Remarks on $C^{1,\gamma}$ - regularity of weak solutions to elliptic systems with BMO gradients

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Abstract

The interior $C^{1,\gamma}$ - regularity for a weak solution with BMO-gradient of a nonlinear nonautonomous second order elliptic systems is investigated.

1 Introduction.

In this paper we give conditions guaranteeing that the BMO first derivatives of weak solutions to a nonlinear elliptic system

$$-D_\alpha a_i^\alpha(x, Du) = -D_\alpha f_i^\alpha(x) \quad \text{on} \quad \Omega \subset \mathbb{R}^n, i = 1, \ldots, N, \alpha = 1, \ldots, n \tag{1.1}$$

belong to $C^{0,\gamma}(\Omega, \mathbb{R}^{nN})$. [1]

The system (1.1) has been extensively studied. S. Campanato in [2],[3] proved that (under suitable assumptions) $Du \in \mathcal{L}^{2,\lambda}_{loc}(\Omega, \mathbb{R}^{nN})$ with $\lambda < n$, and $u \in C^{0,\gamma}_{loc}(\Omega, \mathbb{R}^N)$ for some $\gamma < 1$ if $n = 3, 4$. If a_i^α are differentiable and have controllable growth then there is a positive ϵ such that $Du \in W^{2,2+\epsilon}_{loc}(\Omega, \mathbb{R}^{nN})$ which implies that Du is locally Hölder continuous on Ω for $n = 2$ (see [11], [7], [12]). For this reason we will concentrate on the case $n > 2$. From a series of counterexamples starting from famous De Giorgi work (see [6]) it is well known that Du need not be continuous or even bounded (see [8], [10], [13], [16], [17]) for $n > 2$. Higher smoothness of coefficients does not improve the smoothness of a solution, as there are examples (see [14]) where the coefficients are real analytic while Du is bounded and discontinuous. On the other hand, it follows immediately from so called direct proof of partial regularity (see [7], [5])that if modulus of continuity of $\frac{\partial a_i^\alpha}{\partial p_\beta^j}$ is small enough then Du is Hölder continuous. For this reason we concentrate on conditions that do not require smallness of L^∞ norm of the modulus of continuity and they imply that solutions with BMO gradients are $C^{1,\gamma}_{loc}\Omega$.

By a weak solution to (1.1) we understand $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ such that

$$\int_\Omega a_i^\alpha\left(x, Du(x)\right) D_\alpha \varphi^i(x)\, dx = \int_\Omega f_i^\alpha(x) D_\alpha \varphi^i(x)\, dx, \quad \forall\, \varphi \in W^{1,2}_0(\Omega, \mathbb{R}^N).$$

Here $\Omega \subset \mathbb{R}^n$ is an open set and, as we are interested in the interior regularity, we do not assume that u solves a boundary value problem nor any smoothness of $\partial\Omega$.

On the coefficients we suppose

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[1]Throughout the whole text we use the summation convention over repeated indices.

(i) (Smoothness) $a_i^\alpha(x,p)$ are differentiable functions in x and p with continuous derivatives.

(ii) (Growth) For all $(x,p) \in \Omega \times \mathbb{R}^{nN}$ denote $A_{ij}^{\alpha\beta}(x,p) = \frac{\partial a_i^\alpha}{\partial p_\beta^j}(x,p)$ and suppose

$$|a_i^\alpha(x,p)|, \left|\frac{\partial a_i^\alpha}{\partial x_s}(x,p)\right| \leq M(1+|p|), \tag{1.2}$$

$$\left|A_{ij}^{\alpha\beta}(x,p)\right| \leq M, \tag{1.3}$$

where $M > 0$,

(iii) (Ellipticity) There exist $\nu > 0$ such that for every $x \in \Omega$ and $p,\ \xi \in \mathbb{R}^{nN}$

$$\nu|\xi|^2 \leq A_{ij}^{\alpha\beta}(x,p)\xi_\alpha^i\xi_\beta^j, \tag{1.4}$$

(iv) (Oscillation of coefficients) There is a real function ω continuous on $[0,\infty)$, which is bounded, nondecreasing, concave, $\omega(0) = 0$ and such that for all $x \in \Omega$ and $p,\ q \in \mathbb{R}^{nN}$

$$\left|A_{ij}^{\alpha\beta}(x,p) - A_{ij}^{\alpha\beta}(x,q)\right| \leq \omega\left(|p-q|\right) \tag{1.5}$$

We set $\omega_\infty = \lim_{t\to\infty}\omega(t) \leq 2M$.

(v) $f_i^\alpha \in W^{1,2}, \frac{\partial f_i^\alpha}{\partial x_\beta} \in L^{2,\delta-2}$ for $\delta = n+2\gamma, \gamma \in (0,1), i = 1,...,N$.

It is well known (see [7], p.169) that for uniformly continuous $A_{ij}^{\alpha\beta}$ there exists a real function ω satisfying (iv) and, viceversa, (iv) implies uniform continuity of $A_{ij}^{\alpha\beta}$.

In what follows we will understand by pointwise derivative ω' of ω the right derivative which is finite on $(0,\infty)$.

For $p,p' \in (1,\infty)$ such that $\frac{1}{p}+\frac{1}{p'} = 1$ denote

$$J_p = \int_0^\infty \frac{\frac{d}{dt}(\omega^{2p'})(t)}{t}\,dt, \tag{1.6}$$

$$S_p = \sup_{t\in(0,\infty)} \frac{d}{dt}(\omega^{2p'})(t) \tag{1.7}$$

and

$$P_p = \min\{J_p, S_p\}. \tag{1.8}$$

Now we formulate the result

Theorem 1.1 *Let u be a weak solution to (1.1) such that $Du \in BMO(\Omega,\mathbb{R}^N)$ and coefficients a_i^α satisfy the hypotheses (i), (ii), (iii), (iv) with the constants M, ν, a modulus of continuity ω and a right hand side f satisfying (v). Assume that there is a $p \in (1, \frac{n}{n-2}]$ such that $P_p < \infty$. Then the inequality*

$$\left(P_p^2||Du||_{BMO}\right)^{\frac{1}{2p'}} \leq \nu^2 C \tag{1.9}$$

implies that $Du \in C_{loc}^{0,\gamma}(\Omega,\mathbb{R}^N)$. Here $C = \frac{1}{4C(p,n\frac{M}{\nu})}\frac{\tau^\mu}{2+\tau^\mu}$, L is given in Lemma 2.4, $\mu \in (n,n+2)$, $\tau = (4L)^{\frac{-1}{n+2-\mu}}$ and $C(p,n,\frac{M}{\nu})$ is given in (2.9).

2 Preliminaries and Notations.

Let $n, N \in \mathbb{N}$, $n \geq 3$. We will consider an open set $\Omega \subset \mathbb{R}^n$ with points $x = (x_1, \dots x_n)$.

For a vector-valued function $u : \Omega \to \mathbb{R}^N$, $u(x) = \big(u^1(x), \dots, u^N(x)\big)$, $N \geq 1$ put $Du = (D_1 u, \dots, D_n u)$, $D_\alpha = \partial / \partial x_\alpha$.

If $x \in \mathbb{R}^n$ and r is a positive real number, we set $B(x,r) = \{y \in \mathbb{R}^n : |y - x| < r\}$. Denote by $u_{x,r} = (\kappa_n r^n)^{-1} \int_{B(x,r)} u(y)\, dy$ the mean value of the function $u \in L^1(B(x,r), \mathbb{R}^N)$ over the set $B(x,r)$ (κ_n being the volume of unit ball in $\mathbb{R}^n$).

Moreover, we set $\phi(x,r) = \int_{B(x,r)} |Du(y) - (Du)_{x,r}|^2\, dy$, $U(x,r) = r^{-n}\phi(x,r)$.

Beside the usually used space $C_0^\infty(\Omega, \mathbb{R}^N)$, Hölder space $C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^N)$ and Sobolev spaces $W^{k,p}(\Omega, \mathbb{R}^N)$, $W_0^{k,p}(\Omega, \mathbb{R}^N)$ we use Campanato spaces $\mathcal{L}^{q,\lambda}(\Omega, \mathbb{R}^N)$, Morrey spaces $L^{q,\lambda}(\Omega, \mathbb{R}^N)$ and space of functions with bounded mean oscillations $BMO(\Omega, \mathbb{R}^N)$(see, e.g.[9]). By function space $X_{loc}(\Omega, \mathbb{R}^N)$ we understand the space of all functions which belong to $X(\tilde{\Omega}, \mathbb{R}^N)$ for any bounded subdomain $\tilde{\Omega}$ with smooth boundary which is compactly embedded in Ω.
For definitions and more details see [1], [7], [9] and [12]. In particular, we will use:

Proposition 2.1 *For a bounded domain $\Omega \subset \mathbb{R}^n$ with a Lipschitz boundary we have the following*

(a) For $q \in (1, \infty), 0 < \lambda < \mu < \infty$ we have

$$L^{q,\mu}(\Omega, \mathbb{R}^N) \subset L^{q,\lambda}(\Omega, \mathbb{R}^N),$$
$$\mathcal{L}^{q,\mu}(\Omega, \mathbb{R}^N) \subset \mathcal{L}^{q,\lambda}(\Omega, \mathbb{R}^N),$$

(b) $\mathcal{L}^{q,\lambda}(\Omega, \mathbb{R}^N)$ is isomorphic to the $C^{0,(\lambda-n)/q}(\overline{\Omega}, \mathbb{R}^N)$, for $n < \lambda \leq n + q$,

(c) $L^{q,n}(\Omega, \mathbb{R}^N)$ is isomorphic to the $L^\infty(\Omega, \mathbb{R}^N)$,
$\mathcal{L}^{q,n}(\Omega, \mathbb{R}^N)$ is isomorphic to $BMO(\Omega, \mathbb{R}^N)$,

(d) $L^{q,\lambda}(\Omega, \mathbb{R}^N)$ is isomorphic to the $\mathcal{L}^{q,\lambda}(\Omega, \mathbb{R}^N)$, for $0 < \lambda < n$.

By means of Nirenberg's difference quotients method we obtain

Lemma 2.2 *Let u be a weak solution to (1.1) and coefficients a_i^α satisfy the hypotheses (i), (ii), (iii), (iv) with the constants M, ν and a right hand side $f \in W^{1,2}(\Omega, \mathbb{R}^{nN})$. Then $u \in W^{2,2}_{loc}(\Omega, \mathbb{R}^N)$ and for any $x \in \Omega$ and $R \in (0, 1/2 dist(x, \partial\Omega))$ it holds*

$$\begin{aligned} \int_{B(x,R)} |D^2 u|^2 dx \quad &\leq \quad C(\frac{M}{\nu})(\frac{1}{R^2} \int_{B(x,2R)} |Du - (Du)_{x,2R}|^2 dx \\ &+ \quad R^n + \int_{B(x,2R)} |Du|^2 dx + \int_{B(x,2R)} |Df|^2 dx). \end{aligned} \tag{2.1}$$

In what follows we will use an algebraic lemma due to S. Campanato. We start with recalling it.

Lemma 2.3 *(see [1]) Let α, d be positive numbers, $A > 0$, $\beta \in [0, \alpha)$. Then there exist ϵ_0, C positive so that for any nonnegative, nondecreasing function ϕ defined on $[0, d]$ and satisfying the inequality*

$$\phi(\sigma) \leq \left(A \left(\frac{\sigma}{R} \right)^\alpha + K \right) \phi(R) + BR^\beta \quad \forall \sigma, R : 0 < \sigma < R \leq d, \tag{2.2}$$

with $K \in (0, \epsilon_0]$ and $B \in [0, \infty)$ it holds

$$\phi(\sigma) \leq C\sigma^\beta \left(d^{-\beta}\phi(d) + B \right), \quad \forall \sigma : 0 < \sigma \leq d. \tag{2.3}$$

Remark. Note that we can take any $\mu \in (\beta, \alpha)$,

$$\tau = \min(1/2, (2A)^{\frac{1}{\mu-\alpha}}), \epsilon_0 = \frac{1}{2}\tau^{\mu}.$$

For the statement of following Lemma see e.g.[2], [7], [12].

Lemma 2.4 *Consider system of the type (1.1) with* $a_i^{\alpha}(x,p) = A_{ij}^{\alpha\beta} p_{\beta}^j$, $A_{ij}^{\alpha\beta} \in \mathbb{R}$ *(i.e. linear system with constant coefficients) satisfying (i), (ii) and (iii). Then there exists a constant* $L = L(n, M/\nu) \geq 1$ *such that for every weak solution* $v \in W^{1,2}(\Omega, \mathbb{R}^N)$, *for every* $x \in \Omega$ *and* $0 < \sigma \leq R \leq \operatorname{dist}(x, \partial\Omega)$ *the following estimate*

$$\int\limits_{B(x,\sigma)} |Dv(y) - (Dv)_{x,\sigma}|^2 \, dy \leq L \left(\frac{\sigma}{R}\right)^{n+2} \int\limits_{B(x,R)} |Dv(y) - (Dv)_{x,R}|^2 \, dy$$

holds.

Lemma 2.5 *([18], p.37) Let* $\psi : [0,\infty] \to [0,\infty]$ *be non decreasing function which is absolutely continuous on every closed interval of finite length,* $\psi(0) = 0$. *If* $w \geq 0$ *is measurable,* $E(t) = \{y \in \mathbb{R}^n : w(y) > t\}$ *and* μ *is n-dimensional Lebesgue measure then*

$$\int\limits_{\mathbb{R}^n} \psi \circ w \, dy = \int\limits_0^{\infty} \mu\left(E(t)\right) \psi'(t) \, dt.$$

Proof of theorem 1.1. Let x_0 be any fixed point of Ω. We prove that $Du \in \mathcal{L}^{2,\delta}$ on a neighborhood of x_0. Let $R \leq 1/2\operatorname{dist}(x_0, \partial\Omega)$. Where no confusion can result, we will use the notation $B(R)$, $U(R)$, $\phi(R)$ and $(Du)_R$ instead of $B(x_0,R)$, $U(x_0,R)$, $\phi(x_0,R)$ and $(Du)_{x_0,R}$.

Denote $A_{ij,0}^{\alpha\beta} = A_{ij}^{\alpha\beta}(x_0, (Du)_R)$,

$$\tilde{A}_{ij}^{\alpha\beta}(x) = \int\limits_0^1 A_{ij}^{\alpha\beta}\left(x_0, (Du)_R + t\left(Du(x) - (Du)_R\right)\right) dt.$$

Hence

$$a_i^{\alpha}(x_0, Du(x)) - a_i^{\alpha}(x_0, (Du)_R) = \tilde{A}_{ij}^{\alpha\beta}(x)\left(D_\beta u^j(x) - \left(D_\beta u^j\right)_R\right).$$

Thus we can rewrite the system (1.1) as

$$\begin{aligned} -D_\alpha\left(A_{ij,0}^{\alpha\beta} D_\beta u^j\right) = &- D_\alpha\left(\left(A_{ij,0}^{\alpha\beta} - \tilde{A}_{ij}^{\alpha\beta}\right)\left(D_\beta u^j - \left(D_\beta u^j\right)_R\right)\right) \\ &- D_\alpha\left(a_i^{\alpha}(x_0, Du) - a_i^{\alpha}(x, Du)\right) - D_\alpha(f_i^{\alpha}(x) - (f_i^{\alpha})_R). \end{aligned}$$

Split u as $v + w$ where v is the solution of the Dirichlet problem

$$\begin{aligned} -D_\alpha\left(A_{ij,0}^{\alpha\beta} D_\beta v^j\right) = 0 \quad \text{in} \quad B(R) \\ v - u \in W_0^{1,2}\left(B(R), \mathbb{R}^N\right). \end{aligned}$$

and $w \in W_0^{1,2}(B(R), \mathbb{R}^N)$ is the weak solution of the system

$$\begin{aligned} -D_\alpha\left(A_{ij,0}^{\alpha\beta} D_\beta w^j\right) = \quad & -D_\alpha\left(\left(A_{ij,0}^{\alpha\beta} - \tilde{A}_{ij}^{\alpha\beta}\right)\left(D_\beta u^j - \left(D_\beta u^j\right)_R\right)\right) \\ & -D_\alpha\left(a_i^{\alpha}(x_0, Du) - a_i^{\alpha}(x, Du)\right) - D_\alpha\left(f_i^{\alpha}(x) - (f_i^{\alpha})_R\right). \qquad (2.4) \end{aligned}$$

For every $0 < \sigma \leq R$ from Lemma 2.4 it follows

$$\int_{B(\sigma)} |Dv - (Dv)_\sigma|^2 \, dx \leq L \left(\frac{\sigma}{R}\right)^{n+2} \int_{B(R)} |Dv - (Dv)_R|^2 \, dx$$

hence

$$\begin{aligned}
\int_{B(\sigma)} |Du - (Du)_\sigma|^2 \, dx &\leq 2L \left(\frac{\sigma}{R}\right)^{n+2} \int_{B(R)} |Dv - (Dv)_R|^2 \, dx + 4 \int_{B(R)} |Dw|^2 \, dx \\
&\leq 4L \left(\frac{\sigma}{R}\right)^{n+2} \int_{B(R)} |Du - (Du)_R|^2 \, dx \\
&+ 4 \left(1 + 2L \left(\frac{\sigma}{R}\right)^{n+2}\right) \int_{B(R)} |Dw|^2 \, dx. \qquad (2.5)
\end{aligned}$$

Now as $w \in W_0^{1,2}(B_R, \mathbb{R}^N)$ we can choose test function $\varphi = w$ in (2.4) and get

$$\begin{aligned}
\nu^2 \int_{B(R)} |Dw|^2 \, dx &\leq \int_{B(R)} \omega^2 \left(|Du - (Du)_R|\right) |Du - (Du)_R|^2 \, dx \\
&+ \int_{B(R)} |a_i^\alpha(x_0, Du) - a_i^\alpha(x, Du)|^2 \, dx \\
&+ \int_{B(R)} |f_i^\alpha(x) - (f_i^\alpha)_R|^2 \, dx. \qquad (2.6)
\end{aligned}$$

From (2.5),(2.6) and Poincare's inequality we have

$$\begin{aligned}
\phi(\sigma) &= \int_{B(\sigma)} |Du - (Du)_\sigma|^2 \, dx \leq 4L \left(\frac{\sigma}{R}\right)^{n+2} \int_{B(R)} |Du - (Du)_R|^2 \, dx \\
&+ \frac{4\left(1 + 2L\left(\frac{\sigma}{R}\right)^{n+2}\right)}{\nu^2} \left[\int_{B(R)} \omega^2 \left(|Du - (Du)_R|\right) |Du - (Du)_R|^2 \, dx \right. \\
&+ \left. \int_{B(R)} |a_i^\alpha(x_0, Du) - a_i^\alpha(x, Du)|^2 \, dx + R^2 \int_{B(R)} |Df|^2 \, dx \right] \\
&\leq 4L \left(\frac{\sigma}{R}\right)^{n+2} \phi(R) + \frac{4\left(1 + 2L\left(\frac{\sigma}{R}\right)^{n+2}\right)}{\nu^2} \left[(I_1 + I_2) + R^\delta ||Df||^2_{L^{2,\delta-2}} \right]. \qquad (2.7)
\end{aligned}$$

Then using Hölder inequality with the exponent p from the assumptions of the Theorem, embedding and Lemma 2.2 we have

$$
\begin{aligned}
I_1 &\le \left(\int\limits_{B(R)} |Du-(Du)_R|^{2p}\,dx\right)^{1/p}\left(\int\limits_{B(R)} \omega^{2p'}\left(|Du-(Du)_R|\right)\,dx\right)^{1/p'}\\
&\le C_p^2R^{2-n/p'}\int\limits_{B(R)}|D^2u|^2\,dx\left(\int\limits_{B(R)} \omega^{2p'}\left(|Du-(Du)_R|\right)\,dx\right)^{1/p'}\\
&\le C(p,n,M/\nu)\left(\frac{1}{\kappa_nR^n}\int\limits_{B(R)} \omega^{2p'}\left(|Du-(Du)_R|\right)\,dx\right)^{1/p'}(\phi(2R)\\
&+ R^{n+2}+R^2||Du||^2_{L^2(B(2R))}+R^\delta||Df||^2_{L^{2,\delta-2}(\Omega)})
\end{aligned}
\tag{2.8}
$$

where C_p stands for embedding constant from $W^{1,2}(B(1),\mathbb{R}^{nN})$ into $L^{2p}(B(1),\mathbb{R}^{nN})$ and

$$C(p,n,M/\nu)=C_p^2\times C(\frac{M}{\nu}), \tag{2.9}$$

$C(\frac{M}{\nu})$ is the constant from Lemma 2.2.

Taking in Lemma 2.5 $\psi(t)=\omega^{2p'}(t)$, $w=|Du-(Du)_R|$ on $B(R)$ and $w=0$ otherwise, we have $E_R(t)=\{y\in B(R)\,:\,|Du-(Du)_R|>t\}$ and for the last integral we get

$$\int\limits_{B(R)} \omega^{2p'}\left(|Du-(Du)_R|\right)\,dx=\int\limits_0^\infty\left[\frac{d}{dt}(\omega^{2p'})(t)\right]\mu\left(E_R(t)\right)\,dt.$$

Now we can estimate the integral on the right hand side according to assumptions of the theorem. In the first case we assume that

$$P_p=J_p=\int\limits_0^\infty\frac{\frac{d}{dt}(\omega^{2p'})(t)}{t}\,dt<\infty.$$

As $\mu\left(E_R(t)\right)$ is nonnegative, non-increasing it holds $\mu\left(E_R(t)\right)\le\frac{1}{t}\int\limits_0^t\mu\left(E_R(s)\right)\,ds$ and we have

$$
\begin{aligned}
\int\limits_0^\infty\left[\frac{d}{dt}(\omega^{2p'})(t)\right]\mu\left(E_R(t)\right)\,dt &\le \int\limits_0^\infty\frac{d}{dt}(\omega^{2p'})(t)\left(\frac{1}{t}\int\limits_0^t\mu\left(E_R(s)\right)\,ds\right)\,dt\\
&\le \int\limits_0^\infty\frac{\frac{d}{dt}(\omega^{2p'})(t)}{t}\,dt\int\limits_{B(R)}|Du-(Du)_R|\,dx\\
&\le J_pR^{n/2}\phi^{1/2}(R).
\end{aligned}
\tag{2.10}
$$

If $P_p=S_p=\sup_{0<t<\infty}\frac{d}{dt}(\omega^{2p'})(t)<\infty$ we have

$$\int\limits_0^\infty\left[\frac{d}{dt}(\omega^{2p'})(t)\right]\mu\left(E_R(t)\right)\,dt\le S_pR^{n/2}\phi^{1/2}(R) \tag{2.11}$$

Denoting

$$K = C(p, n, \frac{M}{\nu}) P_p^{1/p'} \|Du\|_{BMO}^{1/2p'} \tag{2.12}$$

and using (2.8), (2.10), (2.11) for the estimate of I_1 we get

$$I_1 \leq K\phi(2R) + K(R^{n+2} + R^2||Du||^2_{L^2} + R^\delta ||Df||^2_{L^{2,\delta-2}}) \tag{2.13}$$

As we suppose that $Du \in BMO(\Omega)$ we have from Proposition 2.1 that $Du \in L^{2,\lambda}$ for any $\lambda < n$. Set $\lambda = \delta - 2, R < 1$. Hence

$$\begin{aligned} I_1 &\leq K\phi(2R) + K(1 + ||Du||^2_{BMO} + ||Df||^2_{L^{2,\delta-2}})R^\delta, \\ I_2 &\leq M^2 R^2 \int\limits_{B(R)} \left(1 + |Du|^2\right) dx \leq M^2 \left(\kappa_n R^{n+2} + R^2 \int\limits_{B(R)} |Du|^2 \, dx \right) \\ &\leq M^2 \left(\kappa_n + \|Du\|^2_{BMO(\Omega)}\right) R^\delta \end{aligned} \tag{2.14}$$

for any $\lambda < n$, $R < 1$.

We get from (2.7) by means of (2.13) and (2.14)

$$\begin{aligned} \phi(\sigma) \leq & \left[4L \left(\frac{\sigma}{R}\right)^{n+2} + \frac{4\left(1 + 2L\left(\frac{\sigma}{R}\right)^{n+2}\right)}{\nu^2} K \right] \phi(2R) \\ + & \frac{4\left(1 + 2L\left(\frac{\sigma}{R}\right)^{n+2}\right)}{\nu^2} (K + M^2) \left(\kappa_n + \|Du\|^2_{BMO(\Omega)} + 2||Df||^2_{L^{2,\delta-2}(\Omega)}\right) R^\delta. \end{aligned} \tag{2.15}$$

If the assumptions of Theorem are satisfied then

$$\frac{4\left(1 + 2L(\tau)^{n+2}\right)}{\nu^2} K < \epsilon_0$$

and we can use Lemma 2.3 with $A = 4L$ and τ, ϵ_0 given in the remark after Lemma 2.3 to get

$$\phi(\sigma) \leq C\sigma^\delta. \tag{2.16}$$

The thesis follows from Proposition 2.1, part (b).

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