STRUCTURE OF THE SET OF NORM-ATTAINING FUNCTIONALS ON STRICTLY CONVEX SPACES

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ABSTRACT. Let X be a separable non-reflexive Banach space. We show that there is no Borel class which contains the set of norm-attaining functionals for every strictly convex renorming of X.

R. Kaufman proved in [3] that every non-reflexive Banach space admits an equivalent norm such that the set of norm-attaining functionals is not Borel. He also observed that the set of norm-attaining functionals is Borel in the case that the space is separable and strictly convex. G. Debs, G. Godefroy and J. Saint Raymond asked in [1] whether there exist strictly convex norms with the set of norm-attaining functionals of arbitrarily high Borel class. We answer this question affirmatively in Theorem 1.

Let $(X, \|\cdot\|)$ be a real normed linear space. We denote by B_X and by S_X the closed unit ball and the unit sphere of X and we recall that the set of norm-attaining functionals with respect to the norm $\|\cdot\|$ is

$$\mathrm{NA}(\|\cdot\|) = \{f \in X^* : \exists x \in B_X \; f(x) = \|f\|\}.$$

The main result follows. Its proof is given at the end of the paper.

Theorem 1. *Let X be a separable non-reflexive Banach space and $\alpha < \omega_1$. Then there exists an equivalent strictly convex norm $|||\cdot|||$ on X such that $\mathrm{NA}(|||\cdot|||)$ is not of the additive Borel class α.*

Of course, it is not essential whether we consider additive or multiplicative class.

One of the ingredients of our construction of the new unit ball is the following result of R. Kaufman. By the Baire space we mean the countable topological product $\mathbb{N}^{\mathbb{N}}$ of natural numbers endowed with the discrete topology.

Proposition 2 ([3, 4])**.** *Let X, Y be non-reflexive Banach spaces such that $Y \subset\subset X$. Then there exists a continuous mapping $\psi : \mathbb{N}^{\mathbb{N}} \to B_Y$ such that*

(i) *if $(\lambda_m)_{m\in\mathbb{N}}$ is a sequence of probability measures on $\mathbb{N}^{\mathbb{N}}$ such that the integrals $\int_{\mathbb{N}^{\mathbb{N}}} \psi d\lambda_m, m \in \mathbb{N}$, belong to a compact subset of Y, then the sequence $(\lambda_m)_{m\in\mathbb{N}}$ is uniformly tight, i.e., for every $\varepsilon > 0$, there is a compact set $K \subset \mathbb{N}^{\mathbb{N}}$ such that $\lambda_m(K) > 1 - \varepsilon$ for all m,*

(ii) *if $F \subset \mathbb{N}^{\mathbb{N}}$ is closed, $\varrho : F \to X$ is a continuous mapping with $\varrho(F)$ relatively compact and θ denotes $\psi|_F + \varrho$, then, for every $x \in \overline{\mathrm{co}}\,\theta(F)$, there is a probability measure λ_x on F such that*

$$x = \int_F \theta d\lambda_x.$$

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In fact, (ii) is a consequence of (i). Since the mappings are continuous and $\mathbb{N}^{\mathbb{N}}$ is separable, it is not essential whether the integrals are understood in the Pettis or in the Bochner sense. We do not distinguish the Baire space and the Polish space of all infinite sets of natural numbers (denoted by J in [3] and by Σ in [4]) because they are homeomorphic (the topology on the space of all infinite sets of natural numbers is induced by the topology on $2^{\mathbb{N}}$).

The proof of the following proposition is given in the form of a series of claims. There are some connections between it and the main result from [4] (more details are discussed in Remark 8).

By an analytic set we mean a continuous image of a Polish space F (i.e., separable completely metrizable topological space). By [5, Theorem 7.9], we can consider F to be a closed subset of $\mathbb{N}^{\mathbb{N}}$.

Proposition 3. *Let X be a non-reflexive Banach space and $\varphi, \phi \in X^*$ be linearly independent. Let $M \subset [0, \pi/2]$ be analytic and dense in $[0, \pi/2]$. Then there is an absolutely convex closed bounded set $R \subset X$ such that, for every $t \in [0, \pi/2]$, $(\cos t)\varphi + (\sin t)\phi$ has the supremum 1 on R, and it is attained if and only if $t \in M$.*

Since M is analytic, there are a closed subset F of $\mathbb{N}^{\mathbb{N}}$ and a continuous mapping $p : F \to [0, \pi/2]$ such that $p(F) = M$.

Notation 4. We denote

$$Y = \operatorname{Ker} \varphi \cap \operatorname{Ker} \phi.$$

The space X can be viewed as

$$X = Y \oplus \mathbb{R}^2,$$

where

$$\varphi(0; 1, 0) = 1, \quad \varphi(0; 0, 1) = 0,$$

$$\phi(0; 1, 0) = 0, \quad \phi(0; 0, 1) = 1$$

(for $y \in Y, r, s \in \mathbb{R}$, we use $(y; r, s)$ instead of $(y, (r, s))$). We put

$$u_t = (\cos t)\varphi + (\sin t)\phi \quad \text{for } t \in [0, 2\pi).$$

Since X is not reflexive, Y is not reflexive, too. Let $\psi : \mathbb{N}^{\mathbb{N}} \to B_Y$ be as in Proposition 2. We define

$$\theta(\eta) = \big(\psi(\eta); \cos p(\eta), \sin p(\eta)\big) \quad \text{for } \eta \in F,$$

$$P = \theta(F), \quad R = \overline{\text{co}}\,\big(P \cup (-P)\big).$$

Further on, we consider the Euclidean norm on $\mathbb{R}^n (n = 2, 3)$ and we denote it by $|\cdot|$.

Claim 5. *Let R' be such that $P \subset R' \subset Y \times B_{\mathbb{R}^2}$. If $t \in [0, \pi/2]$, then u_t has the supremum 1 on R', and it is attained if $t \in M$.*

Proof. For $x = (y; r\cos\alpha, r\sin\alpha) \in Y \times B_{\mathbb{R}^2}$, we have $u_t(x) = r(\cos\alpha\cos t + \sin\alpha\sin t) = r\cos(\alpha - t) \le 1$. Since $R' \subset Y \times B_{\mathbb{R}^2}$, the inequality $\sup u_t(R') \le 1$ holds. On the other hand, for $\eta \in F$, we have $\theta(\eta) \in P \subset R'$ and $u_t(\theta(\eta)) = u_t(\psi(\eta); \cos p(\eta), \sin p(\eta)) = \cos p(\eta)\cos t + \sin p(\eta)\sin t = \cos(p(\eta) - t)$. The inequality $\sup u_t(R') \ge 1$ follows from the fact that $M = p(F)$ is dense in $[0, \pi/2]$.

Now, let $t \in M = p(F)$. For $\eta \in p^{-1}(t)$, we have $\theta(\eta) \in P \subset R'$ and $u_t(\theta(\eta)) = u_t(\psi(\eta); \cos p(\eta), \sin p(\eta)) = \cos^2 t + \sin^2 t = 1 = \sup u_t(R')$. $\square$

Claim 6. *Let $t \in [0, 2\pi)$.*

(a) *If $x \in \overline{\mathrm{co}}\, P$ satisfies $u_t(x) \geq 1$, then $x \in \overline{\mathrm{co}}\, \theta(p^{-1}(t))$.*

(b) *If $t \notin M$, then $u_t(x) < 1$ for every $x \in \overline{\mathrm{co}}\, P$.*

Proof. (a) Clearly, the image of the mapping $\varrho : \eta \in F \mapsto (0; \cos p(\eta), \sin p(\eta))$ is relatively compact. By the choice of ψ and P, there is a probability measure λ_x on F such that $x = \int_F \theta d\lambda_x$. We obtain $1 \leq u_t(x) = \int_F u_t(\theta(\eta)) d\lambda_x = \int_F (\cos p(\eta) \cos t + \sin p(\eta) \sin t) d\lambda_x = \int_F \cos(p(\eta) - t) d\lambda_x$, and thus $\lambda_x(\{\eta \in F : \cos(p(\eta) - t) = 1\}) = 1$. Since $p(\eta) - t \in (-2\pi, \pi/2]$ for $\eta \in F$, $\cos(p(\eta) - t) = 1$ is the same as $p(\eta) = t$, i.e. $\eta \in p^{-1}(t)$. We get $x = \int_F \theta d\lambda_x = \int_{p^{-1}(t)} \theta d\lambda_x \in \overline{\mathrm{co}}\, \theta(p^{-1}(t))$.

(b) If $t \notin M = p(F)$, then $\overline{\mathrm{co}}\, \theta(p^{-1}(t))$ is empty. Considering (a), we see that $u_t(x) < 1$ for every $x \in \overline{\mathrm{co}}\, P$. □

Claim 7. (a) $R \cap (Y \times S_{\mathbb{R}^2}) = (\overline{\mathrm{co}}\, P \cup (-\overline{\mathrm{co}}\, P)) \cap (Y \times S_{\mathbb{R}^2})$.

(b) *If $t \in [0, \pi/2] \setminus M$, then $u_t(x) < 1$ for every $x \in R$.*

Proof. One can easily prove that

$$R = \{\lambda x : \lambda \in [-1, 1], x \in \overline{\mathrm{co}}\, P\}.$$

(a) We have $(\lambda\, \overline{\mathrm{co}}\, P) \cap (Y \times S_{\mathbb{R}^2}) \subset (Y \times \lambda B_{\mathbb{R}^2}) \cap (Y \times S_{\mathbb{R}^2})$, which is empty in the case that $|\lambda| < 1$. We get $R \cap (Y \times S_{\mathbb{R}^2}) = \bigcup_{\lambda \in [-1,1]} (\lambda\, \overline{\mathrm{co}}\, P) \cap (Y \times S_{\mathbb{R}^2}) = \bigcup_{\lambda \in \{-1,1\}} (\lambda\, \overline{\mathrm{co}}\, P) \cap (Y \times S_{\mathbb{R}^2}) = (\overline{\mathrm{co}}\, P \cup (-\overline{\mathrm{co}}\, P)) \cap (Y \times S_{\mathbb{R}^2})$.

(b) It is enough to prove that $-1 < u_t(x) < 1$ for every $x \in \overline{\mathrm{co}}\, P$ (and thus $u_t(\lambda x) < 1$ for $x \in \overline{\mathrm{co}}\, P$ and $\lambda \in [-1, 1]$). The inequality $u_t(x) < 1$ was proved in Claim 6(b). As $M \subset [0, \pi/2]$, we have $t + \pi \in [0, 2\pi) \setminus M$, and thus $-u_t(x) = u_{t+\pi}(x) < 1$ by Claim 6(b) again. □

Now, Proposition 3 follows from Claim 5 and Claim 7(b).

Remark 8. (a) If $\varepsilon > 0$ is small enough, then $\overline{\mathrm{co}}\, (R \cup \varepsilon B_X)$ has the same property as R. Taking $|||\cdot|||$ as the norm which has $\overline{\mathrm{co}}\, (R \cup \varepsilon B_X)$ for its unit ball, we get a norm such that, for every $t \in [0, \pi/2]$, $(\cos t)\varphi + (\sin t)\phi \in \mathrm{NA}(|||\cdot|||)$ if and only if $t \in M$. Considering $M \subset [0, \pi/2]$ to be dense, analytic and non-Borel, we obtain the result from [3].

(b) Proposition 3 (and also Proposition 9 below) can be generalized as follows. It holds: *Let $(X, \|\cdot\|)$ be a non-reflexive Banach space and $\varphi_1, \varphi_2, \ldots, \varphi_n \in X^*$ be linearly independent. Let $M \subset \mathrm{co}\{\varphi_1, \ldots, \varphi_n\}$ be analytic. Then there is an equivalent norm $|||\cdot|||$ on X such that, for every $f \in \mathrm{co}\{\varphi_1, \ldots, \varphi_n\}$, $f \in \mathrm{NA}(|||\cdot|||)$ if and only if $f \in M$.* Assuming that M is dense in $\mathrm{co}\{\varphi_1, \ldots, \varphi_n\}$, we can prove this in a similar way as Proposition 3. In the general case, we realize that $M \cup (\mathrm{co}\{\varphi_1, \ldots, \varphi_n, \varphi_{n+1}\} \setminus \mathrm{co}\{\varphi_1, \ldots, \varphi_n\})$ is dense in $\mathrm{co}\{\varphi_1, \ldots, \varphi_n, \varphi_{n+1}\}$, where $\varphi_{n+1} \in X^*$ is chosen so that $\varphi_1, \ldots, \varphi_n, \varphi_{n+1}$ are linearly independent.

(c) In [1], the authors also ask whether every separable non-reflexive Banach space with separable dual admits a Fréchet smooth norm such that the set of norm-attaining functionals is not Borel. This question is answered affirmatively in [4]. There is a simple way how to give the positive answer with use of Proposition 3. We can proceed as follows. Let X be a separable non-reflexive Banach space with separable dual. We choose $M \subset [0, \pi/2]$ to be analytic, non-Borel and dense in $[0, \pi/2]$ and $\varphi, \phi \in X^*$ to be linearly independent. As M is not Borel, it is enough to find an equivalent Fréchet smooth norm $|||\cdot|||$ on X such that, for every $t \in [0, \pi/2]$, $(\cos t)\varphi + (\sin t)\phi \in \mathrm{NA}(|||\cdot|||)$ if and only if $t \in M$.

By [2, Theorem II.2.6], there is an equivalent norm $\|\cdot\|$ on X such that the dual norm $\|\cdot\|$ is l.u.r. on X^*. Also, there is an equivalent norm $\|\cdot\|'$ on X such that the dual norm $\|\cdot\|'$ is l.u.r. on X^*, too, and, for every $t \in [0, \pi/2]$, $(x_n)_{n\in\mathbb{N}}$ is convergent in X whenever $\|x_n\|' \leq 1$ for $n \in \mathbb{N}$ and $((\cos t)\varphi + (\sin t)\phi)(x_n) \to \|(\cos t)\varphi + (\sin t)\phi\|'$. Indeed, this can be shown for the norm $\|(y; r, s)\|' = |(\|y\|, r, s)|$, $(y; r, s) \in Y \times \mathbb{R}^2$, where Y is as in Notation 4.

Let R be as in Proposition 3. We define $|||\cdot|||$ to satisfy

$$B_{(X,|||\cdot|||)} = \overline{B_{(X,\|\cdot\|')} + R}.$$

For $u \in X^*$, we have $|||u||| = \|u\|' + \sup_{x\in R} u(x)$. From here, it can be shown that $|||\cdot|||$ is l.u.r. on X^*. Consequently, $|||\cdot|||$ is Fréchet smooth ([2, Proposition II.1.5]). It is straightforward to check that, for every $t \in [0, \pi/2]$, $(\cos t)\varphi + (\sin t)\phi \in \mathrm{NA}(|||\cdot|||)$ if and only if $t \in M$. So the norm $|||\cdot|||$ works.

(d) In fact, this method is a simple analogy of the method from [4]. Our method allows us to choose which analytic subset of an arc will be the intersection of this arc with the set of norm-attaining functionals. In [4], these functionals are chosen from a considerably greater set. It is proved: *If X is a separable non-reflexive Banach space with separable dual, then there is a set $H \subset X^*$, homeomorphic to the Hilbert cube $[-1, 1]^{\mathbb{N}}$, such that, for every analytic subset M of H, there is an equivalent Fréchet smooth norm $|||\cdot|||$ on X such that $H \cap \mathrm{NA}(|||\cdot|||) = M$.* In this case, to find the norm corresponding to our norm $\|\cdot\|'$ (mentioned in (c)) is much more complicated. One of the reasons is that the analogy of our space Y above has infinite codimension, and thus it does not have to be complemented.

Proposition 9. *Let $(X, \|\cdot\|)$ be a strictly convex non-reflexive Banach space and $\varphi, \phi \in X^*$ be linearly independent. Let $M \subset [0, \pi/2]$ be Borel and dense in $[0, \pi/2]$. Then there is an equivalent strictly convex norm $|||\cdot|||$ on X such that, for every $t \in [0, \pi/2]$, $(\cos t)\varphi + (\sin t)\phi \in \mathrm{NA}(|||\cdot|||)$ if and only if $t \in M$.*

The proof of the proposition is also given in the form of a series of claims.

Since M is Borel, there are closed subset F of $\mathbb{N}^{\mathbb{N}}$ and a one-to-one continuous mapping $p : F \to [0, \pi/2]$ such that $p(F) = M$ ([5, Theorem 13.7]). We define $Y, u_t, \psi, \theta, P, R$ as in Notation 4. Clearly, Claims 5 – 7 hold. The condition that p is a one-to-one mapping makes the situation more concrete and allows us to improve some of them.

Claim 10. $(\overline{\mathrm{co}}\, P) \cap (Y \times S_{\mathbb{R}^2}) = P$.

Proof. It is enough to prove $(\overline{\mathrm{co}}\, P) \cap (Y \times S_{\mathbb{R}^2}) \subset P$ because the other inclusion is obvious. Let $x \in (\overline{\mathrm{co}}\, P) \cap (Y \times S_{\mathbb{R}^2})$. There are $y \in Y$ and $t \in [0, 2\pi)$ such that $x = (y; \cos t, \sin t)$. We have $u_t(x) = \cos^2 t + \sin^2 t = 1$. By Claim 6(a), $x \in \overline{\mathrm{co}}\, \theta(p^{-1}(t))$. Let η denote the only element of $p^{-1}(t)$. We obtain $x \in \overline{\mathrm{co}}\, \theta(p^{-1}(t)) = \overline{\mathrm{co}}\, \{\theta(\eta)\} = \{\theta(\eta)\} \subset P$. □

Claim 11. $R \cap (Y \times S_{\mathbb{R}^2}) = P \cup (-P)$.

Proof. It follows immediately from Claims 10 and 7(a). □

In the proof of the following claim, we need a continuous function $f : [0, 2] \times [0, 1] \to [0, 1]$ with properties

(a) $f(x, y) \leq 1 - y$ for $(x, y) \in [0, 2] \times [0, 1]$,

(b) $f(\lambda a + (1-\lambda)b) > \lambda f(a) + (1-\lambda) f(b)$ for $a, b \in [0, 2] \times [0, 1)$, $a \neq b$, $\lambda \in (0, 1)$,

(c) $f(x_1, y) > f(x_2, y)$ when $x_1 < x_2$ and $y < 1$, $f(x, y_1) > f(x, y_2)$ when $y_1 < y_2$.

An explicit example of such a function is

$$f(x,y) = 1 - y - (1-y)^2 \left[\frac{1}{6} + \frac{1}{6-x}\right].$$

It is easy to check that the partial derivatives of f are negative on $[0,2] \times [0,1)$ and that

$$\frac{\partial^2 f}{\partial (r,s)^2}(x,y) = -\frac{2}{6-x}\left[s - \frac{1-y}{6-x} r\right]^2 - \frac{1}{3}s^2,$$

which is negative on $[0,2] \times [0,1)$ (by $\frac{\partial^2 f}{\partial (r,s)^2}(x,y)$ we mean the second derivative of f at (x,y) in the direction (r,s)).

Claim 12. *There is a continuous function $\rho : 2B_Y \times B_{\mathbb{R}^2} \to [0,1]$ with properties*

(a) $\rho(y; r, s) \le 1 - |(r,s)|$ *for* $(y; r, s) \in 2B_Y \times B_{\mathbb{R}^2}$,

(b) $\rho(\lambda a + (1-\lambda) b) > \lambda \rho(a) + (1-\lambda)\rho(b)$ *for* $a, b \in 2B_Y \times (B_{\mathbb{R}^2} \setminus S_{\mathbb{R}^2}), a \neq b, \lambda \in (0,1)$,

(c) $\rho(x) = \rho(-x)$ *for* $x \in 2B_Y \times B_{\mathbb{R}^2}$.

Proof. We put

$$\rho(y; r, s) = f\big(\|y\|, |(r,s)|\big), \quad (y; r, s) \in 2B_Y \times B_{\mathbb{R}^2}.$$

Properties (a), (c) are obvious, let us check (b). Let $(y_1, z_1), (y_2, z_2) \in 2B_Y \times B_{\mathbb{R}^2}, (y_1, z_1) \neq (y_2, z_2), |z_1| < 1, |z_2| < 1, \lambda \in (0,1)$. We need to check the inequality

$$f\big(\|\lambda y_1 + (1-\lambda) y_2\|, |\lambda z_1 + (1-\lambda) z_2|\big) > \lambda f\big(\|y_1\|, |z_1|\big) + (1-\lambda) f\big(\|y_2\|, |z_2|\big).$$

If $\|y_1\| \neq \|y_2\|$ or $|z_1| \neq |z_2|$, then $f(\|\lambda y_1 + (1-\lambda) y_2\|, |\lambda z_1 + (1-\lambda) z_2|) \ge f(\lambda \|y_1\| + (1-\lambda)\|y_2\|, \lambda |z_1| + (1-\lambda)|z_2|) > \lambda f(\|y_1\|, |z_1|) + (1-\lambda) f(\|y_2\|, |z_2|)$ by the properties of the function f. If $\|y_1\| = \|y_2\|$ and $|z_1| = |z_2|$, then, by the strict convexity of $\|\cdot\|, |\cdot|$ and by $(y_1, z_1) \neq (y_2, z_2)$, we have $\|\lambda y_1 + (1-\lambda) y_2\| < \lambda \|y_1\| + (1-\lambda)\|y_2\|$ or $|\lambda z_1 + (1-\lambda) z_2| < \lambda |z_1| + (1-\lambda)|z_2|$, and thus $f(\|\lambda y_1 + (1-\lambda) y_2\|, |\lambda z_1 + (1-\lambda) z_2|) > f(\lambda \|y_1\| + (1-\lambda)\|y_2\|, \lambda |z_1| + (1-\lambda)|z_2|) = \lambda f(\|y_1\|, |z_1|) + (1-\lambda) f(\|y_2\|, |z_2|)$. □

Let us take the function ρ from Claim 12. We denote

$$\|(y,z)\|_\infty = \max\{\|y\|, |z|\}, \quad (y,z) \in Y \oplus \mathbb{R}^2,$$

$$B(x, r) = \big\{(y,z) \in Y \oplus \mathbb{R}^2 : \|x - (y,z)\|_\infty \le r\big\}, \quad x \in Y \oplus \mathbb{R}^2,\ r \ge 0.$$

We choose a sequence of positive numbers $(\varepsilon_i)_{i \in \mathbb{N}}$ such that

$$\sum_{i=1}^\infty \varepsilon_i \le 1, \quad \prod_{i=1}^\infty (1 - \varepsilon_i) > 0, \quad \lim_{n\to\infty} \frac{1}{\varepsilon_n} \sum_{i=n}^\infty \varepsilon_i = 1,$$

and define

$$R_0 = R,$$

$$R_n = \bigcup_{x \in R_{n-1}} B(x, \varepsilon_n \rho(x)), \quad n \in \mathbb{N},$$

$$R_\infty = \overline{\bigcup_{n=0}^\infty R_n}.$$

It is easy to verify by the induction that $R_n \subset (1 + \sum_{i=1}^n \varepsilon_i) B_Y \times B_{\mathbb{R}^2}$, and thus $R_n, n \in \mathbb{N}$, are well-defined. Besides this, the sets $R_n, n \in \mathbb{N}$, are absolutely convex.

Further on, by dist we mean the distance with respect to $\|\cdot\|_\infty$.

Claim 13. $R_\infty \cap (Y \times S_{\mathbb{R}^2}) = P \cup (-P)$.

Proof. Using Claim 11, we have $P \cup (-P) = R \cap (Y \times S_{\mathbb{R}^2}) \subset R_\infty \cap (Y \times S_{\mathbb{R}^2})$. It is enough to show that if $(y, z) \in Y \times S_{\mathbb{R}^2}$ and $(y, z) \notin R$, then $(y, z) \notin R_\infty$.

Let $(y, z) \in (Y \times S_{\mathbb{R}^2}) \setminus R$. We denote

$$d = \mathrm{dist}((y, z), R).$$

As $(y, z) \notin R$ and R is closed, $d > 0$. Let $n \in \mathbb{N}$. Given $x = (y', z') \in R_{n-1}$ and $(y'', z'') \in B(x, \varepsilon_n \rho(x))$, we have $\|(y'', z'') - (y, z)\|_\infty \geq \|x - (y, z)\|_\infty - \|x - (y'', z'')\|_\infty \geq \|x-(y, z)\|_\infty - \varepsilon_n\rho(x) \geq \|x-(y, z)\|_\infty - \varepsilon_n(1-|z'|) = \|x-(y, z)\|_\infty - \varepsilon_n(|z| - |z'|) \geq \|x - (y, z)\|_\infty(1 - \varepsilon_n)$. It means that $\mathrm{dist}((y, z), B(x, \varepsilon_n\rho(x))) \geq (1-\varepsilon_n)\|x-(y, z)\|_\infty$ for every $x \in R_{n-1}$. By the definition of R_n, $\mathrm{dist}((y, z), R_n) \geq (1 - \varepsilon_n)\mathrm{dist}((y, z), R_{n-1})$. By an easy induction argument,

$$\mathrm{dist}((y, z), R_n) \geq d \prod_{i=1}^{n} (1 - \varepsilon_i), \quad n = 0, 1, \ldots,$$

$$\mathrm{dist}((y, z), R_\infty) \geq d \prod_{i=1}^{\infty} (1 - \varepsilon_i).$$

So $(y, z) \notin R_\infty$ by the choice of the sequence $(\varepsilon_i)_{i\in\mathbb{N}}$. □

Claim 14. *If a, b are two distinct points of R_∞, then $\lambda a + (1 - \lambda)b$ is an element of the interior of R_∞ for every $\lambda \in (0, 1)$.*

Proof. Given such a, b, λ, we denote $x = \lambda a + (1 - \lambda)b$. Let us realize that $x \notin Y \times S_{\mathbb{R}^2}$. Assume that $x \in Y \times S_{\mathbb{R}^2}$. Since $a, b \in R_\infty \subset Y \times B_{\mathbb{R}^2}$, there is $z \in S_{\mathbb{R}^2}$ such that $a, b \in Y \times \{z\}$. By Claim 13, we have $a, b \in P \cup (-P)$. By the definition of P and by the fact that p is a one-to-one mapping, the set $(P \cup (-P)) \cap (Y \times \{z\})$ has at most one element. Thus $a = b$, which is a contradiction.

So $x \in Y \times (B_{\mathbb{R}^2} \setminus S_{\mathbb{R}^2})$. We may suppose that $a, b \in Y \times (B_{\mathbb{R}^2} \setminus S_{\mathbb{R}^2})$, too (we may take $(1/2)(a + x), (1/2)(b + x)$ instead of a, b). We have

$$\rho(x) = \rho(\lambda a + (1 - \lambda)b) > \lambda\rho(a) + (1 - \lambda)\rho(b).$$

We choose $r' > r > \rho(a)$ and $s' > s > \rho(b)$ such that

$$\rho(x) > \lambda r' + (1 - \lambda)s'.$$

Since ρ is continuous, we can choose $u > 0$ and $v > 0$ such that $\rho \leq r$ on $B(a, u)$ and $\rho \leq s$ on $B(b, v)$. Let us prove that, for $n \in \mathbb{N}$,

$$\mathrm{dist}(a, R_n) \geq \min\left\{u - \varepsilon_n, \mathrm{dist}(a, R_{n-1}) - r\varepsilon_n\right\}.$$

If $y \in R_{n-1} \setminus B(a, u)$ and $z \in B(y, \varepsilon_n\rho(y))$, then $\|a - z\|_\infty \geq \|a - y\|_\infty - \|y - z\|_\infty \geq u - \varepsilon_n\rho(y) \geq u - \varepsilon_n$. If $y \in R_{n-1} \cap B(a, u)$ and $z \in B(y, \varepsilon_n\rho(y))$, then $\|a - z\|_\infty \geq \|a - y\|_\infty - \|y - z\|_\infty \geq \mathrm{dist}(a, R_{n-1}) - \varepsilon_n\rho(y) \geq \mathrm{dist}(a, R_{n-1}) - r\varepsilon_n$.

Now, since $\mathrm{dist}(a, R_n) \to 0$ and $u - \varepsilon_n \to u > 0$, there is n_0 such that $\mathrm{dist}(a, R_n) \geq \mathrm{dist}(a, R_{n-1}) - r\varepsilon_n$ for every $n \geq n_0$. For $n \geq n_0$, we have

$$\begin{aligned}\mathrm{dist}(a, R_n) &\leq \mathrm{dist}(a, R_{n+1}) + r\varepsilon_{n+1} \leq \mathrm{dist}(a, R_{n+2}) + r\varepsilon_{n+1} + r\varepsilon_{n+2} \\ &\leq \cdots \leq r \sum_{i=n+1}^{\infty} \varepsilon_i.\end{aligned}$$

By the same way, we can find m_0 such that $\operatorname{dist}(b, R_n) \le s \sum_{i=n+1}^{\infty} \varepsilon_i$ for $n \ge m_0$. We put $N = \max\{n_0, m_0\}$ and, for every $n \ge N$, we choose $a_n, b_n \in R_n$ such that $\|a - a_n\|_\infty \le r' \sum_{i=n+1}^{\infty} \varepsilon_i$ and $\|b - b_n\|_\infty \le s' \sum_{i=n+1}^{\infty} \varepsilon_i$. For $n \ge N$, we put $x_n = \lambda a_n + (1 - \lambda) b_n$. Since ρ is continuous, we have $\rho(x_n) \to \rho(x)$. Since

$$\frac{\lambda r' + (1-\lambda)s'}{\rho(x_n)} \frac{1}{\varepsilon_{n+1}} \sum_{i=n+1}^{\infty} \varepsilon_i \to \frac{\lambda r' + (1-\lambda)s'}{\rho(x)} < 1,$$

we can choose $n \ge N$ such that $(\lambda r' + (1-\lambda)s') \sum_{i=n+1}^{\infty} \varepsilon_i < \rho(x_n)\varepsilon_{n+1}$. We have

$$\|x - x_n\|_\infty \le \lambda \|a - a_n\|_\infty + (1-\lambda)\|b - b_n\|_\infty \le (\lambda r' + (1-\lambda)s') \sum_{i=n+1}^{\infty} \varepsilon_i < \rho(x_n)\varepsilon_{n+1}.$$

So x is an element of the interior of $B(x_n, \varepsilon_{n+1}\rho(x_n))$, which is a subset of R_{n+1}. □

Claim 15. *If $t \in [0, \pi/2]$, then u_t attains its supremum on R_∞ if and only if $t \in M$.*

Proof. Considering Claim 5, it remains to prove that $u_t(x) < 1$ for every $x \in R_\infty$ in the case that $t \notin M$. Suppose that $t \notin M, x = (y; r\cos\alpha, r\sin\alpha) \in R_\infty$ and $u_t(x) = 1$. We have $1 = u_t(x) = r(\cos\alpha\cos t + \sin\alpha\sin t) = r\cos(\alpha - t)$, which is possible only if $r = 1$ and $\alpha = t$, i.e. $x \in Y \times \{(\cos t, \sin t)\}$. By Claim 13, $x \in P \cup (-P) \subset R$. By Claim 7(b), $u_t(x) < 1$, which is a contradiction. □

Now, we define $|||\cdot|||$ as the norm with the unit ball R_∞. Proposition 9 follows from Claims 14 and 15.

Proof of Theorem 1. We take $M \subset [0, \pi/2]$, dense in $[0, \pi/2]$, which is Borel, but not of the additive Borel class α ([5, Theorem 22.4]). It is known that there is an equivalent strictly convex norm $\|\cdot\|$ on X ([2, Theorem II.2.6]). By Proposition 9, there is a strictly convex norm $|||\cdot|||$ on X such that, for every $t \in [0, \pi/2]$, $(\cos t)\varphi + (\sin t)\phi \in \mathrm{NA}(|||\cdot|||)$ if and only if $t \in M$. Since M is not of the additive Borel class α, $\mathrm{NA}(|||\cdot|||)$ is not of the additive Borel class α, too ($t \in [0, \pi/2] \mapsto (\cos t)\varphi + (\sin t)\phi$ is a continuous mapping). □

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References

[1] G. Debs, G. Godefroy and J. Saint Raymond: *Topological properties of the set of norm-attaining linear functionals*, Can. J. Math. **47** (1995), 318–329.
[2] R. Deville, G. Godefroy and V. Zizler: *Smoothness and renormings in Banach spaces*, Pitman Monographs and Surveys in Pure and Applied Mathematics **64** (1993).
[3] R. Kaufman: *Topics on analytic sets*, Fund. Math. **139** (1991), 215–229.
[4] R. Kaufman: *On smooth norms and analytic sets*, Isr. J. Math. **116** (2000), 21–27.
[5] A. S. Kechris: *Classical descriptive set theory*, Grad. Texts in Math. **156**, Springer-Verlag, New York (1995).

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