Deadline: Thursday 7th January 2021

Consider a homogeneous isotropic elastic solid with the Helmholtz free energy in the form

$$\psi = \psi(\theta, \mathbb{B}). \tag{1}$$

We already know that the Cauchy stress $\mathbb{T}$ tensor is related to the derivative of the Helmholtz free energy via the formula

$$\mathbb{T} = 2\rho \frac{\partial \psi}{\partial \mathbb{B}} \mathbb{B}. \tag{2}$$

1. Since the material is isotropic, the Helmholtz free energy must be in fact a function of the invariants of $\mathbb{B}$,

$$\psi = \psi(\theta, \mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3), \tag{3}$$

where the invariants are given by the formulae

$$\mathrm{I}_1 =_{\mathrm{def}} \operatorname{Tr} \mathbb{B}, \qquad \mathrm{I}_2 =_{\mathrm{def}} \frac{1}{2}\left((\operatorname{Tr} \mathbb{B})^2 - \operatorname{Tr} \mathbb{B}^2 \right), \qquad \mathrm{I}_3 =_{\mathrm{def}} \det \mathbb{B}.$$

(Recall that we have the *representation theorem for scalar isotropic functions.*) Show that if we use (3) in (2), then the formula for the Cauchy stress tensor reads

$$\mathbb{T} = \beta_0 \mathbb{I} + \beta_1 \mathbb{B} + \beta_{-1} \mathbb{B}^{-1}, \tag{4a}$$

where

$$\beta_0 =_{\mathrm{def}} 2\rho \left(\mathrm{I}_2 \frac{\partial \psi}{\partial \mathrm{I}_2} + \mathrm{I}_3 \frac{\partial \psi}{\partial \mathrm{I}_3} \right), \tag{4b}$$

$$\beta_1 =_{\mathrm{def}} 2\rho \frac{\partial \psi}{\partial \mathrm{I}_1}, \tag{4c}$$

$$\beta_{-1} =_{\mathrm{def}} -2\rho \, \mathrm{I}_3 \frac{\partial \psi}{\partial \mathrm{I}_2}. \tag{4d}$$

Note that the corresponding formulae are in the most textbooks on solid mechanics written in a slightly different form. The reason is that the Helmholtz free energy is frequently defined as the Helmholtz free energy per unit *volume*, while we have defined the Helmholtz free energy as the energy per unit *mass*.

2. Some people prefer to write (1) as

$$\psi = \psi(\theta, J, \overline{\mathbb{B}}),$$

where $J =_{\mathrm{def}} \det \mathbb{F}$ and

$$\overline{\mathbb{B}} =_{\mathrm{def}} \frac{\mathbb{B}}{J^{\frac{2}{3}}}.$$

This decomposition is motivated by the fact that J is related to the volume-changing part of the deformation, while $\overline{\mathbb{B}}$ characterises the volume-preserving part of the deformation. (Check that $\det \overline{\mathbb{B}} = 1$.) Show that in this case the counterpart of (2) is

$$\mathbb{T} = \rho J \frac{\partial \psi}{\partial J} \mathbb{I} + 2\rho \left(\frac{\partial \psi}{\partial \overline{\mathbb{B}}} \overline{\mathbb{B}} \right)_{\delta},$$

where $\mathbb{A}_{\delta} =_{\mathrm{def}} \mathbb{A} - \frac{1}{3} (\operatorname{Tr} \mathbb{A}) \mathbb{I}$ denotes the traceless part of the corresponding tensorial quantity.

Deadline: Thursday 17th December 2020

1. Let $\mathbb{T}_{\mathrm{R}}$ and $\mathbb{T}$ denote the first Piola–Kirchhoff tensor and the Cauchy stress tensor respectively. Show that

$$\operatorname{Div} \mathbb{T}_{\mathrm{R}} = (\det \mathbb{F}) \operatorname{div} \mathbb{T},$$

or, in detail, that

$$\operatorname{Div}_{\boldsymbol{X}} \mathbb{T}_{\mathrm{R}}(\boldsymbol{X}, T) = (\det \mathbb{F}(\boldsymbol{X}, t)) \left. (\operatorname{div}_{\boldsymbol{x}} \mathbb{T}(\boldsymbol{x}, t)) \right|_{\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)},$$

where $\mathbb{F}$ denotes the deformation gradient and $\boldsymbol{\chi}(\boldsymbol{X}, t)$ is the deformation. (Hint: Direct differentiation is not a good idea, try the intergal formulation thereof!) Show that if we fix $\mathbb{T} = \mathbb{I}$, then the identity reads

$$\operatorname{Div} \operatorname{cof} \mathbb{F} = \mathbf{0}.$$

This identity is usually referred to as the Piola's identity.

2. [**Optional**] Show that the equation $\operatorname{Div} \operatorname{cof} \mathbb{F} = \mathbf{0}$ is the Euler–Lagrange equation for the functional $\Psi(\boldsymbol{\chi}) = \int_{V(t_0)} \det \mathbb{F} \, \mathrm{dV}$.

Deadline: Thursday 10th December 2020

1. We know that the stress–strain relation for an isotropic solid in the linearised setting reads

$$\mathbb{T} = \lambda \left(\operatorname{Tr} \mathbb{e}\right) \mathbb{I} + 2\mu \mathbb{e},$$

where the linearised strain tensor is defined as $\mathbb{e} = \frac{1}{2}\left(\nabla \boldsymbol{U} + \nabla \boldsymbol{U}^{\top}\right)$. Show that the balance of momentum is in this setting reduced to the following evolution equation for the displacement $\boldsymbol{U}$,

$$\rho_{\mathrm{R}} \frac{\partial^2 \boldsymbol{U}}{\partial t^2} = (\lambda + \mu) \nabla \operatorname{div} \boldsymbol{U} + \mu \Delta \boldsymbol{U}.$$

(We do not consider body forces.)

Deadline: Thursday 3rd December 2020

1. We have investigated the deformation of a right circular cylinder of length L and radius R by the force $\boldsymbol{F}$, see Figure 1. Using the already known formula for the displacement field, show that

$$-\frac{\frac{\Delta R}{R}}{\frac{\Delta L}{L}} = \frac{\lambda}{2(\lambda + \mu)},$$

where ΔL denotes the change of the length of the cylinder in the direction of the z-axis, and ΔR denotes the change of the radius of the cylinder. The constant $\nu =_{\mathrm{def}} \frac{\lambda}{2(\lambda+\mu)}$ is referred to as the Poisson's ratio.

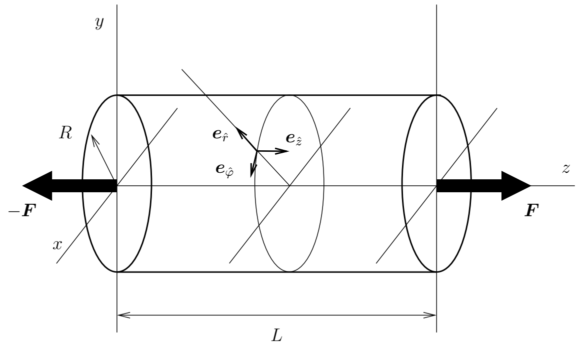

Figure 1: Deformation of a right circular cylinder.

Deadline: Thursday 26th November 2020

1. Assume that the specific Helmholtz free energy is a function of the principal invariants of $\mathbb{B}$, that is

$$\psi = \psi(\theta, \mathrm{I}_1(\mathbb{B}), \mathrm{I}_2(\mathbb{B}), \mathrm{I}_3(\mathbb{B})).$$

Show that this structural assumption implies that

$$\frac{\partial \psi}{\partial \mathbb{B}} \mathbb{B} = \mathbb{B} \frac{\partial \psi}{\partial \mathbb{B}}.$$

Deadline: Thursday 19th November 2020

1. Consider the energetic equation of state for the calorically perfect ideal gas $e(\eta, \rho)$. (The equation of state has been identified in the previous homework, I recall that we have obtained $e(\eta, \rho) = c_{\mathrm{V,ref}} \theta_{\mathrm{ref}} \left(\frac{\rho}{\rho_{\mathrm{ref}}}\right)^{\gamma-1} \mathrm{e}^{\frac{\eta}{c_{\mathrm{V,ref}}}}$.) Show that the specific Helmholtz free energy ψ for the calorically perfect ideal gas is given by the formula

$$\psi(\theta, \rho) = -c_{\mathrm{V,ref}} \theta \left(\ln\left(\frac{\theta}{\theta_{\mathrm{ref}}}\right) - 1 \right) + c_{\mathrm{V,ref}} \theta (\gamma - 1) \ln\left(\frac{\rho}{\rho_{\mathrm{ref}}}\right), \tag{5}$$

where ρ_{ref} and θ_{ref} are some constants. (Temperature and density at a reference state.)

2. [**Optional** – Easy but useful for those of you who are starting with thermodynamics.] Show that the general formulae

$$p_{\mathrm{th}}(\theta, \rho) = \rho^2 \frac{\partial \psi}{\partial \rho}(\theta, \rho),$$
$$c_{\mathrm{V}}(\theta, \rho) = -\theta \frac{\partial^2 \psi}{\partial \theta^2}(\theta, \rho),$$

and the Helmholtz free energy in the form (5) give the expected formulae for the thermodynamic pressure p_{th} and for the specific heat at constant volume c_{V} (as functions of temperature and density), namely the formulae

$$p_{\mathrm{th}} = c_{\mathrm{V,ref}} (\gamma - 1) \rho \theta,$$
$$c_{\mathrm{V}} = c_{\mathrm{V,ref}}.$$

3. [**Optional** – A little bit more difficult exercise, just for those of you who are really interested in the subject matter.] Consider a viscous compressible heat conducting fluid in a thermodynamically isolated vessel Ω. (Not necessarily a calorically prefect ideal gas.) The governing equations are

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \operatorname{div} \boldsymbol{v} &= 0,\\ \rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} &= \operatorname{div}\left(-p_{\mathrm{th}}\mathbb{I} + \lambda \operatorname{div}\boldsymbol{v} + 2\mu\mathbb{D}\right),\\ \rho c_{\mathrm{V}} \frac{\mathrm{d}\theta}{\mathrm{d}t} &= -\theta \frac{\partial p_{\mathrm{th}}}{\partial \theta} \operatorname{div}\boldsymbol{v} + 2\mu \mathbb{D} : \mathbb{D} + \lambda \left(\operatorname{div}\boldsymbol{v}\right)^2 + \operatorname{div}\left(\kappa \nabla \theta\right),\end{aligned}$$

and the boundary conditions are

$$\begin{aligned}\boldsymbol{v}|_{\partial\Omega} &= \mathbf{0},\\ \kappa\nabla\theta \bullet \boldsymbol{n} &= 0.\end{aligned}$$

Clearly, one solution of these governing equations (with the given boundary conditions) is the spatially homogeneous rest state $[\widehat{\rho}, \widehat{\theta}, \widehat{\boldsymbol{v}}]$, where $\widehat{\rho}$ and $\widehat{\theta}$ are some constants and $\widehat{\boldsymbol{v}} = \mathbf{0}$.

a) Write down the governing equations in the neighborhood of this spatially homogeneous rest state, and *linerise these governing equations*. More specifically, rewrite the density, temperature and velocity as

$$\begin{aligned}\rho &= \widehat{\rho} + \widetilde{\rho},\\ \boldsymbol{v} &= \widehat{\boldsymbol{v}} + \widetilde{\boldsymbol{v}},\\ \theta &= \widehat{\theta} + \widetilde{\theta},\end{aligned}$$

and formally find the linearised equations for the perturbation $[\widetilde{\rho}, \widetilde{\boldsymbol{v}}, \widetilde{\theta}]$. (That is neglect all terms that are nonlinear in $[\widetilde{\rho}, \widetilde{\boldsymbol{v}}, \widetilde{\theta}]$.) For example, the balance of mass is manipulated as follows,

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \operatorname{div} \boldsymbol{v} = \frac{\partial \rho}{\partial t} + \boldsymbol{v} \bullet \nabla \rho + \rho \operatorname{div} \boldsymbol{v} &= \frac{\partial}{\partial t}\left(\widehat{\rho} + \widetilde{\rho}\right) + \widehat{\boldsymbol{v}} + \widetilde{\boldsymbol{v}} \bullet \nabla\left(\widehat{\rho} + \widetilde{\rho}\right) + \left(\widehat{\rho} + \widetilde{\rho}\right) \operatorname{div}\left(\widehat{\boldsymbol{v}} + \widetilde{\boldsymbol{v}}\right)\\ &= \frac{\partial \widetilde{\rho}}{\partial t} + \widetilde{\boldsymbol{v}} \bullet \nabla \widetilde{\rho} + \left(\widehat{\rho} + \widetilde{\rho}\right) \operatorname{div} \widetilde{\boldsymbol{v}} \approx \frac{\partial \widetilde{\rho}}{\partial t} + \widehat{\rho} \operatorname{div} \widetilde{\boldsymbol{v}},\end{aligned}$$

where in the last equality we have neglected the terms that are nonlinear in the perturbation.
At the end you should obtain the following system of linearised governing equations for the perturbation.

$$\begin{aligned}\frac{\partial \widetilde{\rho}}{\partial t} + \widehat{\rho} \operatorname{div} \widetilde{\boldsymbol{v}} &= 0,\\ \widehat{\rho} \frac{\partial \widetilde{\boldsymbol{v}}}{\partial t} &= \operatorname{div}\left(-\left(\left.\frac{\partial p_{\mathrm{th}}}{\partial \theta}(\theta,\rho)\right|_{\theta=\widehat{\theta},\,\rho=\widehat{\rho}} \widetilde{\theta} + \left.\frac{\partial p_{\mathrm{th}}}{\partial \rho}(\theta,\rho)\right|_{\theta=\widehat{\theta},\,\rho=\widehat{\rho}} \widetilde{\rho}\right)\mathbb{I} + \lambda \operatorname{div} \widetilde{\boldsymbol{v}} + 2\mu \widetilde{\mathbb{D}}\right),\\ \widehat{\rho} c_{\mathrm{V}}\left(\widehat{\theta}, \widehat{\rho}\right) \frac{\partial \widetilde{\theta}}{\partial t} &= -\widehat{\theta} \left.\frac{\partial p_{\mathrm{th}}}{\partial \theta}(\theta,\rho)\right|_{\theta=\widehat{\theta},\,\rho=\widehat{\rho}} \operatorname{div} \widetilde{\boldsymbol{v}} + \operatorname{div}\left(\kappa \nabla \widetilde{\theta}\right).\end{aligned}$$

b) Using the governing equations for the perturbation, show that the following equality holds

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\widehat{\rho}\int_{\Omega} \left|\widetilde{\boldsymbol{v}}\right|^2 \,\mathrm{dv} + \frac{1}{\widehat{\rho}} \left.\frac{\partial p_{\mathrm{th}}}{\partial \rho}(\theta,\rho)\right|_{\theta=\widehat{\theta},\,\rho=\widehat{\rho}} \int_{\Omega} \widetilde{\rho}^2 \,\mathrm{dv} + \widehat{\rho}\frac{c_{\mathrm{V}}\left(\widehat{\theta},\widehat{\rho}\right)}{\widehat{\theta}} \int_{\Omega} \widetilde{\theta}^2 \,\mathrm{dv}\right)\\ = -\int_{\Omega} \lambda \left(\operatorname{div} \widetilde{\boldsymbol{v}}\right)^2 \,\mathrm{dv} - \int_{\Omega} 2\mu \widetilde{\mathbb{D}} : \widetilde{\mathbb{D}} \,\mathrm{dv} - \int_{\Omega} \kappa \nabla \widetilde{\theta} \bullet \nabla \widetilde{\theta} \,\mathrm{dv}.\end{aligned}$$

(The equality is obtained by "testing" the governing equations with $\widetilde{\boldsymbol{v}}$ and $\widetilde{\theta}$ and the integration by parts. Recall the boundary conditions!)

c) Assume that want the spatially homogeneous rest state to be stable, that is we want the perturbations to decay in $\left\|\cdot\right\|_{L^2(\Omega)}$ as $t \to +\infty$. (This is definitely a good property – any pertrubation to a rest state in a themodynamically isolated vessel should eventually die out.) Show that this behaviour *can be expected* provided that we require

$$\begin{aligned}c_{\mathrm{V}}\left(\widehat{\theta}, \widehat{\rho}\right) &> 0,\\ \left.\frac{\partial p_{\mathrm{th}}}{\partial \rho}(\theta,\rho)\right|_{\theta=\widehat{\theta},\,\rho=\widehat{\rho}} &> 0.\end{aligned}$$

d) Show that the conditions introduced above can be rewritten in terms of the following conditions on the specific Helmhotz free energy,

$$\frac{\partial^2 \psi}{\partial \theta^2}\left(\theta, \frac{1}{\rho}\right) < 0,$$
$$\frac{\partial^2 \psi}{\partial \left(\frac{1}{\rho}\right)^2}\left(\theta, \frac{1}{\rho}\right) > 0.$$

This observation shows us that the **stability of the rest state is intimately related to the convexity/concavity of the specific Helmholtz free energy**.

Deadline: Thursday 12th November 2020

1. Using the assumption that the internal energy e is a function of the density ρ and the yet unknown quantity referred to as the *entropy* η, that is $e = e(\eta, \rho)$, we were able to show that the quantity η has all the desired properties provided that $e = e(\eta, \rho)$ satisfies partial differential equations

$$\frac{\partial e}{\partial \eta}(\eta, \rho) = \theta, \tag{6a}$$
$$\rho^2 \frac{\partial e}{\partial \rho}(\eta, \rho) = p_{\mathrm{th}}. \tag{6b}$$

Let us assume that our substance of interest is the *calorically perfect ideal gas*. We know that the equation of state for this substance reads

$$p_{\mathrm{th}} = c_{\mathrm{V,ref}}\,(\gamma - 1)\,\rho\theta, \tag{7a}$$

where $c_{\mathrm{V,ref}}$ is a positive constant (specific heat at constant volume) and γ is a positive constant greater that one (adiabatic exponent). Symbol θ denotes the absolute temperature. (This is the way how one writes the equation of state $PV = nRT$ in the continuum mechanics setting. We pretend that we do not know anything about atoms, hence we can hardly use concepts such as “molar mass” or “molar gas constant” and so forth.) Further, we also know that the internal energy of the substance is proportional to the temperature

$$e = c_{\mathrm{V,ref}}\theta. \tag{7b}$$

Note that in the last equation we write e as a *function of the density and the temperature*, that is it should read $e(\rho, \theta) = c_{\mathrm{V,ref}}\theta$. As you probably know, the relations (7) are outcomes of experiments, see Gay-Lussac, Boyle, Charles and Joule expansion.

Now use characterisations (7) and solve the partial differential equations (6) for the internal energy e as a *function of the entropy and the density*. Once you find the function $e(\eta, \rho)$, find also explicit formula for the entropy as a function of the temperature and the density, $\eta(\theta, \rho)$. You should obtain something like

$$\eta(\theta, \rho) = c_{\mathrm{V,ref}} \ln\left[\frac{\theta}{\theta_{\mathrm{ref}}}\left(\frac{\rho}{\rho_{\mathrm{ref}}}\right)^{1-\gamma}\right].$$

Deadline: Thursday 5th November 2020

Let $\boldsymbol{q}(\boldsymbol{x}, t) \in \mathbb{R}^3$ be a vector field in $\mathbb{R}^3$. Let $\boldsymbol{\chi} : \mathcal{B}_0 \subset \mathbb{R}^3 \mapsto \mathcal{B}_t \subset \mathbb{R}^3$ be the deformation function, and let $s(t)$ be a surface transported by the medium, that is $s(t) = \boldsymbol{\chi}(S(t_0), t_0)$, where $S(t_0)$ denotes the position of the surface at time $t = t_0$.

1. Show that the time derivative of the flux of $\boldsymbol{q}$ through a material surface $s(t)$, that is of the integral $\int_{s(t)} \boldsymbol{q} \bullet \boldsymbol{n}\,\mathrm{d}s$, where $\boldsymbol{n}$ denotes the normal to the surface $s(t)$, reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{s(t)} \boldsymbol{q} \bullet \boldsymbol{n}\,\mathrm{d}s = \int_{s(t)} \left(\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} + \boldsymbol{q}\,\mathrm{div}\,\boldsymbol{v} - \mathbb{L}\boldsymbol{q}\right) \bullet \boldsymbol{n}\,\mathrm{d}s,$$

where the symbol $\frac{\mathrm{d}}{\mathrm{d}t}$ denotes the material time derivative, and $\mathbb{L}$ denotes the gradient of the Eulerian velocity field $\mathbb{L} =_{\mathrm{def}} \nabla \boldsymbol{v}$.

2. Show that the formula for the time derivative of the flux of $\boldsymbol{q}$ through a material surface $s(s)$ can be equivalently rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{s(t)} \boldsymbol{q} \bullet \boldsymbol{n}\,\mathrm{d}s = \int_{s(t)} \left(\frac{\partial \boldsymbol{q}}{\partial t} + \mathrm{rot}\,(\boldsymbol{q} \times \boldsymbol{v}) + \boldsymbol{v}\,\mathrm{div}\,\boldsymbol{q}\right) \bullet \boldsymbol{n}\,\mathrm{d}s.$$

Hint: All you need to do is to repeat the derivation of Reynolds transport theorem, but this time for the surface integral.

Deadline: Thursday 29th October 2020

1. Consider the deformation $\boldsymbol{\chi}$ given by the following formulae

$$
\begin{aligned}
r &= f(R), \\
\varphi &= \Phi, \\
z &= Z.
\end{aligned}
$$

This means that the deformation $\boldsymbol{\chi}$ is given as a function that takes a point with the coordinates $[R, \Phi, Z]$ in the reference configuration—with respect to the cylindrical coordinate system—and returns the position of that point in terms of cylindrical coordinates in the current configuration, see Figure 2.

Show that the deformation gradient $\mathbb{F}$ is given by the formula

$$
\mathbb{F} = \frac{\mathrm{d}f}{\mathrm{d}R} \boldsymbol{e}_{\hat{r}} \otimes \boldsymbol{E}_{\hat{R}} + \frac{f}{R} \boldsymbol{e}_{\hat{\varphi}} \otimes \boldsymbol{E}_{\hat{\Phi}} + \boldsymbol{e}_{\hat{z}} \otimes \boldsymbol{E}_{\hat{Z}}.
$$

that is

$$
\mathbb{F} = \begin{bmatrix} \frac{\mathrm{d}f}{\mathrm{d}R} & 0 & 0 \\ 0 & \frac{f}{R} & 0 \\ 0 & 0 & 1 \end{bmatrix}.
$$

In other words, show that if we have a vector $\boldsymbol{V} = \mathrm{V}^{\hat{R}} \boldsymbol{E}_{\hat{R}} + \mathrm{V}^{\hat{\Phi}} \boldsymbol{E}_{\hat{\Phi}} + \mathrm{V}^{\hat{Z}} \boldsymbol{E}_{\hat{Z}}$ in the reference configuration and a corresponding vector $\boldsymbol{c} = \mathrm{v}^{\hat{r}} \boldsymbol{e}_{\hat{r}} + \mathrm{v}^{\hat{\varphi}} \boldsymbol{e}_{\hat{\varphi}} + \mathrm{v}^{\hat{z}} \boldsymbol{e}_{\hat{z}}$ in the current configuration, then the relation between the components of the vectors reads

$$
\begin{bmatrix} \mathrm{v}^{\hat{r}} \\ \mathrm{v}^{\hat{\varphi}} \\ \mathrm{v}^{\hat{z}} \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}f}{\mathrm{d}R} & 0 & 0 \\ 0 & \frac{f}{R} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{V}^{\hat{R}} \\ \mathrm{V}^{\hat{\Phi}} \\ \mathrm{V}^{\hat{Z}} \end{bmatrix}.
$$

(Recall that by the vector we mean an infinitesimal line segment placed at the given point or a tangent vector to a curve.) Since we have already found the formula using an insight into the geometrical meaning of $\mathbb{F}$, **I would like to ask you to solve the problem by a brute force approach**.

Since the formula for the deformation gradient in the Cartesian coordinate system reads

$$
\mathbb{F} = \frac{\partial \chi^{\hat{x}}}{\partial X} \boldsymbol{e}_{\hat{x}} \otimes \boldsymbol{E}^{\hat{X}} + \frac{\partial \chi^{\hat{x}}}{\partial Y} \boldsymbol{e}_{\hat{x}} \otimes \boldsymbol{E}^{\hat{Y}} + \frac{\partial \chi^{\hat{x}}}{\partial Z} \boldsymbol{e}_{\hat{x}} \otimes \boldsymbol{E}^{\hat{Z}} + \frac{\partial \chi^{\hat{y}}}{\partial X} \boldsymbol{e}_{\hat{y}} \otimes \boldsymbol{E}^{\hat{X}} + \frac{\partial \chi^{\hat{y}}}{\partial Y} \boldsymbol{e}_{\hat{y}} \otimes \boldsymbol{E}^{\hat{Y}} + \frac{\partial \chi^{\hat{y}}}{\partial Z} \boldsymbol{e}_{\hat{y}} \otimes \boldsymbol{E}^{\hat{Z}} + \frac{\partial \chi^{\hat{z}}}{\partial X} \boldsymbol{e}_{\hat{z}} \otimes \boldsymbol{E}^{\hat{X}} + \frac{\partial \chi^{\hat{z}}}{\partial Y} \boldsymbol{e}_{\hat{z}} \otimes \boldsymbol{E}^{\hat{Y}} + \frac{\partial \chi^{\hat{z}}}{\partial Z} \boldsymbol{e}_{\hat{z}} \otimes \boldsymbol{E}^{\hat{Z}},
$$

then in principle you need to use the chain rule to transform the derivatives, for example

$$
\frac{\partial \chi^{\hat{x}}}{\partial X} = \frac{\partial}{\partial X} (r \cos\varphi) = \frac{\partial r}{\partial X} \cos\varphi - r \sin\varphi \frac{\partial \varphi}{\partial X} = \left(\frac{\partial r}{\partial R} \frac{\partial R}{\partial X} + \frac{\partial r}{\partial \Phi} \frac{\partial \Phi}{\partial X} + \frac{\partial r}{\partial Z} \frac{\partial Z}{\partial X} \right) \cos\varphi - r \sin\varphi \left(\frac{\partial \varphi}{\partial R} \frac{\partial R}{\partial X} + \frac{\partial \varphi}{\partial \Phi} \frac{\partial \Phi}{\partial X} + \frac{\partial \varphi}{\partial Z} \frac{\partial Z}{\partial X} \right),
$$

and then also transform the basis accordingly, for example

$$
\begin{aligned}
\boldsymbol{E}^{\hat{X}} &= \cos\Phi \boldsymbol{E}^{\hat{R}} - \sin\Phi \boldsymbol{E}^{\hat{\Phi}}, \\
\boldsymbol{E}^{\hat{Y}} &= \sin\Phi \boldsymbol{E}^{\hat{R}} + \cos\Phi \boldsymbol{E}^{\hat{\Phi}}, \\
\boldsymbol{E}^{\hat{Z}} &= \boldsymbol{E}^{\hat{Z}}.
\end{aligned}
$$

At the end you should recover the formula we have already derived at last lecture.

Deadline: Thursday 22nd October 2020

1. Show that the formula for the rate of change of infinitesimal surface element $\mathrm{d}\mathbf{s}$ reads as follows,

$$
\dot{\overline{\mathrm{d}\mathbf{s}}} = \left((\operatorname{div} \boldsymbol{v}) \mathbb{I} - \mathbb{L}^{\top} \right) \mathrm{d}\mathbf{s}.
$$

2. Show that

$$
\begin{aligned}
\frac{\partial \mathrm{I}_1(\mathbb{A})}{\partial \mathbb{A}} [\mathbb{B}] &= \operatorname{Tr} \mathbb{B}, \\
\frac{\partial \mathrm{I}_2(\mathbb{A})}{\partial \mathbb{A}} [\mathbb{B}] &= (\operatorname{Tr} \mathbb{A}) (\operatorname{Tr} \mathbb{B}) - \operatorname{Tr} (\mathbb{A}\mathbb{B}), \\
\frac{\partial^2 \mathrm{I}_1(\mathbb{A})}{\partial \mathbb{A}^2} [\mathbb{B}, \mathbb{C}] &= 0, \\
\frac{\partial^2 \mathrm{I}_2(\mathbb{A})}{\partial \mathbb{A}^2} [\mathbb{B}, \mathbb{C}] &= (\operatorname{Tr} \mathbb{C}) (\operatorname{Tr} \mathbb{B}) - \operatorname{Tr} (\mathbb{C}\mathbb{B}), \\
\frac{\partial^2 \mathrm{I}_3(\mathbb{A})}{\partial \mathbb{A}^2} [\mathbb{B}, \mathbb{C}] &= (\det \mathbb{A}) \left(\operatorname{Tr} \left(\mathbb{A}^{-1} \mathbb{B} \right) \operatorname{Tr} \left(\mathbb{A}^{-1} \mathbb{C} \right) - \operatorname{Tr} \left(\mathbb{A}^{-1} \mathbb{B} \mathbb{A}^{-1} \mathbb{C} \right) \right),
\end{aligned}
$$

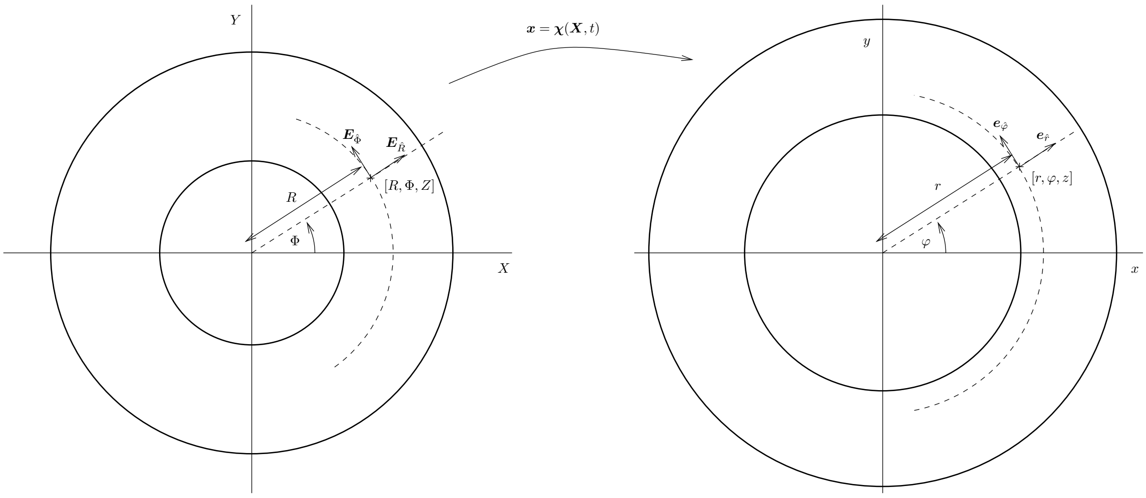

Figure 2: Problem geometry.

where $\mathrm{I}_1(\mathbb{A})$, $\mathrm{I}_2(\mathbb{A})$ and $\mathrm{I}_3(\mathbb{A})$ denote the principal invariants of matrix $\mathbb{A}$, that is

$$
\begin{aligned}
\mathrm{I}_1(\mathbb{A}) &=_{\mathrm{def}} \operatorname{Tr}\mathbb{A}, \\
\mathrm{I}_2(\mathbb{A}) &=_{\mathrm{def}} \frac{1}{2}\left((\operatorname{Tr}\mathbb{A})^2 - \operatorname{Tr}\left(\mathbb{A}^2\right)\right), \\
\mathrm{I}_3(\mathbb{A}) &=_{\mathrm{def}} \det\mathbb{A}.
\end{aligned}
$$

Deadline: Thursday 15th October 2020

1. Let φ, ψ, $\boldsymbol{u}$, $\boldsymbol{v}$ and $\mathbb{A}$ be smooth scalar, vector and tensor fields in $\mathbb{R}^3$. Show that

$$
\begin{aligned}
\operatorname{div}(\varphi\boldsymbol{v}) &= \boldsymbol{v} \bullet (\nabla\varphi) + \varphi \operatorname{div}\boldsymbol{v}, \\
\operatorname{div}(\boldsymbol{u}\times\boldsymbol{v}) &= \boldsymbol{v} \bullet \operatorname{rot}\boldsymbol{u} - \boldsymbol{u} \bullet \operatorname{rot}\boldsymbol{v}, \\
\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{v}) &= [\nabla\boldsymbol{u}]\,\boldsymbol{v} + \boldsymbol{u}\operatorname{div}\boldsymbol{v}, \\
\operatorname{div}(\varphi\mathbb{A}) &= \mathbb{A}(\nabla\varphi) + \varphi\operatorname{div}\mathbb{A}.
\end{aligned}
$$

   Further, show that

$$
\begin{aligned}
\nabla(\varphi\psi) &= \psi\nabla\varphi + \varphi\nabla\psi, \\
\nabla(\varphi\boldsymbol{v}) &= \boldsymbol{v}\otimes\nabla\varphi + \varphi\nabla\boldsymbol{v}, \\
\nabla(\boldsymbol{u}\bullet\boldsymbol{v}) &= (\nabla\boldsymbol{u})^\top\boldsymbol{v} + (\nabla\boldsymbol{v})^\top\boldsymbol{u}, \\
\operatorname{rot}(\varphi\boldsymbol{v}) &= \varphi\operatorname{rot}\boldsymbol{v} - \boldsymbol{v}\times\nabla\varphi.
\end{aligned}
$$

2. [**Optional** – I recommend this exercise for those who have never worked with the Levi–Civitta symbol] Let $\boldsymbol{v}$ be a smooth vector field. Show that

$$\operatorname{rot}(\operatorname{rot}\boldsymbol{v}) = \nabla(\operatorname{div}\boldsymbol{v}) - \Delta\boldsymbol{v}.$$

3. Let $\mathbb{A}$ be a sufficiently smooth tensor/matrix field, and let $\mathbb{A}$ be (at every point $\boldsymbol{x}$) a *symmetric matrix*. Show that

$$\operatorname{rot}\left((\operatorname{rot}\mathbb{A})^\top\right) = [\Delta\operatorname{Tr}\mathbb{A} - \operatorname{div}(\operatorname{div}\mathbb{A})]\,\mathbb{I} + \left\{\nabla(\operatorname{div}\mathbb{A}) + [\nabla(\operatorname{div}\mathbb{A})]^\top\right\} - \nabla(\nabla\operatorname{Tr}\mathbb{A}) - \Delta\mathbb{A}.$$

   (This identity is the counterpart of the identity $\operatorname{rot}(\operatorname{rot}\boldsymbol{v}) = \nabla(\operatorname{div}\boldsymbol{v}) - \Delta\boldsymbol{v}$ for tensor fields.)

Deadline: Thursday 8th October 2020

Let us assume that we know that

$$\epsilon_{ijk}\epsilon_{lmn} = \det\begin{bmatrix}\delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn}\end{bmatrix}.$$

1. Show that

$$\epsilon_{ijk}\delta_{lm} = \epsilon_{jkm}\delta_{il} + \epsilon_{kim}\delta_{jl} + \epsilon_{ijm}\delta_{kl}$$

2. Let us assume that $\boldsymbol{a} \in \mathbb{R}^3$ is a unit vector, that is $|\boldsymbol{a}| = 1$, and that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{t} \in \mathbb{R}^3$ are arbitrary vectors. Show that

$$(\boldsymbol{u}\times\boldsymbol{v})\bullet(\boldsymbol{t}\times\boldsymbol{w}) = (\boldsymbol{u}\bullet\boldsymbol{t})(\boldsymbol{v}\bullet\boldsymbol{w}) - (\boldsymbol{u}\bullet\boldsymbol{w})(\boldsymbol{v}\bullet\boldsymbol{t}).$$

Note that this identity implies that

$$|\boldsymbol{u} \times \boldsymbol{v}|^2 = (\boldsymbol{u} \times \boldsymbol{v}) \bullet (\boldsymbol{u} \times \boldsymbol{v}) = |\boldsymbol{u}|^2 |\boldsymbol{v}|^2 - (\boldsymbol{u} \bullet \boldsymbol{v})^2 .$$

Show that vectors $(\mathbb{I} - \boldsymbol{a} \otimes \boldsymbol{a}) \boldsymbol{u}$ and $\boldsymbol{a} \times \boldsymbol{u}$ have the same length, that is

$$|(\mathbb{I} - \boldsymbol{a} \otimes \boldsymbol{a}) \boldsymbol{u}| = |\boldsymbol{a} \times \boldsymbol{u}| ,$$

where $\mathbb{I}$ denotes the identity matrix.

3. [**Optional** – I recommend this exercise for those who have never worked with the Levi–Civitta symbol] Show that

$$\begin{aligned} \epsilon_{ijk}\epsilon_{ijn} &= 2\delta_{kn} , \\ \epsilon_{ijk}\epsilon_{imn} &= \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} . \end{aligned}$$