1. Consider the vector field $\boldsymbol{u} = \frac{\boldsymbol{x}}{|\boldsymbol{x}|^n}$, where

$$\boldsymbol{x} =_{\text{def}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad |\boldsymbol{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2},$$

and n is a natural number. Find explicit expressions for

$$\operatorname{rot}\operatorname{rot}\boldsymbol{u}, \qquad \nabla \operatorname{div}\boldsymbol{u}, \qquad \Delta\boldsymbol{u},$$

and verify that

$$\operatorname{rot}\operatorname{rot}\boldsymbol{u} = \nabla \operatorname{div}\boldsymbol{u} - \Delta\boldsymbol{u}.$$

2. [OPTIONAL] Consider the deformation $\boldsymbol{\chi}$ given by the following formulae

$$r = f(R), \tag{1a}$$

$$\varphi = \Phi, \tag{1b}$$

$$z = Z. \tag{1c}$$

This means that the deformation $\boldsymbol{\chi}$ is given as a function that takes a point with the coordinates $[R, \Phi, Z]$ in the reference configuration—with respect to the cylindrical coordinate system—and returns the position of that point in terms of polar coordinates in the current configuration, see Figure 1.

We already know that the deformation gradient $\mathbb{F}$ is given by the formula

$$\mathbb{F} = \frac{\mathrm{d}f}{\mathrm{d}R}\boldsymbol{e}_{\hat{r}} \otimes \boldsymbol{E}_{\hat{R}} + \frac{f}{R}\boldsymbol{e}_{\hat{\varphi}} \otimes \boldsymbol{E}_{\hat{\Phi}} + \boldsymbol{e}_{\hat{z}} \otimes \boldsymbol{E}_{\hat{Z}}.$$

that is

$$\mathbb{F} = \begin{bmatrix} \frac{\mathrm{d}f}{\mathrm{d}R} & 0 & 0 \\ 0 & \frac{f}{R} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In other words, show that if we have a vector $\boldsymbol{V} = \mathrm{V}^{\hat{R}}\boldsymbol{E}_{\hat{R}} + \mathrm{V}^{\hat{\Phi}}\boldsymbol{E}_{\hat{\Phi}} + \mathrm{V}^{\hat{Z}}\boldsymbol{E}_{\hat{Z}}$ in the reference configuration and a corresponding vector $\boldsymbol{c} = \mathrm{v}^{\hat{r}}\boldsymbol{e}_{\hat{r}} + \mathrm{v}^{\hat{\varphi}}\boldsymbol{e}_{\hat{\varphi}} + \mathrm{v}^{\hat{z}}\boldsymbol{e}_{\hat{z}}$ in the current configuration, then the relation between the components of the vectors reads

$$\begin{bmatrix} \mathrm{v}^{\hat{r}} \\ \mathrm{v}^{\hat{\varphi}} \\ \mathrm{v}^{\hat{z}} \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}f}{\mathrm{d}R} & 0 & 0 \\ 0 & \frac{f}{R} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{V}^{\hat{R}} \\ \mathrm{V}^{\hat{\Phi}} \\ \mathrm{V}^{\hat{Z}} \end{bmatrix}.$$

(Recall that by the vector we mean an infinitesimal line segment placed at the given point, or, more precisely it is a tangent vector to the corresponding curve passing through the given point.)

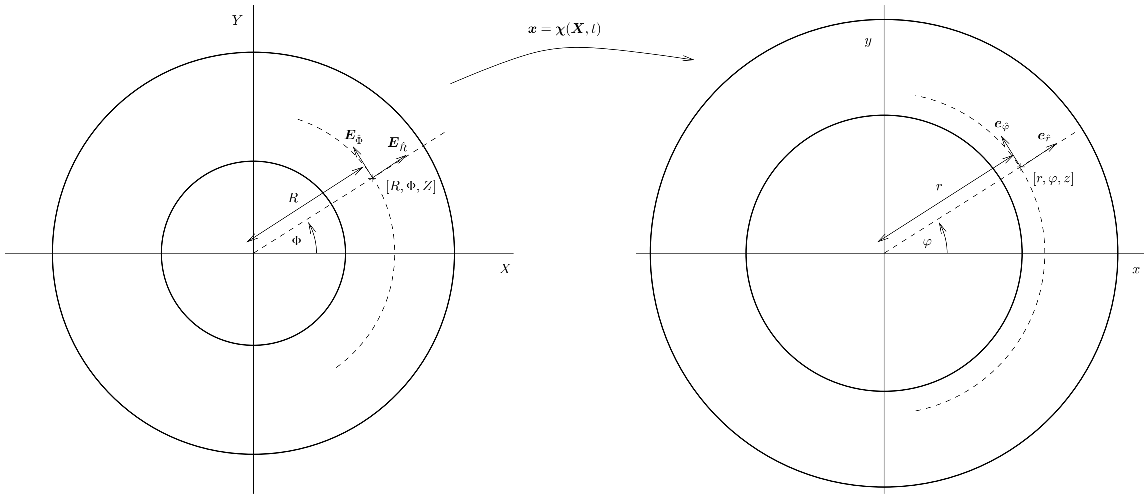

Figure 1: Problem geometry.

In the lecture we have obtained the result via a computational trick. Show that you can recover the result by a "brute force" approach. The "brute force" approach is based on the following steps.

First we have to realise that the formula for the deformation gradient is

$$\mathbb{F} = \frac{\partial \boldsymbol{\chi}(\boldsymbol{X}, t)}{\partial \boldsymbol{X}} = \frac{\partial \chi^i(\boldsymbol{X}, t)}{\partial X_j}\boldsymbol{e}_i \otimes \boldsymbol{E}^j, \tag{2}$$

or in full

$$\mathbb{F} = \frac{\partial \chi^{\hat{x}}}{\partial X} \boldsymbol{e}_{\hat{x}} \otimes \boldsymbol{E}^{\hat{X}} + \frac{\partial \chi^{\hat{x}}}{\partial Y} \boldsymbol{e}_{\hat{x}} \otimes \boldsymbol{E}^{\hat{Y}} + \frac{\partial \chi^{\hat{x}}}{\partial Z} \boldsymbol{e}_{\hat{x}} \otimes \boldsymbol{E}^{\hat{Z}} + \frac{\partial \chi^{\hat{y}}}{\partial X} \boldsymbol{e}_{\hat{y}} \otimes \boldsymbol{E}^{\hat{X}} + \frac{\partial \chi^{\hat{y}}}{\partial Y} \boldsymbol{e}_{\hat{y}} \otimes \boldsymbol{E}^{\hat{Y}} + \frac{\partial \chi^{\hat{y}}}{\partial Z} \boldsymbol{e}_{\hat{y}} \otimes \boldsymbol{E}^{\hat{Z}} + \frac{\partial \chi^{\hat{z}}}{\partial X} \boldsymbol{e}_{\hat{z}} \otimes \boldsymbol{E}^{\hat{X}} + \frac{\partial \chi^{\hat{z}}}{\partial Y} \boldsymbol{e}_{\hat{z}} \otimes \boldsymbol{E}^{\hat{Y}} + \frac{\partial \chi^{\hat{z}}}{\partial Z} \boldsymbol{e}_{\hat{z}} \otimes \boldsymbol{E}^{\hat{Z}}. \tag{3}$$

Now we need to find a relation between the Cartesian basis $\boldsymbol{E}^{\hat{X}}$, $\boldsymbol{E}^{\hat{Y}}$ and $\boldsymbol{E}^{\hat{Z}}$ in the reference configuration, and the cylindrical basis $\boldsymbol{E}^{\hat{R}}$, $\boldsymbol{E}^{\hat{\Phi}}$ and $\boldsymbol{E}^{\hat{Z}}$ in the reference configuration. The sought relation is

$$\boldsymbol{E}^{\hat{X}} = \cos\Phi \boldsymbol{E}^{\hat{R}} - \sin\Phi \boldsymbol{E}^{\hat{\Phi}}, \tag{4a}$$
$$\boldsymbol{E}^{\hat{Y}} = \sin\Phi \boldsymbol{E}^{\hat{R}} + \cos\Phi \boldsymbol{E}^{\hat{\Phi}}, \tag{4b}$$
$$\boldsymbol{E}^{\hat{Z}} = \boldsymbol{E}^{\hat{Z}}. \tag{4c}$$

Further we need to find a relation between the Cartesian basis $\boldsymbol{e}_{\hat{x}}$, $\boldsymbol{e}_{\hat{y}}$ and $\boldsymbol{e}_{\hat{z}}$ in the current configuration, and the cylindrical basis $\boldsymbol{e}_{\hat{r}}$, $\boldsymbol{e}_{\hat{\varphi}}$ and $\boldsymbol{e}_{\hat{z}}$ in the current configuration. The sought relation is

$$\boldsymbol{e}_{\hat{x}} = \cos\varphi \boldsymbol{e}_{\hat{r}} - \sin\varphi \boldsymbol{e}_{\hat{\varphi}}, \tag{5a}$$
$$\boldsymbol{e}_{\hat{y}} = \sin\varphi \boldsymbol{e}_{\hat{r}} + \cos\varphi \boldsymbol{e}_{\hat{\varphi}}, \tag{5b}$$
$$\boldsymbol{e}_{\hat{z}} = \boldsymbol{e}_{\hat{z}}. \tag{5c}$$

Now we are ready to substitute the formulae for $\boldsymbol{E}^{\hat{X}}$, $\boldsymbol{E}^{\hat{Y}}$, $\boldsymbol{E}^{\hat{Z}}$ and $\boldsymbol{e}_{\hat{x}}$, $\boldsymbol{e}_{\hat{y}}$, $\boldsymbol{e}_{\hat{z}}$ into (3).

The next step is to find a suitable expressions for the partial derivatives $\frac{\partial \chi^{\hat{x}}}{\partial X}$, $\frac{\partial \chi^{\hat{x}}}{\partial Y}$, $\frac{\partial \chi^{\hat{x}}}{\partial Z}$ and so on. The aim is to express the derivatives in terms of the derivatives of functions r, φ and z with respect to the reference coordinates R, Φ and Z. Recalling that the relation between the cylindrical coordinates $[r, \varphi, z]$ and the Cartesian coordinates $[x, y, z]$ is

$$x = r\cos\varphi, \tag{6a}$$
$$y = r\sin\varphi, \tag{6b}$$
$$z = z, \tag{6c}$$

and that the relation between the cylindrical coordinates $[R, \Phi, Z]$ and the Cartesian coordinates $[X, Y, Z]$ reads

$$X = R\cos\Phi, \tag{7a}$$
$$Y = R\sin\Phi, \tag{7b}$$
$$Z = Z, \tag{7c}$$

we see that the application of the chain rule yields

$$\frac{\partial \chi^{\hat{x}}}{\partial X} = \frac{\partial}{\partial X}(r\cos\varphi) = \frac{\partial r}{\partial X}\cos\varphi - r\sin\varphi\frac{\partial \varphi}{\partial X} = \left(\frac{\partial r}{\partial R}\frac{\partial R}{\partial X} + \frac{\partial r}{\partial \Phi}\frac{\partial \Phi}{\partial X} + \frac{\partial r}{\partial Z}\frac{\partial Z}{\partial X}\right)\cos\varphi - r\sin\varphi\left(\frac{\partial \varphi}{\partial R}\frac{\partial R}{\partial X} + \frac{\partial \varphi}{\partial \Phi}\frac{\partial \Phi}{\partial X} + \frac{\partial \varphi}{\partial Z}\frac{\partial Z}{\partial X}\right). \tag{8}$$

Formulae (7) imply that

$$\frac{\partial R}{\partial X} = \cos\Phi, \tag{9a}$$
$$\frac{\partial R}{\partial Y} = \sin\Phi, \tag{9b}$$
$$\frac{\partial \Phi}{\partial X} = -\frac{\sin\Phi}{R}, \tag{9c}$$
$$\frac{\partial \Phi}{\partial Y} = \frac{\cos\Phi}{R}, \tag{9d}$$

hence (8) yields—for the particular deformation (1)—

$$\frac{\partial \chi^{\hat{x}}}{\partial X} = \frac{\mathrm{d}f}{\mathrm{d}R}\cos^2\Phi + \frac{f}{R}\sin^2\Phi, \tag{10a}$$

where we have used the fact that partial derivatives $\frac{\partial r}{\partial \Phi}$ and $\frac{\partial r}{\partial \Phi}$ and all partial derivatives with respect to Z vanish, and that $\frac{\partial \varphi}{\partial \Phi} = 1$. Repeating the same procedure for the remaining partial derivatives yields

$$\frac{\partial \chi^{\hat{x}}}{\partial Y} = \left(\frac{\mathrm{d}f}{\mathrm{d}R} - \frac{f}{R} \right) \cos\Phi \sin\Phi, \tag{10b}$$

$$\frac{\partial \chi^{\hat{y}}}{\partial X} = \left(\frac{\mathrm{d}f}{\mathrm{d}R} - \frac{f}{R} \right) \cos\Phi \sin\Phi, \tag{10c}$$

$$\frac{\partial \chi^{\hat{y}}}{\partial Y} = \frac{\mathrm{d}f}{\mathrm{d}R} \sin^2\Phi + \frac{f}{R} \cos^2\Phi, \tag{10d}$$

$$\frac{\partial \chi^{\hat{z}}}{\partial Z} = 1, \tag{10e}$$

and all the other partial derivatives vanish. Finally, substituting (4), (5) and (10) into (3) leads upon tedious but straightforward calculation to formula

$$\mathbb{F} = \frac{\mathrm{d}f}{\mathrm{d}R} \boldsymbol{e}_{\hat{r}} \otimes \boldsymbol{E}^{\hat{R}} + \frac{f}{R} \boldsymbol{e}_{\hat{\varphi}} \otimes \boldsymbol{E}^{\hat{\Phi}} + \boldsymbol{e}_{\hat{z}} \otimes \boldsymbol{E}^{\hat{Z}}. \tag{11}$$