A NOTE ON INTERSECTIONS OF SIMPLICES

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Abstract. — We provide a corrected proof of [1, Théorème 9] stating that any metrizable infinite-dimensional simplex is affinely homeomorphic to the intersection of a decreasing sequence of Bauer simplices.

Résumé (*Sur certains intersections de simplexes*). — Nous exposons une démonstration rectifiée de [1, Théorème 9], montrant ainsi que tout simplexe de Choquet métrisable et de dimension infinie se représente comme intersection d'une suite décroissante de simplexes de Bauer.

1. Introduction

If X is a compact convex subset of a locally convex space over the real numbers, it is called a *Choquet simplex* (briefly *simplex*) if the dual $(A(X))^*$ to the space $A(X)$ of all affine continuous functions is a lattice.

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If, moreover, the set $\operatorname{ext} X$ of all extreme points of X is closed, X is termed a *Bauer simplex* (see [2] for more information on simplices).

The following theorem can be found as [1, Théorème 9]. By (ℓ^1, w^*) we mean ℓ^1 with the topology $\sigma(\ell^1, c_0)$.

THEOREM 1.1. — *Let X be a metrizable infinite-dimensional simplex. Then there exists a decreasing sequence $(T_n)_{n\in\mathbb{N}}$ of Bauer simplices in (ℓ^1, w^*) such that $\bigcap_{n=1}^{\infty} T_n$ is affinely homeomorphic to X.*

Unfortunately, the proof presented in [1] is not entirely correct, since the inclusion

$$S_{n+1} \cup F_{n+1} \subset \big(\operatorname{conv}(S_n \cup \{e^{n+1}\})\big) \cup F_{n+1}$$

on page 237 of [1] need not hold in general. The aim of our note is to indicate how to mend the proof of this theorem.

By [3, Theorem 5.2] (see also [2, Theorem 3.22]), for every metrizable infinite-dimensional simplex X there exists an inverse sequence $(X_n, \varphi_n)_{n\in\mathbb{N}}$ of $(n-1)$–dimensional simplices such that X is affinely homeomorphic to its inverse limit $\varprojlim X_n$. More precisely, every $\varphi_n : X_{n+1} \to X_n$ is an affine continuous surjection and X is affinely homeomorphic to

$$\{(x_n) \in \prod_{n=1}^{\infty} X_n : \varphi_n(x_{n+1}) = x_n, n \in \mathbb{N}\}. \tag{1}$$

Inverse sequences $(X_n, \varphi_n)_{n\in\mathbb{N}}$ and $(Y_n, \psi_n)_{n\in\mathbb{N}}$ are called *equivalent* if there exist affine homeomorphisms $\omega_n : X_n \to Y_n$ such that $\omega_n \circ \varphi_n = \psi_n \circ \omega_{n+1}$, $n \in \mathbb{N}$. Clearly, two equivalent inverse sequences have the same inverse limit up to an affine homeomorphism.

A description of a simplex by an inverse sequence yields a method of representing X by an infinite matrix A that is constructed inductively as follows.

In the first step, let $X_1 = \{u_1^1\}$.

Assume now that $n \in \mathbb{N}$ and $\{u_1^n, \dots, u_n^n\}$ is the enumeration of vertices of X_n chosen in the n–th step. We choose vertices $\{u_1^{n+1}, \dots, u_n^{n+1}\}$ of X_{n+1} such that $\varphi_n(u_i^{n+1}) = u_i^n$, $i = 1, \dots, n$. If $u_{n+1}^{n+1} \in X_{n+1}$ is the remaining vertex, let $a_{1,n}, \dots, a_{n,n}$ be positive numbers with $\sum_{i=1}^n a_{i,n} = 1$ such that

$$\varphi_n(u_{n+1}^{n+1}) = \sum_{i=1}^{n} a_{i,n} u_i^n.$$

Then

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ 0 & a_{2,2} & a_{2,3} & \cdots \\ 0 & 0 & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is the *representing matrix* of X. It is not difficult to see that A is uniquely determined by the inverse sequence $(X_n, \varphi_n)_{n\in\mathbb{N}}$.

Conversely, any such matrix describes a unique inverse sequence of simplices and thus codes a unique metrizable simplex. We refer the reader to [2], [3], [4] and [5] for detailed information on representing matrices.

We need the following observation based upon [4, Theorem 4.7].

Proposition 1.2. — *Let A be a representing matrix for a simplex X. Then there exists a matrix $B = \{b_{i,n}\}_{n=1,2,\dots}^{1\le i\le n}$ representing X such that $b_{i,n} > 0$ for all $1 \le i \le n$ and $n = 1, 2, \dots$.*

Proof. — It follows from [4, Theorem 4.7] that two matrices A and B represent the same simplex if $\sum_{n=1}^{\infty}\sum_{i=1}^{n} |a_{i,n} - b_{i,n}| < \infty$. Thus it is enough to slightly perturb coefficients of A to get the required matrix B. □

2. Proof of Theorem 1.1

We recall some notation from [1]. Let e^n, $n \in \mathbb{N}$, denote the standard basis vectors in ℓ^1 and let $e^0 = 0$. For $n \in \mathbb{N}$, let $E_n = \operatorname{conv}\{e^0, \dots, e^{n-1}\}$ and let $P_n : \ell^1 \to \ell^1$ be the natural projection on the space spanned by vectors $e^0, \dots, e^{n-1}$, precisely

$$P_n : (x_1, x_2, \dots) \mapsto (x_1, \dots, x_{n-1}, 0, 0, \dots), \quad (x_1, x_2, \dots) \in \ell^1.$$

In particular, P_1 maps ℓ^1 onto e^0.

We state an easy observation needed in the proof of Proposition 2.2

Lemma 2.1. — *Let X be a finite dimensional simplex in a vector space E containing 0 and x be a vector not contained in the linear span of X. Then for any y in the relative interior of X there exists $\varepsilon > 0$ such that $y + \varepsilon x \in \operatorname{conv}(X \cup \{x\})$.*

Proof. — If y is in the relative interior of X and $0 \in X$, there exists $\varepsilon \in (0,1)$ such that $(1-\varepsilon)^{-1}y \in X$. Then

$$y + \varepsilon x = (1-\varepsilon)\frac{y}{1-\varepsilon} + \varepsilon x \in \operatorname{conv}(X \cup \{x\}),$$

which finishes the proof. □

Now we start with the proof of Theorem 1.1. Given a metrizable simplex X, Proposition 1.2 provides an inverse sequence $(X_n, \varphi_n)_{n\in\mathbb{N}}$ such that X is its inverse limit and the associated representing matrix A has all entries $a_{i,n} > 0$ for all $n \in \mathbb{N}$ and $1 \le i \le n$.

Proposition 2.2. — *Let X be a metrizable infinite-dimensional simplex with a representing matrix A such that $a_{i,n} > 0$ for all $n \in \mathbb{N}$ and $1 \le i \le n$. Let $(X_n, \varphi_n)_{n\in\mathbb{N}}$ be the inverse sequence associated with A.*

Then there exist $(n-1)$–dimensional simplices $S_n \subset \ell^1$, $n \in \mathbb{N}$, such that

(i) *$S_n \subset E_n$, $n \in \mathbb{N}$,*
(ii) *S_n is a face of S_m, $n < m$,*
(iii) *$P_n S_m = S_n$, $n < m$,*
(iv) *$S_{n+1} \subset \operatorname{conv}(S_n \cup \{e^n\})$, $n \in \mathbb{N}$,*
(v) *the inverse sequences $(X_n, \varphi_n)_{n\in\mathbb{N}}$ and $(S_n, P_n)_{n\in\mathbb{N}}$ are equivalent.*

Proof. — We construct inductively simplices S_n together with mappings $\omega_n : X_n \to S_n$, $n \in \mathbb{N}$, observing that the resulting inverse sequence is equivalent to the original one.

We start the construction by setting $S_1 = E_1 = \{e^0\}$ and $S_2 = E_2 = \operatorname{conv}\{e^0, e^1\}$. Let $\omega_1 : X_1 \to S_1$ and $\omega_2 : X_2 \to S_2$ be the obvious affine homeomorphisms.

We assume that the construction has been completed up to the n–th stage. If $\omega_n : X_n \to S_n$ is the affine homeomorphism guaranteed by the inductive assumption and $\{u_1^n, \dots, u_n^n\}$ are the vertices of X_n, $\{\omega_n(u_1^n), \dots, \omega_n(u_n^n)\}$ are the vertices of S_n.

Let $\{u_1^{n+1}, \dots, u_n^{n+1}\}$ be the vertices of X_{n+1} that are mapped by φ_n onto the vertices $\{u_1^n, \dots, u_n^n\}$ of X_n and let u_{n+1}^{n+1} be the remaining vertex mapped onto the point $\sum_{i=1}^n a_{i,n} u_i^n$. Since all numbers $a_{1,n}, \dots, a_{n,n}$ are strictly positive, the point

$$\omega_n(\varphi_n(u_{n+1}^{n+1})) = \sum_{i=1}^n a_{i,n}\omega_n(u_i^n)$$

is contained in the relative interior of S_n. By Lemma 2.1, there exists $\varepsilon > 0$ such that

$$\omega_n(\varphi_n(u^{n+1}_{n+1})) + \varepsilon e^n \in \operatorname{conv}(S_n \cup \{e^n\}). \tag{2}$$

By defining

$$S_{n+1} = \operatorname{conv}(S_n \cup \{\omega_n(\varphi_n(u^{n+1}_{n+1})) + \varepsilon e^n\}) \tag{3}$$

we get an n–simplex with vertices

$$\{\omega_n(u^n_1), \dots, \omega_n(u^n_n), \omega_n(\varphi_n(u^{n+1}_{n+1})) + \varepsilon e^n\}.$$

We define $\omega_{n+1} : X_{n+1} \to S_{n+1}$ by conditions

$$\omega_{n+1}(u^{n+1}_i) = \omega_n(\varphi_n(u^{n+1}_i)), \quad i = 1, \dots, n,$$
$$\omega_{n+1}(u^{n+1}_{n+1}) = \omega_n(\varphi_n(u^{n+1}_{n+1})) + \varepsilon e^n.$$

By (2) and (3) and the inductive assumption,

$$S_{n+1} \subset \operatorname{conv}(S_n \cup \{e^n\}) \subset E_{n+1}.$$

Further, S_n is a face of S_{n+1}, $P_n S_{n+1} = S_n$ and $\omega_n \circ \varphi_n = P_n \circ \omega_{n+1}$.

Thus conditions (i)–(iv) are satisfied and the mappings ω_n, $n \in \mathbb{N}$, show that the sequences (X_n, φ_n) and (S_n, P_n) are equivalent. This finishes the proof. □

The rest of the proof Theorem 1.1 can proceed as in [1]. To clarify what is going on, we give two more propositions. The proof of Theorem 1.1 follows immediately from them.

Proposition 2.3. — *Let S_n, $n \in \mathbb{N}$, be weak* compact convex subsets of ℓ^1 satisfying conditions* (i), (ii′) *and* (iii), *where* (i) *and* (iii) *are conditions from Proposition 2.2 and*

(ii′) $S_n \subset S_m$ *for* $n \leq m$.

Then the inverse limit of the inverse sequence $(S_n, P_n)_{n\in\mathbb{N}}$ is affinely homeomorphic to the closure of $\bigcup_{n=1}^{\infty} S_n$ in the weak topology.*

Proof. — Let Y denote the weak*-closure of $\bigcup_{n=1}^{\infty} S_n$, and let X be the inverse limit $\varprojlim S_n$ represented in the form given by the formula (1). An affine homeomorphism $\varphi : Y \to X$ can be defined by the equation

$$\varphi(y) = (P_n(y))_{n\in\mathbb{N}}, \qquad y \in Y.$$

To see that φ is well defined, note that by (ii′) and (iii) we have $P_n(y) \in S_n$ whenever $y \in \bigcup_{n=1}^{\infty} S_n$, and hence, by the weak*-continuity of $P_n : \ell^1 \to \ell^1$, that $P_n(y) \in S_n$ for all $y \in Y$. Moreover, φ is clearly affine, continuous and one-to-one. To see that φ is onto, choose any $x = (x_n)_{n\in\mathbb{N}} \in X$. Let $y \in \mathbb{R}^{\mathbb{N}}$ have as n-th coordinate y_n the n-th coordinate of the vector x_{n+1}. Then $(y_1, \dots, y_n, 0 \dots) \in S_n$ for each $n \in \mathbb{N}$, therefore $y \in \ell_1$ by (i), and so $y \in Y$. Moreover, clearly $\varphi(y) = x$. This completes the proof. □

Proposition 2.4. — *Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of simplices in ℓ^1 satisfying conditions* (i)–(iv) *of Proposition 2.2. Set*

$$F_n = \overline{\operatorname{conv}}\{e^0, e^n, e^{n+1}, \dots\}, \quad n \in \mathbb{N},$$

where the bar denotes norm-closure, and

$$T_n = \operatorname{conv}(S_n \cup F_n), \quad n \in \mathbb{N}.$$

Then $(T_n)_{n\in\mathbb{N}}$ is a decreasing sequence of Bauer simplices in (ℓ^1, w^*) whose intersection is the weak-closure of $\bigcup_{n=1}^\infty S_n$.*

Proof. — It is clear that both F_n and S_n are Bauer simplices in (ℓ^1, w^*). Thus T_n is a Bauer simplex in (ℓ^1, w^*) as well. Moreover,

$$\begin{aligned} S_{n+1} \cup F_{n+1} &\subset (\operatorname{conv}(S_n \cup \{e^n\})) \cup F_{n+1} \\ &\subset \operatorname{conv}(S_n \cup F_n), \end{aligned}$$

and hence $T_{n+1} \subset T_n$ for $n \in \mathbb{N}$.

It remains to prove the final equality. Set $T = \bigcap_{n=1}^\infty T_n$ and denote by Y the weak*-closure of $\bigcup_{n=1}^\infty S_n$. Let $n \in \mathbb{N}$ be arbitrary. Then for each $m \geq n$ we have $S_n \subset S_m \subset T_m$. Thus $S_n \subset T$. It follows that $Y \subset T$.

To see the converse inclusion, take any $x \in T$. For each $n \in \mathbb{N}$ we have $x \in T_n$, $0 \in S_n$, and hence $P_n(x) \in S_n$. But the sequence $(P_n(x))_{n\in\mathbb{N}}$ is weak* convergent to x, so $x \in Y$. □

Finally, Theorem 1.1 follows immediately by combining Propositions 1.2, 2.2, 2.3 and 2.4.

Remark 2.5. — We note that it is not essential that we work in the space (ℓ^1, w^*). The norm structure of this space is used only in the definition of F_n, and can be replaced there by weak*-closure. So, it would be possible (and, perhaps, more natural) to work in the locally convex space $\mathbb{R}^{\mathbb{N}}$ equipped with the pointwise topology. Anyway, we decided to keep the setting from [1].

BIBLIOGRAPHY

[1] D. A. Edwards, *Systèmes projectifs d'ensembles convexes compacts*, Bull. Soc. Math. France **103** (1975), 225–240.

[2] V. P. Fonf, J. Lindenstrauss and R. R. Phelps, *Infinite dimensional convexity*, Handbook of the geometry of Banach spaces, vol. 1, Elsevier Science B.V., 2001, 600 – 670.

[3] A. J. Lazar and J. Lindenstrauss, *Banach spaces whose duals are L_1–spaces and their representing matrices*, Acta Math. **126** (1971), 165–193.

[4] J. Lindenstrauss, G. H. Olsen and Y. Sternfeld, *The Poulsen simplex*, Ann. Inst. Fourier (Grenoble) **28** (1978), 91 – 114.

[5] Y. Sternfeld, *Characterization of Bauer simplices and some other classes of Choquet simplices by their representing matrices*, Notes in Banach spaces, Univ. Texas Press, Austin, 1980, 306–358.