Abstract. Let $f : X \to Y$ be a surjection of a zero-dimensional metrizable $X$ onto a metrizable $Y$ which maps clopen sets in $X$ to locally closed (or more generally, resolvable) sets in $Y$. We prove that if $X$ is completely metrizable, or hereditarily Baire, then $Y$ has also the respective property. This strengthens some recent results of A. Ostrovsky [0s] and provides an answer to his question.

We shall strengthen some results from a recent paper by A. Ostrovsky [0s], and in particular, we shall answer a question stated in this paper.

Let us recall that a set $E$ in a metrizable space $M$ is resolvable if for any nonempty closed set $F \subset M$, one of the sets $F \setminus (F \cap E)$, $F \setminus (F \setminus E)$ is nonempty, cf. [Ku], § 12, II and V.

Resolvable sets are simultaneously $F_\sigma$ and $G_\delta$ and the collection of resolvable sets in $M$ is an algebra containing all locally closed sets, i.e., intersections of open and closed sets.

Given a metric space $(X, d)$ we call $C \subset X$ metrically discrete if $\inf\{d(a, b) : a, b \in C, a \neq b\} > 0$.

Theorem 1. Let $f : X \to Y$ be a continuous map of a complete metric space $X$ onto a metrizable space $Y$ such that for each countable metrically discrete $C$ and its neighbourhood $V$ in $X$, there is $L$ such that $C \subset L \subset V$ and $f(L)$ is resolvable. Then $Y$ is completely metrizable.

This result is closely related to some results of E. Michael [Mi] linking the completeness with complete sieves formed by exhaustive covers.

A key element in our proof is Lemma 5, which is based on a variation of a reasoning of N. Ghoussoub and B. Maurey [G-M, Lemma I.1], cf. [Mi, Lemma 6.1]. With this lemma at hand, we are ready to use Theorem 1.6 from [Mi].

One can recover from Theorem 1 a theorem of E. Michael ([Mi, Corollary 1.7]), extending a result of N. Ghoussoub and B. Maurey [G-M] to non separable spaces. We have only to notice that every scattered set is clearly resolvable.

Corollary 2 (Michael; Ghoussoub and Maurey for separable spaces). Let $f : X \to Y$ be a continuous surjection of a complete metric space $X$ onto a metrizable space $Y$. If $f$ takes metrically discrete sets to scattered sets, then $Y$ is completely metrizable.

2000 Mathematics Subject Classification. Primary 54E50, 54C10.

Key words and phrases. metric spaces, completeness, hereditary Baire spaces.

The first author was supported by the research project MSM 0021620839 financed by MSMT, by GAČR 201/06/0198, and by GAČR 201/06/0018.
The following corollary answers (for separable $X$) a question at the end of [0s].

**Corollary 3.** Let $f : X \to Y$ be a continuous map from a completely metrizable zero-dimensional space $X$ onto a metrizable space $Y$. If $f$ takes clopen sets in $X$ to resolvable sets in $Y$, then $Y$ is completely metrizable.

Since every open neighbourhood of a metrically discrete set $D$ in a zero-dimensional space contains a clopen neighbourhood of $D$, the corollary follows readily from Theorem 1.

Let us recall that $X$ is hereditarily Baire (or $F_{\omega 1}$ space, cf. [0s]) if each closed subspace of $X$ is a Baire space. By Hurewicz’s theorem [Hu], a metrizable space is hereditarily Baire if and only if it contains no closed homeomorphic copy of rational numbers. We shall also derive from Theorem 1 the following corollary, cf. [0s, Theorem 2].

**Corollary 4.** Let $f : X \to Y$ be a continuous map from a zero-dimensional hereditarily Baire space onto a metrizable space. If $f$ takes clopen sets in $X$ to resolvable sets in $Y$, then $Y$ is hereditarily Baire.

Let us pass now to proofs of Theorem 1 and Corollary 4.

**Lemma 5.** Let $f : X \to Y$ satisfy the assumptions of Theorem 1 (except possibly for the completeness). Let $U \subset X$ be an open set in $X$, $S$ a nonempty subset of $f(U)$ and $\varepsilon > 0$. There is an open set $M \subset U$, covered by finitely many $\varepsilon$-balls in $X$, such that $f(M) \cap S$ has a nonempty relative interior in $S$.

**Proof.** Aiming at a contradiction assume that for some nonempty set $S \subset f(U)$ and $\varepsilon > 0$, there is no $M$ satisfying the assertion of the lemma.

Let us begin with the following observation. Suppose that we have given a finite set $F \subset U$ and a nonempty relatively open set $W$ in $S$. Then, using the assumption, we can pick $x \in U$ such that $f(x) \in W \setminus \bigcup_{c \in F} f(B(c, \varepsilon) \cap U)$.

Repeating this observation we can choose inductively $a_n \in U$, such that

1. $f(a_n) \not\in \bigcup_{j < n} f(B(a_j, \varepsilon) \cap U)$,

2. for any $n$ and $p$ in $\mathbb{N}$ there is $k \in \mathbb{N}$ such that $\rho(f(a_n), f(a_k)) < \frac{1}{p}$,

where $\rho$ is a fixed metric on $Y$ generating the topology.

More specifically, let us fix a surjection $u : \mathbb{N} \to U$ such that $u(n) < n$ for $n > 1$ and $u^{-1}(n)$ is infinite for $n \in \mathbb{N}$, cf. [Mi, proof of Lemma 6.1]. Choose $a_1 \in U$ arbitrarily. Then, at the $n$th stage of the construction, we set $F = \{a_1, \ldots, a_{n-1}\}$, we let $W$ be the $\frac{1}{n}$-ball in $S$ centered at $a_{u(n)}$, and we use the observation to pick $a_n \in U$ with $f(a_n) \in W \setminus \bigcup_{j < n} f(B(a_j, \varepsilon) \cap U)$.

Having completed the inductive construction, we shall consider the metrically discrete set $A = \{a_n : n \in \mathbb{N}\}$ and we let

3. $V_n = B(a_n, \frac{1}{n}) \cap U \setminus \bigcup \{f^{-1}(a_j) : j < n\}$.

The set $Q = f(A)$ is homeomorphic to the set of rationals, being countable and infinite by (1), and dense-in-itself by (2). Let us take $C \subset A$ such that both $f(C)$ and $f(A \setminus C)$ are dense in $Q$, and let $V = \bigcup \{V_n : a_n \in C\}$, cf. (3). Notice that $f(C) \cap f(A \setminus C) = \emptyset$, cf. (1).
By the assumptions, there is \( L \) such that \( C \subset L \subset V \) and \( f(L) \) is resolvable. But, by (1) and (3), \( f(L) \cap Q = f(C) \), hence in the closure \( F \) of \( Q \) both \( f(L) \cap F \) and \( F \setminus f(L) \) are dense, contradicting the resolvability of \( f(L) \).

**Proof of Theorem 1.** We shall apply Michael’s Theorem 1.6 from [Mi]. Given an open set \( U \subset X \), an \( \varepsilon > 0 \), and a nonempty subset \( S \) of \( f(U) \), we have by Lemma 5 an open set \( M \subset U \) and an open set \( W \subset Y \) such that \( \emptyset \neq W \subset M \subset f(M) \) and \( M \) is covered by finitely many \( \frac{\varepsilon}{2} \)-balls. Then \( V = f^{-1}(W) \cap M \subset U \) is covered by finitely many sets of diameter \( \leq \varepsilon \) and \( f(V) \cap S = W \cap S \) is a nonempty relatively open set in \( S \). Therefore, the assumptions of Michael’s theorem are satisfied for the collection \( U \) of all open sets in \( X \), and by the assertion of this theorem, \( Y \) is completely metrizable.

**Remark 6.** Any continuous surjection \( f : X \to Y \) from a completely metrizable \( X \) onto a metrizable \( Y \) which is either open or closed satisfies the assumptions of Theorem 1. Therefore, Theorem 1 yields the invariance of complete metrizability under open maps (Hausdorff’s theorem) and closed maps (Vainstein’s theorem), cf. [Mi].

**Remark 7.** We arrived at an answer to the question by Ostrovsky independently, and the present note is a result of our further discussions on the topic. The original setting of the
first of the authors concerned non-metrizable spaces. This more general approach requires an explicite use of complete sequences of covers and will be presented separately.

REFERENCES


Charles University, Faculty of Mathematics and Physics, Department of Mathematical Analysis, Sokolovská 83, 186 75 Praha 8, Czech Republic

E-mail address: holicky@karlin.mff.cuni.cz

Institute of Mathematics, University of Warsaw, Banacha 2, 02-097 Warszawa, Poland

E-mail address: R.Pol@mimuw.edu.pl