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On a New Proof of the Faber-Manteuffel Theorem

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A fundamental theorem in the area of iterative methods is the Faber-Manteuffel Theorem [2]. It shows that a short recurrence for orthogonalizing Krylov subspace bases for a matrix A exists if and only if the adjoint of A is a low degree polynomial in A. This result is important, since it characterizes all matrices, for which an optimal Krylov subspace method with short recurrences can be constructed. Here optimal means that the error is minimized in the norm induced by the given inner product. Of course, such methods are highly desirable, due to convenient work and storage requirements for generating the orthogonal basis vectors. Examples are the CG method [3] for solving systems of linear algebraic equations with a symmetric positive definite matrix A, or the MINRES method [6] for solving symmetric but indefinite systems.

Now we briefly describe the result of Faber and Manteuffel. Let A be a nonsingular matrix and v be a vector of grade d (d is the degree of the uniquely determined monic polynomial of smallest degree that annihilates v). For theoretical as well as practical purposes it is often convenient to orthogonalize the basis $v, \ldots, A^{d-1}v$ of the cyclic subspace $\mathcal{K}_d(A, v)$. The classical approach to orthogonalization is to use the Arnoldi method, that produces mutually orthogonal vectors $v_1, \ldots, v_d$ satisfying $\text{span}\{v_1, \ldots, v_n\} = \text{span}\{v, \ldots, A^{n-1}v\}$, $n = 1, \ldots, d$. The algorithm can be written in a matrix form

$$v_1 = v\,, \tag{1}$$

$$A \underbrace{[v_1, \ldots, v_{d-1}]}_{\equiv V_{d-1}} = \underbrace{[v_1, \ldots, v_d]}_{\equiv V_d} \underbrace{\begin{bmatrix} h_{1,1} & \cdots & h_{1,d-1} \\ 1 & \ddots & \vdots \\ & \ddots & h_{d-1,d-1} \\ & & 1 \end{bmatrix}}_{\equiv H_{d,d-1}}, \tag{2}$$

$$(v_i, v_j) = 0 \quad \text{for } i \neq j\,, \quad i, j = 1, \ldots, d\,. \tag{3}$$

As described above, for efficiency reasons, it is desirable to generate such an orthogonal basis with a short recurrence, meaning that in each iteration step only a few of the latest basis vectors are required to generate the new basis vector. This corresponds to the situation when the matrix $H_{d,d-1}$ in (2) is, for each starting vector v_1, low-band Hessenberg matrix. Note that an unreduced upper Hessenberg matrix is called $(s+2)$-band Hessenberg, when its s-th superdiagonal contains at least one nonzero entry, and all its entries above its s-th superdiagonal are zero. We say that A *admits an optimal* $(s+2)$*-term recurrence* if $H_{d,d-1}$ is for each starting vector at most $(s+2)$-band Hessenberg and, moreover, there exists an initial vector such that $H_{d,d-1}$ is exactly $(s+2)$-band Hessenberg (the s-th superdiagonal contains at least one nonzero entry). The fundamental question is, what properties are necessary and sufficient for A to admit an optimal $(s+2)$-term recurrence. This question was answered by Faber and Manteuffel [2].

THEOREM (Faber-Manteuffel) *Let A be a nonsingular matrix with minimal polynomial degree $d_{\min}(A)$. Let s be a nonnegative integer, $s+2 < d_{\min}(A)$. Then A admits an optimal $(s+2)$-term recurrence if and only if $A^* = p(A)$, where p is a polynomial of smallest degree s having this property (i.e. A is normal(s)).*

While the sufficiency of the normal(s) condition is rather easy to prove, the proof of necessity given by Faber and Manteuffel is based on a clever, highly nontrivial construction by using results from mathematical analysis ("continuous function"), topology ("closed set of smaller dimension") or multilinear algebra ("wedge product").

In [5], Liesen and Strakoš discuss and clarify the existing important results in the context of the Faber-Manteuffel Theorem. They suggest that, in light of the fundamental nature of the result, it is desirable to find an alternative, and possibly simpler proof of the necessity part.

In our recent paper [1] we address this issue and give two new proofs of the necessity part, which use more elementary tools. In particular, we avoid topological and analytical arguments, and use linear algebra tools instead. In this talk we will present the proof which employs elementary Givens rotations to prove the necessity part by contradiction. In particular, the proof is based on orthogonal transformations ("rotations") of upper Hessenberg matrices. We will also discuss why the proof of the necessity part is much more difficult than the proof of sufficiency.

References

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