11. Flow near a rotating disk. A further example of an exact solution of the Navier-Stokes equations is furnished by the flow around a flat disk which rotates about an axis perpendicular to its plane with a uniform angular velocity, ω, in a fluid otherwise at rest. The layer near the disk is carried by it through friction and is thrown outwards owing to the action of centrifugal forces. This is compensated by particles which flow in an axial direction towards the disk to be in turn carried and ejected centrifugally. Thus the case is seen to be one of fully three-dimensional flow, i. e., there exist velocity components in the radial direction, r, the circumferential direction, ϕ, and the axial direction, z, which we shall denote respectively by u, v, and w. An axonometric representation of this flow field is shown in Fig. 5.12. At first the calculation will be performed for the case of an infinite rotating plane. It will then be easy to extend the result to include a disk of finite diameter $D = 2\,R$, on condition that the edge effect is neglected.

Taking into account rotational symmetry as well as the notation for the problem we can write down the Navier-Stokes equations (3.36) as:

$$\left.\begin{aligned}
u\frac{\partial u}{\partial r} - \frac{v^2}{r} + w\frac{\partial u}{\partial z} &= -\frac{1}{\varrho}\frac{\partial p}{\partial r} + \nu\left\{\frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{u}{r}\right) + \frac{\partial^2 u}{\partial z^2}\right\},\\
u\frac{\partial v}{\partial r} + \frac{uv}{r} + w\frac{\partial v}{\partial z} &= \nu\left\{\frac{\partial^2 v}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{\partial^2 v}{\partial z^2}\right\},\\
u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} &= -\frac{1}{\varrho}\frac{\partial p}{\partial z} + \nu\left\{\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right\},\\
\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} &= 0\,.
\end{aligned}\right\} \qquad (5.48)$$

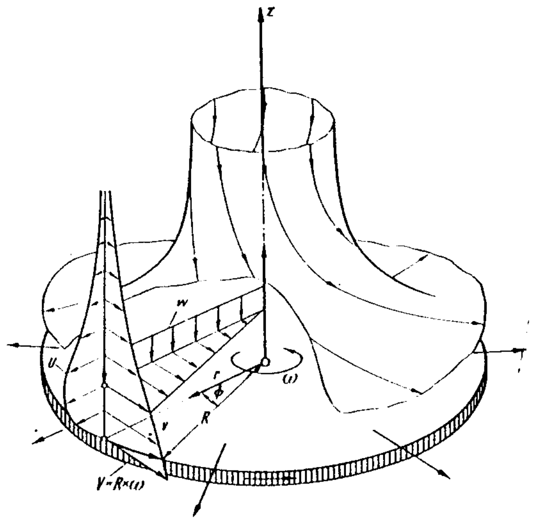

Fig. 5.12. Flow in the neighbourhood of a disk rotating in a fluid at rest

Velocity components: u-radial, v-circumferential, w-axial. A layer of fluid is carried by the disk owing to the action of viscous forces. The centrifugal forces in the thin layer give rise to secondary flow which is directed radially outward

The no-slip condition at the wall gives the following boundary conditions:

$$\left.\begin{aligned}
z = 0:&\quad u = 0, \quad v = r\,\omega, \quad w = 0,\\
z = \infty:&\quad u = 0, \quad v = 0.
\end{aligned}\right\} \qquad (5.49)$$

We shall begin by estimating the thickness, δ, of the layer of fluid 'carried' by the disk [23]. It is clear that the thickness of the layer of fluid which rotates with the disk owing to friction decreases with the viscosity and this view is confirmed when compared with the results of the preceding examples. The centrifugal force per unit volume which acts on a fluid particle in the rotating layer at a distance r from the axis is equal to $\varrho\, r\, \omega^2$. Hence for a volume of area $\mathrm{d}r \cdot \mathrm{d}s$ and height, δ, the centrifugal force becomes: $\varrho\, r\, \omega^2\, \delta\, \mathrm{d}r\, \mathrm{d}s$. The same element of fluid is acted upon by a shearing stress τ_w, pointing in the direction in which the fluid is slipping, and forming an angle, say θ, with the circumferential velocity. The radial component of the shearing stress must now be equal to the centrifugal force, and hence

$$\tau_w \sin\theta\, \mathrm{d}r\, \mathrm{d}s = \varrho\, r\, \omega^2\, \delta\, \mathrm{d}r\, \mathrm{d}s$$

or

$$\tau_w \sin\theta = \varrho\, r\, \omega^2\, \delta\,.$$

On the other hand the circumferential component of the shearing stress must be proportional to the velocity gradient of the circumferential velocity at the wall. This condition gives

$$\tau_w \cos\theta \sim \mu\, r\, \omega/\delta\,.$$

Eliminating τ_w from these two equations we obtain

$$\delta^2 \sim \frac{\nu}{\omega}\tan\theta\,.$$

If it is assumed that the direction of slip in the flow near the wall is independent of the radius, the thickness of the layer carried by the disk becomes

$$\delta \sim \sqrt{\frac{\nu}{\omega}}\,,$$

which is identical with the result obtained in the case of the oscillating wall on p. 94. Further, we can write for the shearing stress at the wall

$$\tau_w \sim \varrho\, r\, \omega^2\, \delta \sim \varrho\, r\, \omega\sqrt{\nu\,\omega}\,.$$

The torque, which is equal to the product of shearing stress at the wall, area and arm becomes

$$M \sim \tau_w R^3 \sim \varrho\, R^4\, \omega\sqrt{\nu\,\omega}\,, \qquad (5.50)$$

R denoting the radius of the disk.

In order to integrate the system of equations (5.48) it is convenient to introduce a dimensionless distance from the wall, $\zeta \sim z/\delta$, thus putting

$$\zeta = z\sqrt{\frac{\omega}{\nu}}\,. \qquad (5.51)$$

Further, the following assumptions are made for the velocity components and pressure

$$u = r\,\omega\,F(\zeta); \qquad v = r\,\omega\,G(\zeta); \qquad w = \sqrt{\nu\,\omega}\;H(\zeta) \tag{5.52}$$
$$p = p\,(z) = \varrho\,\nu\,\omega\,P(\zeta)\,.$$

Inserting these equations into eqns. (5.48) we obtain a system of four simultaneous ordinary differential equations for the functions F, G, H, and P:

$$\left.\begin{aligned} 2\,F + H' &= 0 \\ F^2 + F'\,H - G^2 - F'' &= 0 \\ 2\,F\,G + H\,G' - G'' &= 0 \\ P' + H\,H' - H'' &= 0\,. \end{aligned}\right\} \tag{5.53}$$

The boundary conditions can be calculated from eqn. (5.49) and are:

$$\zeta = 0: \quad F = 0\,, \quad G = 1\,, \quad H = 0\,, \quad P = 0$$
$$\zeta = \infty: \quad F = 0\,, \quad G = 0\,.$$

The first solution of the system of eqns. (5.53) by an approximate method was given by a method of numerical integration†. They are plotted in Fig. 5.13. The starting values of the solution indicated in Table 5.2 were given by E. M. Sparrow and J. L. Gregg [32].

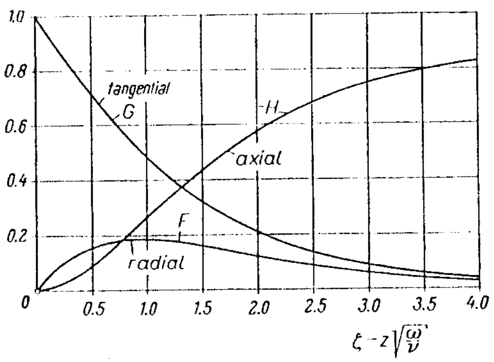

Fig. 5.13. Velocity distribution near a disk rotating in a fluid at rest

In the case under discussion, just as in the example involving a stagnation point, the velocity field is the first to be evaluated from the equation of continuity and the equations of motion parallel to the wall. The pressure distribution is found subsequently from the equation of motion perpendicular to the wall.

† This solution was obtained in the form of a power series near $\zeta = 0$ and an asymptotic series for large values of ζ which were then joined together for moderate values of ζ.

Table 5.2. Values of the functions needed for the description of the flow of a disk rotating in a fluid at rest calculated at the wall and at a large distance from the wall, as calculated by E. M. Sparrow and J. L. Gregg [32]

$\zeta = z\sqrt{\frac{\omega}{\nu}}$	F'	$-G'$	$-H$	P
0	0·510	0·6159	0	0
∞	0	0	0·8845	0·3912

It is seen from Fig. 5.13 that the distance from the wall over which the peripheral velocity is reduced to half the disk velocity is $\delta_{0\cdot5} \approx \sqrt{\nu/\omega}$. It is to be noted from the solution that when $\delta \approx \sqrt{\nu/\omega}$ is small, the velocity components u and v have appreciable values only in a thin layer of thickness $\sqrt{\nu/\omega}$. The velocity component w, normal to the disk is, at any rate, small and of the order $\sqrt{\nu\,\omega}$. The inclination of the relative streamlines near the wall with respect to the circumferential direction, if the wall is imagined at rest and the fluid is taken to rotate at a large distance from the wall, becomes

$$\tan\phi_0 = -\left(\frac{\partial u/\partial z}{\partial v/\partial z}\right)_{z=0} = -\frac{F'(0)}{G'(0)} = \frac{0{\cdot}510}{0{\cdot}616} = 0{\cdot}828,$$

or

$$\phi_0 = 39{\cdot}6^\circ\,.$$

Although the calculation is, strictly speaking, applicable to an infinite disk only, we may utilize the same results for a finite disk, provided that its radius R is large compared with the thickness δ of the layer carried with the disk. We shall now evaluate the turning moment of such a disk. The contribution of an annular disk element of width $\mathrm{d}r$ on radius r is $\mathrm{d}M = -2\,\pi\,r\,\mathrm{d}r\,r\,\tau_{z\phi}$, and hence the moment for a disk wetted on one side becomes

$$M = -2\pi \int_0^R r^2\,\tau_{z\phi}\,\mathrm{d}r\,.$$

Here $\tau_{z\phi} = \mu(\partial v/\partial z)_0$ denotes the circumferential component of the shearing stress. From eqn. (5.52) we obtain

$$\tau_{z\phi} = \varrho\,r\,\nu^{1/2}\,\omega^{3/2}\,G'(0)\,.$$

Hence the moment for a disk wetted on both sides becomes

$$2\,M = -\pi\,\varrho\,R^4\,(\nu\omega^3)^{1/2}\,G'(0) = 0{\cdot}616\,\pi\,\varrho\,R^4\,(\nu\omega^3)^{1/2}\,. \tag{5.54}$$

It is customary to introduce the following dimensionless moment coefficient,

$$C_M = \frac{2\,M}{\frac{1}{2}\,\varrho\,\omega^2\,R^5}\,. \tag{5.55}$$

This gives

$$C_M = -\frac{2\pi G'(0)\,\nu^{1/2}}{R\,\omega^{1/2}},$$

or, defining a Reynolds number based on the radius and tip velocity,

$$\mathsf{R} = \frac{R^2\,\omega}{\nu}$$

and introducing the numerical value $-2\pi\,G'(0) = 3{\cdot}87$, we obtain finally

$$\boxed{C_M = \frac{3{\cdot}87}{\sqrt{\mathsf{R}}}}\,. \tag{5.56}$$

Fig. 5.14 shows a plot of this equation, curve (1), and compares it with measurements [39]. For Reynolds numbers up to about $\mathsf{R} = 3 \times 10^5$ there is excellent agreement between theory and experiment. At higher Reynolds numbers the flow becomes turbulent, and the respective case is considered in Chap. XXI.
Curves (2) and (3) in Fig. 5.14 are obtained from the turbulent flow theory. Older measurements, carried out by G. Kempf [16] and W. Schmidt [31], show tolerable agreement with theoretical results. Prior to these solutions, D. Riabouchinsky [26], [27] established empirical formulae for the turning moment of rotating disks which were based on very careful measurements. These formulae showed very good agreement with the theoretical equations discovered subsequently.

The quantity of liquid which is pumped outwards as a result of the centrifuging action on the one side of a disk of radius R is

$$Q = 2\pi\,R \int_{z=0}^{\infty} u\,\mathrm{d}z\,.$$

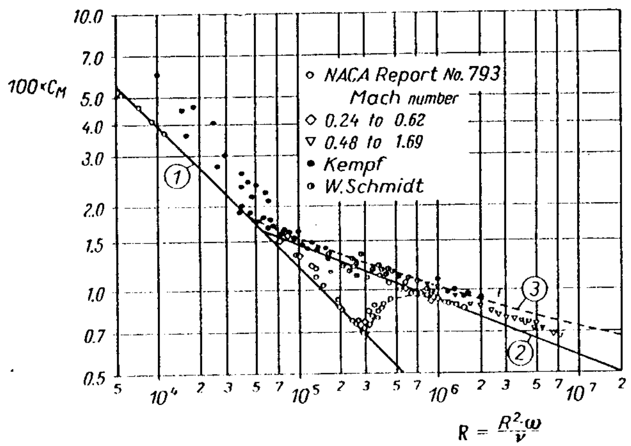

Fig. 5.14. Turning moment on a rotating disk; curve (1) from eqn. (5.56), *laminar*; curves (2) and (3) from eqns. (21.30) and (21.33), *turbulent*

Calculation shows that

$$Q = 0{\cdot}885\,\pi\,R^2\sqrt{\nu\,\omega} = 0{\cdot}885\,\pi\,R^3\,\omega\,\mathsf{R}^{-1/2}\,. \tag{5.57}$$

The quantity of fluid flowing towards the disk in the axial direction is of equal magnitude. It is, further, worthy of note that the pressure difference over the layer carried by the disk is of the order $\varrho\,\nu\,\omega$, i. e., very small for small viscosities. The pressure distribution depends only on the distance from the wall, and there is no radial pressure gradient.

A generalised form of the preceding problem has been studied by M. G. Rogers and G. N. Lance [28] who assumed that the fluid moves with an angular velocity $\Omega = s\,\omega$ at infinity. With this assumption, the second equation (5.53) becomes modified to

$$F^2 + F'\,H - G^2 - F'' + s^2 = 0\,,$$

and the second boundary condition for the function $G(\zeta)$ must be replaced by $G(\infty) = s$. In this connexion a comparison should be made with the case of rotating flow over a fixed disk given in Sec. XIa. Numerical solutions for rotation in the same sense ($s > 0$) can be found in [26]. When the rotations are in opposite senses ($s < 0$), physically meaningful solutions can be obtained for $s < -0{\cdot}2$ only if uniform suction at right angles to the disk is admitted.

The problem of a rotating disk in a housing is discussed in Chap. XXI.

It is particularly noteworthy that the solution for the rotating disk as well as the solutions obtained for the flow with stagnation are, in the first place, exact solutions of the Navier-Stokes equations and, in the second, that they are of a *boundary-layer type*, in the sense discussed in the preceding chapter. In the limiting case of very small viscosity these solutions show that the influence of viscosity extends over a very small layer in the neighbourhood of the solid wall, whereas in the whole of the remaining region the flow is, practically speaking, identical with the corresponding ideal (potential) case. These examples show further that the boundary layer has a thickness of the order $\sqrt{\nu}$. The one-dimensional examples of flow discussed previously display the same boundary-layer character. In this connexion the reader may wish to consult a paper by G. K. Batchelor [2] which discusses the solution of the Navier-Stokes equations for the case of two co-axial, rotating disks placed at a certain distance apart, as well as a paper by K. Stewartson [34]. An extension of the preceding solution to the case of uniform suction is due to J. T. Stuart ([92] in Chap. XIV) and to E. M. Sparrow and J. L. Gregg (see p. 3 in [32]). The latter contains also an analysis of the case with homogeneous blowing. The limiting case of very vigorous blowing was discussed by H. K. Kuiken [18].

12. Flow in convergent and divergent channels. A further class of exact solutions of the Navier-Stokes equations can be obtained in the following way: Let it be assumed that the family of straight lines passing through a point in a plane constitute the streamlines of a flow. Let the velocity differ from line to line, which means that it is assumed to be a function of the polar angle ϕ. The rays along which the velocity vanishes can then be regarded as the solid walls of a convergent or a divergent channel. The continuity equations can be satisfied by assuming that the velocity along every ray is inversely proportional to the distance from the origin. Hence the radial velocity u has the form $u \sim F(\phi)/r$, or, if F is to be dimensionless,