INTERNAL NORMALITY AND INTERNAL COMPACTNESS

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Abstract. This text contains an example which presents a way to modify any Dowker space to get a normal space \( X \) such that \( X \times [0,1] \) is not \( \kappa \)-normal, and a theorem implying the existence of a non-Tychonoff space which is internally compact in a larger regular space. It gives answers to several questions by Arhangel’skii [Ar2].

Systematic study of relative topological properties was begun by A. V. Arhangel’skii and H. M. M. Gennedi in a paper published in Russian [AG]. In 1996 Arhangel’skii wrote a survey article on this topic [Ar1].

Relative topological properties (defined for a pair of spaces \( Y \subset X \)) generalize a global property in the sense that if the subspace \( Y \) coincides with the larger space \( X \), then the relative property should be the same as the global one. In this paper we study a version of relative compactness as well as \( \kappa \)-normality, a classical notion which turned out to be a useful tool in dealing with relative normality (cf. [Ar1], [Ar2]).

We will use standard notation, mainly following [En]. In particular, \( \omega \) is the set of all natural numbers and each \( n \in \omega \) is a set of all smaller elements of \( \omega \). For convenience, we use also \( N = \{1,2,\ldots\} \). An AD system on \( \omega \) is an almost disjoint system of infinite subsets of \( \omega \) and a MAD system is a maximal such system. The closed unit interval is denoted by \( I \). All topological spaces are assumed to be \( T_1 \).

1. ON \( \kappa \)-NORMALITY

Definition 1 ([Sc]). A topological space \( X \) is \( \kappa \)-normal if every two disjoint regular closed subsets of \( X \) can be separated by disjoint open subsets of \( X \).

The existence of a normal space \( X \) such that \( X \times I \) is not normal is well known (the first ZFC example is due to Rudin, [Ru]). Such spaces are called Dowker spaces and are exactly normal not countably paracompact. We will show, how to modify any Dowker space, to give a negative answer to the following questions.

Question 2 ([Ar2], Question 7). Is the product of a normal space \( X \) and the closed interval \( I \) always \( \kappa \)-normal?

Question 3 ([Ar2], Question 8). Let \( X \) be a normal space and \( B \) a compact Hausdorff space. Is then the space \( X \times B \) \( \kappa \)-normal?

Example 4. Let \( Y \) be any Dowker space. On the underlying set \((\omega + 1) \times Y \), refine the product topology by declaring all points in \( \omega \times Y \) to be isolated. The resulting space will be denoted \( X \). As a subspace, the top level \( \{\omega\} \times Y \) is isomorphic to \( Y \) and will be denoted \( Y' \).

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The space $X$ is normal. Indeed, let $A$ and $B$ be two disjoint closed subsets of $X$. Then $A \cap Y'$ and $B \cap Y'$ are two disjoint closed subsets of $Y'$ and, since $Y'$ is normal, there exist disjoint open subsets $U$ and $V$ of $Y'$ separating $A \cap Y'$ and $B \cap Y'$. It follows that $(A \setminus Y') \cup ((\omega + 1) \times U) \setminus B$ and $(B \setminus Y') \cup ((\omega + 1) \times V) \setminus A$ are disjoint open subsets of $X$ separating $A$ and $B$.

The construction of regular closed subsets of $X \times I$ is analogous to the classical one ([Do], see also [En, Chapter 5.2]). Since $Y'$ is not countably paracompact, there exists a sequence $\{F_n : n \in \mathbb{N}\}$ of closed subsets of $Y$ such that $F_{n+1} \subseteq F_n$, $\bigcap \{F_n : n \in \mathbb{N}\} = \emptyset$ and for each sequence $\{G_n : n \in \mathbb{N}\}$ of open sets in $Y$, such that $F_n \subseteq G_n$, $\bigcap \{G_n : n \in \mathbb{N}\}$ is nonempty.

For each $n \in \mathbb{N}$, put

$$B_n = (\omega \setminus n) \times F_n \times \left( \frac{1}{2(n+1)}, \frac{3}{2(n+1)} \right)$$

and

$$S_n = n \times Y \times \left[ 0, \frac{1}{2(n+2)} \right).$$

Note that $B_n$ and $S_n$ are open subsets of $X \times I$ and $B_n \cap S_m = \emptyset$ for each $n, m \in \mathbb{N}$.

Let us define regular closed subsets of $X \times I$:

$$F = \bigcup \{B_n : n \in \mathbb{N}\}$$

and

$$E = \bigcup \{S_n : n \in \mathbb{N}\}.$$

To prove that $E$ and $F$ are disjoint, it is only necessary to show that $(Y' \times \{0\}) \cap F = \emptyset$. Pick any $x \in Y'$, fix $n \in \mathbb{N}$ such that $x \notin (\omega \times F_n)$ and let

$$O = (\omega + 1) \times (Y \setminus F_n) \times \left[ 0, \frac{1}{2(n+1)} \right];$$

$O$ is an open neighborhood of $(x, 0)$. We will show that $O$ is disjoint from $B_m$ for each $m \in \mathbb{N}$ and thus disjoint from $F$. If $m \leq n$, then

$$O \subseteq (\omega + 1) \times Y \times \left[ 0, \frac{1}{2(n+1)} \right]$$

and

$$B_m \subseteq (\omega + 1) \times Y \times \left( \frac{1}{2(n+1)}, 1 \right],$$

hence $O$ and $B_m$ are disjoint. If $n < m$, then $F_m \subseteq F_n$ so $B_m \subseteq (\omega + 1) \times F_n \times I$ and this set is disjoint from $O$.

Now it is clear that

$$E = (Y' \times \{0\}) \cup \bigcup \{S_n : n \in \mathbb{N}\}$$

and

$$F = \bigcup \{B_n : n \in \mathbb{N}\}$$

where

$$\overline{B_n} = ((\omega + 1) \setminus n) \times F_n \times \left( \frac{1}{2(n+1)}, \frac{3}{2(n+1)} \right).$$

The sets $E$ and $F$ cannot be separated by disjoint open neighborhoods. Indeed, if $F \subseteq U$ and $U$ is open then $\{\omega \times F_n \times \{1/(n+1)\} \subseteq U$ for each $n$ and thus $\{G_n : n \in \mathbb{N}\}$, where $G_n = \pi_Y [U \cap (Y \times \{1/(n+1)\})]$, is a sequence of open
subsets of $Y$ such that $F_n \subset G_n$ ($\pi_Y$ is the projection from $\{\omega\} \times Y \times I$ onto $Y$). This implies that there exists some $x \in \bigcap\{G_n : n \in \mathbb{N}\}$. For this $x$ we have $(\omega, x, 0) \in U \cap E$ and therefore $E$ and $F$ cannot be separated. This shows that $X \times I$ is not $\kappa$-normal.

2. Internal compactness

In some cases a relative property implies an absolute property of the smaller space. A well known and easy-to-prove is the fact that if a space $Y$ is normal in a larger space $X$ then $Y$ is a regular space. Let us recall that $Y$ is normal in $X$ if for every pair of closed disjoint subsets $A$ and $B$ of $X$ there are disjoint open sets $U$ and $V$ in $X$ such that $A \cap Y \subset U$ and $B \cap Y \subset V$. It is not known yet if normality of a subspace $Y$ in a larger regular space $X$ can imply $Y$ being Tychonoff. There are only consistent counterexamples: [GG] and recently [Mi]. Moreover, relative compactness of a subspace of a Hausdorff space implies its relative normality. These facts motivated Arhangel’skiii’s questions on a pair of weaker properties.

Definition 5. Let $X$ be a topological space, $Y \subset X$. We say that $Y$ is internally normal in $X$ if for every two disjoint subsets $A$ and $B$ of $Y$ which are closed in $X$, there are disjoint sets $U$ and $V$, open in $X$, such that $A \subset U$ and $B \subset V$. Further, we say that $Y$ is internally compact in $X$ if every $M \subset Y$ closed in $X$ is compact.

Question 6 ([Ar2], Question 9). Let $Y$ be a subspace of a Hausdorff (regular) space $X$ such that $Y$ is internally compact in $X$. Is then true that $Y$ is Tychonoff?

Question 7 ([Ar2], Question 10). Let $Y$ be a subspace of a regular space $X$ such that $Y$ is internally normal in $X$. Is then $Y$ Tychonoff?

First, we show that in the Hausdorff case, the subspace need not be even regular. The following lemma is an easy exercise.

Lemma 8. For each ultrafilter $\mathcal{U}$ on $\omega$ there exists a MAD system $\mathcal{A}$ on $\omega$ such that $\mathcal{A} \cap \mathcal{U} = \emptyset$. □

Example 9. The idea is to construct a Hausdorff space $X = Y \cup Z$ with $Y$ non-regular such that all “nontrivial” infinite subsets of $Y$ have cluster points in $Z$. Then there are only few closed subsets of $X$ contained in $Y$ and these are managed to be compact.

Fix a free ultrafilter $\mathcal{U}$ on $\omega$ and let $\mathcal{A}$ be a MAD system on $\omega$ from Lemma 8. Put $Y = \{\mathcal{U}\} \cup ((\omega + 1) \times \omega)$, $F = \{\omega\} \times \omega \subset Y$. Let us endow the set $X = Y \cup A$ with a topology by declaring each point of $\omega \times \omega$ isolated,

$$\{(\omega + 1) \setminus n_0 \times \{n\} : n_0 \in \omega\}$$

an open base in $(\omega, n) \in F$,

$$\{\mathcal{U}\} \cup (\omega \times \mathcal{U}) : U \in \mathcal{U}\}$$

an open base in $\mathcal{U}$ and

$$\{\mathcal{A}\} \cup ((\omega + 1) \times (A \setminus n_0)) : n_0 \in \omega\}$$

an open base in $A \in \mathcal{A}$. This obviously defines a Hausdorff topology on $X$, while the closed subset $F$ of $Y$ cannot be separated from $\mathcal{U}$, hence $Y$ is not regular.

It remains to show that $Y$ is internally compact in $X$. Consider a closed subset $C$ of $X$, $C \subset Y$ and an infinite $B \subset C$ whose cluster point is to be found in $C$. Since $C$ is closed,
(∀A ∈ A) \{n ∈ A : C ∩ ((ω + 1) × \{n\}) ≠ \emptyset\} is finite.

Thus
\[ N = \{n ∈ \omega : C ∩ ((ω + 1) × \{n\}) ≠ \emptyset\} \]
is almost disjoint from A. It follows that N is finite. As B is infinite, there is an \(n_0\) such that \(B ∩ (\omega × \{n_0\})\) is infinite. Now \(\omega, n_0\) is a cluster point of B.

From now on, our goal is to construct a non-Tychonoff space which is internally compact in a larger regular space.

**Lemma 10.** A subspace \(Y\) is internally compact in \(X\) if and only if for each centered family \(C\) of subsets of \(Y\) which are closed in \(X\) the intersection \(\bigcap C\) is nonempty.

In the next example, notation used within the construction of a Jones space \(J(X)\) over a regular space \(X\) is introduced.

**Example 11.** Let \(X\) be a regular non-normal topological space. The construction of the space \(J(X)\) uses a method called *Jones machine* to build a regular non-Tychonoff space. For details see [Jo].

Pick two closed disjoint subsets \(A_0\) and \(A_1\) of \(X\) such that \(A_0\) and \(A_1\) cannot be separated by disjoint open neighborhoods. Add one new point \(z\) to the product \(X × \omega\). Let an open base at \(z\) consist of the sets of the form
\[ \{z\} \cup (X × (\omega \setminus 2n + i)) \cup ((X \setminus A_i) × \{2n - 1 + i\}) \]
for \(n ∈ N\) and \(i ∈ \{0, 1\}\). The resulting space \((X × \omega) \cup \{z\}\) will be denoted by \(P(X)\).

Finally, identify each point \((a, 2n)\) in the set \(A_0 × \{2n\}\) with the corresponding point \((a, 2n + 1)\) in \(A_0 × \{2n + 1\}\) and each point \((a, 2n + 1)\) in \(A_1 × \{2n + 1\}\) with \((a, 2n + 2)\) in \(A_1 × \{2n + 2\}\) for every \(n ∈ \omega\). The quotient space is the Jones space \(J(X)\) and the quotient mapping will be denoted \(q : P(X) → J(X)\).

Theorem 12 states that the Jones machine preserves internal compactness in the following sense. Let \(Y\) be a non-normal subspace of a regular space \(X\). Suppose, moreover, that \(A_0\) and \(A_1\) are two disjoint closed subsets of \(Y\) which cannot be separated by disjoint open neighborhoods in \(Y\) and such that \(\overline{A_0^X} ∩ \overline{A_1^X} = \emptyset\). In this situation \(J(Y)\) can be considered as a subspace of \(J(X)\) in a natural way; the new point (in Example 11 denoted by \(z\)) is the same for both \(J(Y)\) and \(J(X)\). The two sets whose points are being identified are \(A_0\) and \(A_1\) for \(J(Y)\) and \(\overline{A_0^X}\) and \(\overline{A_1^X}\) for \(J(X)\).

**Theorem 12.** Let \(Y\) be a non-normal subspace of a regular space \(X\) and suppose that the sets \(A_0, A_1\) are as in the previous paragraph. If \(Y\) is internally compact in \(X\), then \(J(Y)\) is internally compact in \(J(X)\).

**Proof.** We will use the notation established in Example 11. Pick any centered system \(C\) of subsets of \(J(Y)\) such that all sets in \(C\) are closed in \(J(X)\). We have to prove that the intersection \(\bigcap C\) is nonempty.

Assume that \(z \notin Z\) for some \(Z ∈ C\); otherwise we are done. Then \(q^{-1}[Z] ⊂ Y × n\) for some \(n ∈ ω\). Since \(q^{-1}[Z] ∩ (X × \{j\})\) is a subset of \(j\)-th copy of \(Y\) and it is closed in \(j\)-th copy of \(X\) for each \(j ∈ n\) and since \(Y\) is internally compact in \(X\), the set \(q^{-1}[Z]\) is a finite sum of compact sets and thus compact. Hence \(\emptyset ≠ \bigcap[q^{-1}[C] : C ∈ C] = q^{-1}[\bigcap C]\) and \(J(Y)\) is internally compact in \(J(X)\). □
Theorem 13. There exists a non-normal space $Y$ which is internally compact in a zero-dimensional space $X$.

Proof. Throughout this proof, all points of the Čech-Stone compactification $\beta D$ of a discrete space $D$ will be identified with ultrafilters on $D$. For any discrete space $D$, let us also define a subspace $\gamma D$ of $\beta D$ as

$$\gamma D = \{ p \in \beta D : (\exists P \in p) |P| \leq \omega \}.$$  

Let $A$ and $B$ be two disjoint sets of size $\omega_2$, put $C = A \times B$ and $\pi_A$, $\pi_B$ will denote the natural projections of $C$ onto $A$ and $B$. The underlying sets for $X$ and $Y$ are

$$Y = A \cup B \cup C$$
and

$$X = \gamma A \cup \gamma B \cup \gamma C$$
and the topology is defined as follows: $\gamma C$ is an open subspace of $X$, other basic open sets of $X$ are

$$O \cup \pi_A^{-1}[O \cap A] \setminus K$$
for $|K| \leq \omega$, $O$ open subset of $\gamma A$ and

$$O \cup \pi_B^{-1}[O \cap B] \setminus K$$
for $|K| \leq \omega$, $O$ open subset of $\gamma B$. It is a routine to check that we have defined a base of a topology on $X$ correctly.

Claim. $X$ is a Hausdorff space.

Proof. We need to show that each two distinct points $a$ and $b$ in $X$ can be separated by disjoint open neighborhoods. If $a, b \in \gamma C$, then $\gamma C \subset \beta C$ implies that these two points can be separated. If $a, b \in \gamma A$, then there are disjoint open sets $U$ and $V$ separating a and b in $\gamma A$ thus

$$U \cup \pi_A^{-1}[U \cap A] \setminus \gamma C$$
and

$$V \cup \pi_A^{-1}[V \cap A] \setminus \gamma C$$
separate a and b in X. Case $a, b \in \gamma B$ is similar. If $a \in \gamma A$ and $b \in \gamma B$, then fix countable sets $U \subset A$ and $V \subset B$ such that $a \in U^\gamma A$ and $b \in V^\gamma B$. The sets

$$U \cup \pi_A^{-1}[U] \setminus (U \times V) \setminus \gamma C$$
and

$$V \cup \pi_B^{-1}[V] \setminus (U \times V) \setminus \gamma C$$
separate a and b in $X$. And if $a \in \gamma A$, $b \in \gamma C$, then fix countable sets $U \subset A$ and $V \subset C$ such that $a \in U^\gamma A$ and $b \in V^\gamma C$. The sets

$$U \cup \pi_A^{-1}[U] \setminus \gamma C$$
and

$$V \cup \pi_C^{-1}[V] \setminus \gamma C$$
separate a and b in $X$. □

Claim. $X$ is a zero-dimensional space.
Proof. For each \( x \in \gamma C \) there is an open base at \( x \) which consists of the sets of the form \( \gamma K \) where \( K \subset C \) is such that \( |K| \leq \omega \), and for such \( K \) is \( \gamma K = \overline{K}^X \). For \( x \in \gamma A \) there is an open base at \( x \) which consists of the sets of the form

\[
B = \gamma O \cup \pi^{-1}_A [O \cap A] \setminus K^C
\]

where \( K \subset C \), \( |K| \leq \omega \) and \( O \subset A \) is such that \( |O| \leq \omega \). For such \( O \) and \( K \), \( B \) is closed in \( X \). The case \( x \in \gamma B \) is similar. \( \square \)

Claim. \( A \) and \( B \) are closed subsets of \( Y \) which cannot be separated by disjoint open sets in \( Y \). Moreover, \( \overline{A}^X \cap \overline{B}^X = \emptyset \).

Proof. Let \( U \) be open in \( Y \) and let \( A' \subset U \cap A \) be some set of size \( \omega_1 \). We will show that \( \overline{U} \cap B \) is nonempty. For each \( a \in A' \) fix a countable \( K_a \subset C \) such that \( \pi^{-1}_A [\{a\}] \setminus K_a \subset U \). Hence

\[
\pi^{-1}_A [A'] \setminus K \subset U
\]

where

\[
K = \bigcup \{K_a : a \in A'\}
\]

and notice that \( |K| \leq \omega_1 \). Each

\[
b \in B \setminus \pi_B [K]
\]

(and such clearly exists) is an element of \( \overline{U} \) because

\[
\pi^{-1}_B [\{b\}] \cap U \supset A' \times \{b\}
\]

and the product \( A' \times \{b\} \) has cardinality \( \omega_1 \).

\[
\overline{A}^X \cap \overline{B}^X = \emptyset \text{ is a consequence of } \overline{A}^X = \gamma A \text{ and } \overline{B}^X = \gamma B. \square
\]

Claim. If \( G \subset Y \) is closed in \( X \) then \( |G| < \omega \).

Proof. Suppose \( G \subset Y \), \( \omega \leq |G| \). Then at least one of the sets \( G \cap A, G \cap B \) and \( G \cap C \) must be infinite. Assume that \( \omega \leq |G \cap C| \). Then \( \emptyset \neq \overline{G \cap C}^C \setminus (G \cap C) \subset \overline{Y} \setminus Y \). Thus \( G \) is not closed. Cases \( \omega \leq |G \cap A| \) and \( \omega \leq |G \cap B| \) work similarly. \( \square \)

The last claim implies that \( Y \) is internally compact in \( X \) and the Theorem is proved.

Corollary 14. There exists a non-Tychonoff space \( Y \) which is internally compact in a regular space \( X \).

Proof. Apply Theorem 13 and Theorem 12. \( \square \)

Corollary 15. There exists a non-Tychonoff space \( Y \) which is internally normal in a regular space \( X \).

Proof. Notice that if \( Y \) is internally compact in a Hausdorff space \( X \) then \( Y \) is internally normal in \( X \). \( \square \)
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