A NOTE ON INTERSECTIONS OF SIMPLICIES

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Abstract. — We provide a corrected proof of [1, Théorème 9] stating that any metrizable infinite-dimensional simplex is affinely homeomorphic to the intersection of a decreasing sequence of Bauer simplices.

Résumé (Sur certains intersections de simplexes). — Nous exposons une démonstration rectifiée de [1, Théorème 9], montrant ainsi que tout simplexe de Choquet métrisable et de dimension infinie se représente comme intersection d’une suite décroissante de simplexes de Bauer.

1. Introduction

If $X$ is a compact convex subset of a locally convex space over the real numbers, it is called a Choquet simplex (briefly simplex) if the dual $(A(X))^*$ to the space $A(X)$ of all affine continuous functions is a lattice.
If, moreover, the set \( \text{ext} X \) of all extreme points of \( X \) is closed, \( X \) is termed a \textit{Bauer simplex} (see \cite{2} for more information on simplices).

The following theorem can be found as \cite[Théorème 9]{1}. By \((\ell^1, w^*)\) we mean \( \ell^1 \) with the topology \( \sigma(\ell^1, c_0) \).

**Theorem 1.1.** — \textit{Let \( X \) be a metrizable infinite-dimensional simplex. Then there exists a decreasing sequence \((T_n)_{n \in \mathbb{N}}\) of Bauer simplices in \((\ell^1, w^*)\) such that \( \bigcap_{n=1}^{\infty} T_n \) is affinely homeomorphic to \( X \).}

Unfortunately, the proof presented in \cite{1} is not entirely correct, since the inclusion

\[
S_{n+1} \cup F_{n+1} \subset \left( \text{conv}(S_n \cup \{e^{n+1}\}) \right) \cup F_{n+1}
\]

on page 237 of \cite{1} need not hold in general. The aim of our note is to indicate how to mend the proof of this theorem.

By \cite[Theorem 5.2]{3} (see also \cite[Theorem 3.22]{2}), for every metrizable infinite-dimensional simplex \( X \) there exists an inverse sequence \((X_n, \varphi_n)_{n \in \mathbb{N}}\) of \((n-1)\)-dimensional simplices such that \( X \) is affinely homeomorphic to its inverse limit \( \lim_{\leftarrow} X_n \). More precisely, every \( \varphi_n : X_{n+1} \to X_n \) is an affine continuous surjection and \( X \) is affinely homeomorphic to

\[
\{(x_n) \in \prod_{n=1}^{\infty} X_n : \varphi_n(x_{n+1}) = x_n, n \in \mathbb{N}\}.
\]

Inverse sequences \((X_n, \varphi_n)_{n \in \mathbb{N}}\) and \((Y_n, \psi_n)_{n \in \mathbb{N}}\) are called \textit{equivalent} if there exist affine homeomorphisms \( \omega_n : X_n \to Y_n \) such that \( \omega_n \circ \varphi_n = \psi_n \circ \omega_{n+1}, n \in \mathbb{N} \). Clearly, two equivalent inverse sequences have the same inverse limit up to an affine homeomorphism.

A description of a simplex by an inverse sequence yields a method of representing \( X \) by an infinite matrix \( A \) that is constructed inductively as follows.

In the first step, let \( X_1 = \{u_1^1\} \).

Assume now that \( n \in \mathbb{N} \) and \( \{u_1^n, \ldots, u_n^n\} \) is the enumeration of vertices of \( X_n \) chosen in the \( n \)-th step. We choose vertices \( \{u_1^{n+1}, \ldots, u_n^{n+1}\} \) of \( X_{n+1} \) such that \( \varphi_n(u_i^{n+1}) = u_i^n, i = 1, \ldots, n \). If \( u_{n+1}^{n+1} \in X_{n+1} \) is the remaining vertex, let \( a_{1,n}, \ldots, a_{n,n} \) be positive numbers with \( \sum_{i=1}^{n} a_{i,n} = 1 \) such that

\[
\varphi_n(u_{n+1}^{n+1}) = \sum_{i=1}^{n} a_{i,n} u_i^n.
\]
Then

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ 0 & a_{2,2} & a_{2,3} & \cdots \\ 0 & 0 & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is the representing matrix of $X$. It is not difficult to see that $A$ is uniquely determined by the inverse sequence $(X_n, \varphi_n)_{n \in \mathbb{N}}$.

Conversely, any such matrix describes a unique inverse sequence of simplices and thus codes a unique metrizable simplex. We refer the reader to [2], [3], [4] and [5] for detailed information on representing matrices.

We need the following observation based upon [4, Theorem 4.7].

**Proposition 1.2.** — Let $A$ be a representing matrix for a simplex $X$. Then there exists a matrix $B = \{b_{i,n}\}_{1 \leq i \leq n}^{n=1,2,\ldots}$ representing $X$ such that $b_{i,n} > 0$ for all $1 \leq i \leq n$ and $n = 1, 2, \ldots$.

**Proof.** — It follows from [4, Theorem 4.7] that two matrices $A$ and $B$ represent the same simplex if $\sum_{n=1}^{\infty} \sum_{i=1}^{n} |a_{i,n} - b_{i,n}| < \infty$. Thus it is enough to slightly perturb coefficients of $A$ to get the required matrix $B$.

\[ \square \]

### 2. Proof of Theorem 1.1

We recall some notation from [1]. Let $e^n, n \in \mathbb{N}$, denote the standard basis vectors in $\ell^1$ and let $e^0 = 0$. For $n \in \mathbb{N}$, let $E_n = \text{conv}\{e^0, \ldots, e^{n-1}\}$, and let $P_n : \ell^1 \to \ell^1$ be the natural projection on the space spanned by vectors $e^0, \ldots, e^{n-1}$, precisely

$$P_n : (x_1, x_2, \ldots) \mapsto (x_1, \ldots, x_{n-1}, 0, 0, \ldots), \quad (x_1, x_2, \ldots) \in \ell^1.$$  

In particular, $P_1$ maps $\ell^1$ onto $e^0$.

We state an easy observation needed in the proof of Proposition 2.2.

**Lemma 2.1.** — Let $X$ be a finite dimensional simplex in a vector space $E$ containing $0$ and $x$ be a vector not contained in the linear span of $X$. Then for any $y$ in the relative interior of $X$ there exists $\varepsilon > 0$ such that $y + \varepsilon x \in \text{conv}(X \cup \{x\})$. 


Proof. — If $y$ is in the relative interior of $X$ and $0 \in X$, there exists $\varepsilon \in (0,1)$ such that $(1-\varepsilon)^{-1}y \in X$. Then

$$y + \varepsilon x = (1-\varepsilon)\frac{y}{1-\varepsilon} + \varepsilon x \in \text{conv}(X \cup \{x\}),$$

which finishes the proof.

Now we start with the proof of Theorem 1.1. Given a metrizable simplex $X$, Proposition 1.2 provides an inverse sequence $(X_n, \varphi_n)_{n \in \mathbb{N}}$ such that $X$ is its inverse limit and the associated representing matrix $A$ has all entries $a_{i,n} > 0$ for all $n \in \mathbb{N}$ and $1 \leq i \leq n$.

PROPOSITION 2.2. — Let $X$ be a metrizable infinite-dimensional simplex with a representing matrix $A$ such that $a_{i,n} > 0$ for all $n \in \mathbb{N}$ and $1 \leq i \leq n$. Let $(X_n, \varphi_n)_{n \in \mathbb{N}}$ be the inverse sequence associated with $A$.

Then there exist $(n-1)$-dimensional simplices $S_n \subset \ell^1$, $n \in \mathbb{N}$, such that

(i) $S_n \subset E_n$, $n \in \mathbb{N}$,
(ii) $S_n$ is a face of $S_m$, $n < m$,
(iii) $P_nS_m = S_n$, $n < m$,
(iv) $S_{n+1} \subset \text{conv}(S_n \cup \{e^n\})$, $n \in \mathbb{N}$,
(v) the inverse sequences $(X_n, \varphi_n)_{n \in \mathbb{N}}$ and $(S_n, P_n)_{n \in \mathbb{N}}$ are equivalent.

Proof. — We construct inductively simplices $S_n$ together with mappings $\omega_n : X_n \to S_n$, $n \in \mathbb{N}$, observing that the resulting inverse sequence is equivalent to the original one.

We start the construction by setting $S_1 = E_1 = \{e^0\}$ and $S_2 = E_2 = \text{conv}\{e^0, e^1\}$. Let $\omega_1 : X_1 \to S_1$ and $\omega_2 : X_2 \to S_2$ be the obvious affine homeomorphisms.

We assume that the construction has been completed up to the $n$-th stage. If $\omega_n : X_n \to S_n$ is the affine homeomorphism guaranteed by the inductive assumption and $\{u_{1,n}, \ldots, u_{n,n}\}$ are the vertices of $X_n$, $\{\omega_n(u_{1,n}), \ldots, \omega_n(u_{n,n})\}$ are the vertices of $S_n$.

Let $\{u_{1,1}, \ldots, u_{n,1}\}$ be the vertices of $X_{n+1}$ that are mapped by $\varphi_n$ onto the vertices $\{u_{1,n}, \ldots, u_{n,n}\}$ of $X_n$ and let $u_{n+1,n}$ be the remaining vertex mapped onto the point $\sum_{i=1}^{n} a_{i,n} u_{i,n}$. Since all numbers $a_{1,n}, \ldots, a_{n,n}$ are strictly positive, the point

$$\omega_n(\varphi_n(u_{n+1,n})) = \sum_{i=1}^{n} a_{i,n} \omega_n(u_{i,n})$$
is contained in the relative interior of $S_n$. By Lemma 2.1, there exists $\varepsilon > 0$ such that
\begin{equation}
\omega_n(\varphi_n(u_{n+1}^n)) + \varepsilon e^n \in \text{conv}(S_n \cup \{e^n\}).
\end{equation}
By defining
\begin{equation}
S_{n+1} = \text{conv}(S_n \cup \{\omega_n(\varphi_n(u_{n+1}^n)) + \varepsilon e^n\})
\end{equation}
we get an $n$–simplex with vertices
\[\{\omega_n(u_1^n), \ldots, \omega_n(u_n^n), \omega_n(\varphi_n(u_{n+1}^n)) + \varepsilon e^n\}.
\]
We define $\omega_{n+1} : X_{n+1} \to S_{n+1}$ by conditions
\[\omega_{n+1}(u_i^{n+1}) = \omega_n(\varphi_n(u_i^{n+1})), \quad i = 1, \ldots, n,
\]
\[\omega_{n+1}(u_{n+1}^{n+1}) = \omega_n(\varphi_n(u_{n+1}^{n+1})) + \varepsilon e^n.
\]
By (2) and (3) and the inductive assumption,
\[S_{n+1} \subset \text{conv}(S_n \cup \{e^n\}) \subset E_{n+1}.
\]
Further, $S_n$ is a face of $S_{n+1}$, $P_nS_{n+1} = S_n$ and $\omega_n \circ \varphi_n = P_n \circ \omega_{n+1}$.
Thus conditions (i)–(iv) are satisfied and the mappings $\omega_n$, $n \in \mathbb{N}$, show that the sequences $(X_n, \varphi_n)$ and $(S_n, P_n)$ are equivalent. This finishes the proof.

The rest of the proof Theorem 1.1 can proceed as in [1]. To clarify what is going on, we give two more propositions. The proof of Theorem 1.1 follows immediately from them.

**Proposition 2.3.** — Let $S_n$, $n \in \mathbb{N}$, be weak* compact convex subsets of $\ell^1$ satisfying conditions (i), (ii') and (iii), where (i) and (iii) are conditions from Proposition 2.2 and (ii') $S_n \subset S_m$ for $n \leq m$.

Then the inverse limit of the inverse sequence $(S_n, P_n)_{n \in \mathbb{N}}$ is affinely homeomorphic to the closure of $\bigcup_{n=1}^{\infty} S_n$ in the weak* topology.

**Proof.** — Let $Y$ denote the weak*–closure of $\bigcup_{n=1}^{\infty} S_n$, and let $X$ be the inverse limit $\lim_{\leftarrow} S_n$ represented in the form given by the formula (1). An affine homeomorphism $\varphi : Y \to X$ can be defined by the equation
\[\varphi(y) = (P_n(y))_{n \in \mathbb{N}}, \quad y \in Y.
\]
To see that $\varphi$ is well defined, note that by (ii') and (iii) we have $P_n(y) \in S_n$ whenever $y \in \bigcup_{n=1}^{\infty} S_n$, and hence, by the weak*–continuity of $P_n : \ell^1 \to \ell^1$, that $P_n(y) \in S_n$ for all $y \in Y$. Moreover, $\varphi$ is clearly
affine, continuous and one-to-one. To see that \( \varphi \) is onto, choose any \( x = (x_n)_{n \in \mathbb{N}} \in X \). Let \( y \in \mathbb{R}^N \) have as \( n \)-th coordinate \( y_n \) the \( n \)-th coordinate of the vector \( x_{n+1} \). Then \( (y_1, \ldots, y_n, 0 \ldots) \in S_n \) for each \( n \in \mathbb{N} \), therefore \( y \in \ell_1 \) by (i), and so \( y \in Y \). Moreover, clearly \( \varphi(y) = x \). This completes the proof.

**Proposition 2.4.** — Let \( (S_n)_{n \in \mathbb{N}} \) be a sequence of simplices in \( \ell^1 \) satisfying conditions (i)–(iv) of Proposition 2.2. Set

\[
F_n = \overline{\text{conv}} \{e_0, e_n, e_{n+1}, \ldots\}, \quad n \in \mathbb{N},
\]

where the bar denotes norm-closure, and

\[
T_n = \text{conv}(S_n \cup F_n), \quad n \in \mathbb{N}.
\]

Then \( (T_n)_{n \in \mathbb{N}} \) is a decreasing sequence of Bauer simplices in \( (\ell^1, w^*) \) whose intersection is the weak*-closure of \( \bigcup_{n=1}^\infty S_n \).

**Proof.** — It is clear that both \( F_n \) and \( S_n \) are Bauer simplices in \( (\ell^1, w^*) \). Thus \( T_n \) is a Bauer simplex in \( (\ell^1, w^*) \) as well. Moreover,

\[
S_{n+1} \cup F_{n+1} \subset \text{conv}(S_n \cup \{e_n\}) \cup F_{n+1} 
\subset \text{conv}(S_n \cup F_n),
\]

and hence \( T_{n+1} \subset T_n \) for \( n \in \mathbb{N} \).

It remains to prove the final equality. Set \( T = \bigcap_{n=1}^\infty T_n \) and denote by \( Y \) the weak*-closure of \( \bigcup_{n=1}^\infty S_n \). Let \( n \in \mathbb{N} \) be arbitrary. Then for each \( m \geq n \) we have \( S_n \subset S_m \subset T_m \). Thus \( S_n \subset T \). It follows that \( Y \subset T \).

To see the converse inclusion, take any \( x \in T \). For each \( n \in \mathbb{N} \) we have \( x \in T_n, 0 \in S_n \), and hence \( P_n(x) \in S_n \). But the sequence \( (P_n(x))_{n \in \mathbb{N}} \) is weak*-convergent to \( x \), so \( x \in Y \).

Finally, Theorem 1.1 follows immediately by combining Propositions 1.2, 2.2, 2.3 and 2.4.

**Remark 2.5.** — We note that it is not essential that we work in the space \( (\ell^1, w^*) \). The norm structure of this space is used only in the definition of \( F_n \), and can be replaced there by weak*-closure. So, it would be possible (and, perhaps, more natural) to work in the locally convex space \( \mathbb{R}^N \) equipped with the pointwise topology. Anyway, we decided to keep the setting from [1].
BIBLIOGRAPHY


