\ F_s^k &\to F_s \\ F_d^k &\to F_d \end{aligned}\right\} \quad \text{weakly in } L^2(0, T).$$

This is sufficient to obtain the limit in (2.1). The proof will be finished once we handle (2.2); the equation (2.3) is done similarly.

We employ monotonicity of f in a standard way. Observing that $f_k \to f$ locally uniformly in $\mathbb{R}$, while F_s^k are bounded, we have that

$$f(F_s^k) = \underbrace{f(F_s^k) - f_k(F_s^k)}_{\rightrightarrows 0} + \underbrace{f_k(F_s^k)}_{x^k} \to x \quad \text{uniformly on } [0, T].$$

Now, we can write

$$\int_0^T (f(F_s^k) - f(w))(F_s^k - w)\,\mathrm{d}t \geq 0,$$

where $w \in L^2(0,T)$ will be specified in a moment. Letting $k \to \infty$, we deduce

$$\int_0^T (x - f(w))(F_s - w)\,\mathrm{d}t \geq 0.$$

Taking now $w = F_s \pm \lambda\chi$, where $\lambda > 0$ is constant and $\chi \in L^2(0,T)$ is arbitrary function, we divide by λ and then let $\lambda \to 0$ to finally obtain

$$\int_0^T (x - f(F_s))\chi\,\mathrm{d}t = 0.$$

By the arbitrariness of χ, it follows that $x = f(F_s)$ almost everywhere. □

3 Uniqueness for the autonomous problem

We will now impose additional structural assumptions on the function f, namely that there exists a finite number of disjoint closed intervals I_k and real constants ξ_k such that

$$f \equiv \xi_k \qquad \text{in } I_k,$$

while f is strictly increasing outside $\cup_k I_k$. We set

$$\phi := \left(f|_{\mathbb{R}\setminus\cup_k I_k}\right)_{-1};$$

our final requirement is that ϕ is locally Lipschitz on its domain of definition, i.e., on $\mathbb{R} \setminus \{\xi_1, \xi_2, \dots\}$. Note that ϕ is a strictly increasing function.

The subsequent analysis is closely related to problem of inverting (1.6). If $x \neq \xi_k$, this is indeed equivalent to $F_s = \phi(x)$. On the other hand, one has

$$\begin{gathered} \dot{x}(t) = 0 \quad \text{for a.e. } t \in M, \\ \text{where} \quad M = \left\{t \in [0,T];\ \exists k \text{ such that } x(t) = \xi_k\right\}. \end{gathered} \tag{3.1}$$

This follows from the fact the derivative of x is zero in every Lebesgue point of the set $\{t \in [0,T]; x(t) = \xi_k\}$. We conclude by recalling the well-known fact that Lebesgue points are the set of full measure, see for example Rudin [4, Theorem 7.7].

Finally, we set

$$\mathcal{F}(x) := \int_0^x 2\phi(\xi)d\xi.$$

Observe that $\mathcal{F}$ is locally Lipschitz, strictly convex and has a global strict minimum at $\mathcal{F}(0) = 0$. One also has

$$\mathcal{F}'(x) = 2\phi(x), \quad \forall x \neq \xi_k. \tag{3.2}$$

An important step towards uniqueness is the energy (in)equality.

Lemma 1. *Every solution to the system* (1.1), (1.6), (1.7) *satisfies*

$$\frac{d}{dt}\big[(\dot{x})^2 + \mathcal{F}(x)\big] + 2F_d\dot{x} = 2F\dot{x} \tag{3.3}$$

almost everywhere on $[0, T]$.

Proof. We have $\frac{d}{dt}(\dot{x})^2 = 2\ddot{x}\dot{x}$, and

$$\frac{d}{dt}\mathcal{F}(x) = \mathcal{F}'(x)\dot{x} = 2\phi(x)\dot{x}$$

almost everywhere – this follows simply by chain rule if $x \neq \xi_k$, and it is a consequence of (3.1) if $t \in M$. Thus, (3.3) is equivalent to

$$\big(\ddot{x} + F_d + \phi(x)\big)2\dot{x} = 2F\dot{x};$$

for $x \neq \xi_k$ this follows from the equation (since $F_s = \phi(x)$); while for $t \in M$ we again invoke (3.1). $\square$

For the sake of brevity, we will say that the solution is unique at the point (x_0, x_1), if two arbitrary solutions satisfying the same initial condition

$$x(t_0) = x_0, \qquad \dot{x}(t_0) = x_1 \tag{3.4}$$

coincide on $[t_0, t_0 + \delta]$ for some $\delta > 0$. By the usual continuation argument, uniqueness at every point implies global (forward) uniqueness.

Lemma 2. *Let h be nondecreasing on the neighborhood V of x_1, and let ψ be Lipschitz on the neighborhood U of x_0. Then the solution of the system*

$$\begin{aligned} \ddot{x} + F_d + \psi(x) &= F, \\ \dot{x} &= h(F_d) \end{aligned}$$

is unique at the point (x_0, x_1).

Proof. Let x^1, x^2 be two solutions with the initial condition (x_0, x_1); we can assume $t_0 = 0$. Let F_d^1, F_d^2, be the corresponding forces. By continuity, we can find $\delta > 0$ such that x^i and $\dot{x}^i$ remain in the U and V for $t \in [0, \delta]$, respectively.

Subtracting the equations for x^1 and x^2 yields

$$\frac{d}{dt}\big(\dot{x}^1 - \dot{x}^2\big) + F_d^1 - F_d^2 = \psi(x^2) - \psi(x^1);$$

we multiply by $2(\dot{x}^1 - \dot{x}^2)$ and note that

$$(F_d^1 - F_d^2)(\dot{x}^1 - \dot{x}^2) = (F_d^1 - F_d^2)(g(F_d^1) - g(F_d^2)) \geq 0.$$

Hence, introducing $z = x^1 - x^2$, we arrive at

$$\frac{d}{dt}(\dot{z})^2 \leq 2Lz^2,$$

and integration yields (by the same token as in (2.5))

$$(\dot{z}(t))^2 \leq 2L \int_0^t z^2 ds \leq C \int_0^t (\dot{z})^2 ds,$$

for any $t \in [0, \delta]$. By Gronwall's lemma we deduce $z \equiv 0$. □

We can now prove the main result of this section.

Theorem 2. *Let $F \equiv F_0$, and let f satisfies the structural assumptions from the beginning of the section. Then the solution of* (1.1), (1.6), (1.7) *is unique at every point* (x_0, x_1).

Proof. The proof is split into several cases.

Case 0. If $x_0 \neq \xi_k$, the conclusion follows from Lemma 2 (with $\psi = \phi$).

Case 1. If $(x_0, x_1) = (0, 0)$ and $F_0 = 0$, we deduce from (3.3) (note that $F_d \dot{x} \geq 0$) that

$$(\dot{x}(t))^2 + \mathcal{F}(x(t)) \leq 0;$$

for all $t \geq 0$, hence $x \equiv 0$.

Case 2. Assume that $(x_0, x_1) = (\xi_k, 0)$, and the force can be written as $F_0 = \psi + \varphi$, where $g(\psi) = 0$, $f(\varphi) = \xi_k$. In other words, the external force can be compensated by the forces in the dashpot and the spring that keep the system at the equilibrium.

Replacing x by $x - \xi_k$, g by $g(\cdot + \psi)$ and f by $f(\cdot + \varphi) - \xi_k$, we deduce that $x \equiv \xi_k$ is the unique solution by reduction to the argument of case 1.

It remains to handle the following situations:

Case 3i. $(x_0, x_1) = (x_k, x_1)$ with $x_1 \neq 0$;

Case 3ii. $(x_0, x_1) = (\xi_k, 0)$, where $F_0 \notin g^{-1}(0) + f^{-1}(\xi_k)$.

In both the situations, we first claim that (any possible) solution is strictly monotone on $[0, \delta]$ for some $\delta > 0$. This is obvious if $\dot{x}(0) = x_1 \neq 0$, while in the case 3ii, by the continuity argument, $\ddot{x} = F_0 - F_d - F_s$ is either positive or negative a.e. on $(0, \delta)$. Together with the initial condition $\dot{x}(0) = 0$ this yields the desired monotonicity.

Assume that x^1, x^2 are two solutions that do not coincide on $[0, \delta]$ for any $\delta > 0$. Without loss of generality, x^i are strictly increasing.

Set $z := x^1 - x^2$, and let $z(t_1) > 0$ for some $t_1 \in (0, \delta)$. We further define

$$t_2 := \inf \{\tau \in [0, t_1);\ z > 0 \text{ on } (\tau, t_1)\}.$$

It follows that $t_2 \in [0, t_1)$ and $z > 0$ on (t_1, t_2). By the mean value theorem, there exists $\tau \in (t_2, t_1)$ such that $\dot{z}(\tau) > 0$. By continuity, z, $\dot{z} > 0$ even on some $(\tau - \eta, \tau)$, $\eta > 0$.

On the other hand, we have the equation

$$\ddot{z} + F_d^1 - F_d^2 + F_s^1 - F_s^2 = 0;$$

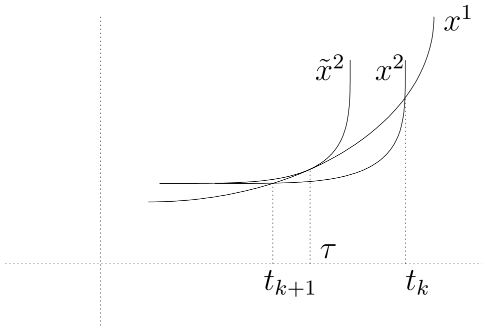

Figure 4: End of the proof of Theorem 2.

since $z = f(F_s^1) - f(F_s^2) > 0$, $\dot{z} = g(F_d^1) - g(F_d^2) > 0$, we deduce that $\ddot{z} < 0$ a.e. on $(\tau - \eta, \tau)$. Hence $\dot{z}(\tau - \eta) > \dot{z}(\tau) > 0$.

Repeating the argument, we eventually deduce that $\dot{z}(t_2) > \dot{z}(t_1) > 0$. Since $\dot{z}(0) = 0$, obviously $t_2 > 0$, and $z < 0$ on some $(t_2 - \eta, t_2)$. By the same token, there is a sequence of points $t_k \searrow 0$ such that $z(t_k) = 0$, and $(-1)^k \dot{z}(t_k) > 0$.

We will bring this to contradiction. Recall that x^i are strictly increasing, and assume that $t_{k+1} < t_k$ are close enough to zero so that $x^i \neq \xi_k$ for all $t \in (t_{k+1}, t_k)$. Now, it is possible to find c such that x^1 and $\tilde{x}^2 = x^2(\cdot + c)$ have the same value and derivative at some $\tau \in (t_{k+1}, t_k)$. See Figure 4; rigorous proof is obtained by application of Lagrange theorem to inverse functions to x^i and is left to the pedantic reader.

However, $\tilde{x}^2$ solves the same autonomous equation; and hence, $\tilde{x} \equiv x^1$ on $[\tau, t_k]$ in virtue of the uniqueness proven in case 0. On the other hand, the construction implies that $\tilde{x}^2(t_k) > x^1(t_k)$ – a contradiction.

□

Remark 1. *It is interesting to note that the forces are NOT determined uniquely. For example, if f, $g \equiv 0$ on some neighborhood of zero, then $x \equiv 0$ is a solution, while F_s, F_d can change arbitrarily as long as $F_d + F_s \equiv 0$. Nonetheless, no possible combinations of these forces would yield any motion.*

4 Construction of solutions for particular models

The aim of this section is a more detailed analysis of solutions for some simple particular models.

4.1 Linear spring and Bingham dashpot

Assume Hooke's law for the spring, i.e.,

$$x = \frac{F_s}{k}$$

and Bingham fluid in the dashpot; that is to say $\dot{x} = g(F_d)$, where

$$g(u) = \begin{cases} 0, & |u| \le \gamma_0, \\ a(u - \gamma_0), & u > \gamma_0, \\ a(u + \gamma_0), & u < -\gamma_0, \end{cases}$$

where γ_0, a are some positive constants. Our problem is reduced to the pair of equations

$$\ddot{x} + F_d + kx = F, \tag{4.1}$$

$$\dot{x} = g(F_d). \tag{4.2}$$

We have global existence by Theorem 1; note that we also have global (forward) uniqueness by the argument of Lemma 2 – the proof obviously works for an arbitrary (non-constant) right-hand side, with a globally Lipschitz function $\phi(u) = ku$.

If $\dot{x} \neq 0$, we can invert (4.2) and our problem reduces to a single equation

$$\ddot{x} + \frac{\dot{x}}{a} + kx = F - \gamma_0 \operatorname{sgn}(\dot{x}). \tag{4.3}$$

This is a linear ODE with constant coefficients.

On the other hand, the systems admits equilibria of the form

$$x \equiv x_e,$$
$$F_d = F - kx_e,$$
$$|F_d| \le \gamma_0.$$

In other words: if $x(t_0) = x_e$ and $\dot{x}(t_0) = 0$ for some t_0, then the solution remains at $x = x_e$ as long as $|F - kx_e| \le \gamma_0$ is satisfied.

4.1.1 The case with constant right-hand side

Assume that $F \equiv F_0$. We have a strip of equilibria

$$\frac{F_0 - \gamma_0}{k} \le x_e \le \frac{F_0 + \gamma_0}{k}. \tag{4.4}$$

If $\dot{x} > 0$, the solutions are governed by

$$\ddot{x} + ax + kx = F_0 - \gamma_0;$$

while for $\dot{x} < 0$, we have

$$\ddot{x} + a x + k x = F_0 + \gamma_0.$$

Solutions of these equations (exponentially) stabilize to $(F_0 - \gamma_0)/k$, $(F_0 + \gamma_0)/k$, respectively. One deduces that any solution of the whole system is eventually (i.e., after a finite number of oscillations) trapped by one of the equilibria (4.4).

To be more precise: if the system is overdamped ($1/a^2 > 4k$), one can also have solutions that are monotone and reach one of the equilibria $(F_0 \pm \gamma_0)/k$ in infinite time.

4.2 Linear dashpot and a rigid-elastic spring

We set $\dot{x} = aF_d$, and $x = f(F_s)$, where

$$f(u) = \begin{cases} 0, & |u| \leq \phi_0 \\ a(u - \phi_0), & u > \phi_0, \\ a(u + \phi_0), & u < -\phi_0, \end{cases}$$

where ϕ_0, $k > 0$ are constants. Our problem can be recast as

$$\ddot{x} + \frac{\dot{x}}{a} + F_s = F, \tag{4.5}$$

$$x = f(F_s). \tag{4.6}$$

Observe that the system fits into our general scheme; in particular, f satisfies the structural assumptions given in Section 3. We thus have global existence and uniqueness (in case of constant right-hand side). There is a trivial equilibrium $x = 0$, which can be maintained as long as $F_s = F \in [-\phi_0, \phi_0]$. If $x \neq 0$, we can invert $f(\cdot)$, and reduce our problem to a single (constant coefficient) equation

$$\ddot{x} + \frac{\dot{x}}{a} + kx = F - \phi_0 \operatorname{sgn}(x).$$

4.2.1 The case with constant right-hand side

Let $F \equiv F_0$. If $x > 0$, we have

$$\ddot{x} + \frac{\dot{x}}{a} + kx = F_0 - \phi_0;$$

the solutions of this subproblem stabilize to $x_- := (F_0 - \phi_0)/k$. For $x < 0$, we have

$$\ddot{x} + \frac{\dot{x}}{a} + kx = F_0 + \phi_0;$$

which implies that x tends to $x_+ := (F_0 + \phi_0)/k$.

Concerning the asymptotic behavior of the full system (4.5 - 4.6), we can distinguish three cases:

1. If $x_- \leq 0 \leq x_+$, zero is a possible equilibrium and solutions infinitely oscillate closer and closer around it.

2. If $x_- > 0$, solutions are positive for t large enough and stabilize to x_-.

3. If $x_+ < 0$, solutions are negative for t large and stabilize to x_+.

5 Concluding Remarks

In this short paper we have considered the existence of a solution for an interesting class of problems concerning vibrating systems whose governing equations reduce to a system of differential-algebraic equations. In such vibrating systems one cannot express the forces in the components of the lumped parameter system as a function of kinematical quantities. On the other hand we have the kinematics being defined as a function of the forces. We have sought solutions under several special assumptions concerning the function that expresses the kinematical quantity in terms of the forces such as it being monotone. A great deal of work remains open, especially when the components of the lumped parameter system are such that one can only provide implicit relationship between the forces and its derivatives and the kinematical quantities.

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