1. Consider a hollow cylinder of initial inner radius R_{in} and outer radius R_{out}, see Figure 1, and assume that the cylinder is in this configuration in a stress free state. Further, assume that the material of which is the cylinder made is a homogeneous isotropic *incompressible* elastic material specified by constitutive relation

$$\mathbb{T} = -p\mathbb{I} + \mu\left(\mathbb{B} - \mathbb{I}\right),$$

where μ is a positive constant and $\mathbb{B}$ denotes the left Cauchy–Green tensor with respect to the initial configuration with the inner radius R_{in} and the outer radius R_{out}.

Let us now apply a pressure P_{in} inside the cylinder and a pressure P_{out} outside the cylinder. If the inner pressure is higher than the outer pressure, then the cylinder inflates. The task is to find a relation between the relative change in the void area

$$c =_{\mathrm{def}} \frac{r_{\mathrm{in}}^2 - R_{\mathrm{in}}^2}{R_{\mathrm{in}}^2}$$

and the pressure difference $P_{\mathrm{in}} - P_{\mathrm{out}}$.

Find the answer using *linearised elasticity* theory, that is use the governing equations in the form

$$\begin{aligned} \operatorname{div} \mathbb{\tau} &= \mathbf{0}, \\ \operatorname{Tr}\left(\nabla \boldsymbol{U}\right) &= 0, \end{aligned}$$

where $\mathbb{\tau} =_{\mathrm{def}} -p\mathbb{I} + 2\mu\mathbb{\epsilon}$ and $\mathbb{\epsilon} =_{\mathrm{def}} \frac{1}{2}\left(\nabla \boldsymbol{U} + \left(\nabla \boldsymbol{U}\right)^{\top}\right)$. The boundary conditions read

$$\begin{aligned} \mathbb{\tau}\boldsymbol{E}_{\hat{Z}}\big|_{R=R_{\mathrm{in}}} &= P_{\mathrm{in}}\boldsymbol{E}_{\hat{Z}}\big|_{R=R_{\mathrm{in}}}, \\ \mathbb{\tau}\boldsymbol{E}_{\hat{Z}}\big|_{R=R_{\mathrm{out}}} &= P_{\mathrm{out}}\boldsymbol{E}_{\hat{Z}}\big|_{R=R_{\mathrm{out}}}, \end{aligned}$$

while the displacement $\boldsymbol{U}$ is assumed to take the form

$$\begin{aligned} \mathrm{U}^{\hat{r}} &= g(R), \\ \mathrm{U}^{\hat{\varphi}} &= 0, \\ \mathrm{U}^{\hat{z}} &= 0. \end{aligned}$$

The result should be identical to the result

$$P_{\mathrm{out}} - P_{\mathrm{in}} \approx \mu \int_{r=R_{\mathrm{in}}}^{R_{\mathrm{out}}} \frac{2cR_{\mathrm{in}}^2}{r^3}\,\mathrm{d}r$$

that we have already obtained via linearisation of the solution to the complete system of nonlinear governing equations.

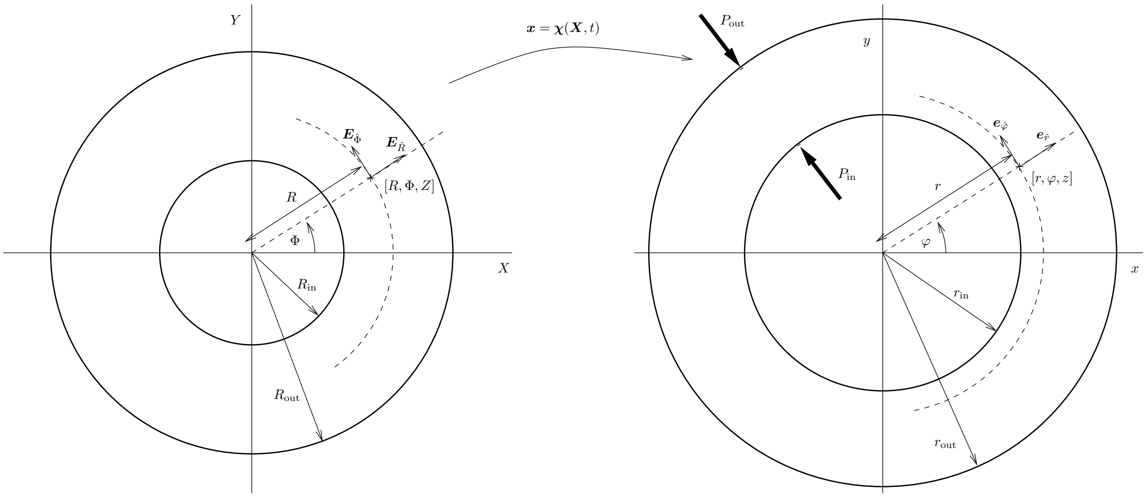

Figure 1: Inflation of a hollow cylinder made of an incompressible elastic material.

The formula for the divergence of a tensorial quantity $\mathbb{A}$ in the cylindrical coordinate system reads

$$\operatorname{div} \mathbb{A} = \begin{bmatrix} \frac{\partial \mathrm{A}^{\hat{r}}{}_{\hat{r}}}{\partial r} + \frac{1}{r}\left(\frac{\partial \mathrm{A}^{\hat{r}}{}_{\hat{\varphi}}}{\partial \varphi} - \mathrm{A}^{\hat{\varphi}}{}_{\hat{\varphi}} + \mathrm{A}^{\hat{r}}{}_{\hat{r}}\right) + \frac{\partial \mathrm{A}^{\hat{r}}{}_{\hat{z}}}{\partial z} \\ \frac{\partial \mathrm{A}^{\hat{\varphi}}{}_{\hat{r}}}{\partial r} + \frac{1}{r}\left(\frac{\partial \mathrm{A}^{\hat{\varphi}}{}_{\hat{\varphi}}}{\partial \varphi} + \mathrm{A}^{\hat{r}}{}_{\hat{\varphi}} + \mathrm{A}^{\hat{\varphi}}{}_{\hat{r}}\right) + \frac{\partial \mathrm{A}^{\hat{\varphi}}{}_{\hat{z}}}{\partial z} \\ \frac{\partial \mathrm{A}^{\hat{z}}{}_{\hat{r}}}{\partial r} + \frac{1}{r}\left(\frac{\partial \mathrm{A}^{\hat{z}}{}_{\hat{\varphi}}}{\partial \varphi} + \mathrm{A}^{\hat{z}}{}_{\hat{r}}\right) + \frac{\partial \mathrm{A}^{\hat{z}}{}_{\hat{z}}}{\partial z} \end{bmatrix},$$

while the formula for the gradient of a vector field $\boldsymbol{v}$ reads

$$\nabla \boldsymbol{v} = \begin{bmatrix} \frac{\partial \mathrm{v}^{\hat{r}}}{\partial r} & \frac{1}{r}\frac{\partial \mathrm{v}^{\hat{r}}}{\partial \varphi} - \frac{\mathrm{v}^{\hat{\varphi}}}{r} & \frac{\partial \mathrm{v}^{\hat{r}}}{\partial z} \\ r\frac{\partial}{\partial r}\left(\frac{\mathrm{v}^{\hat{\varphi}}}{r}\right) + \frac{\mathrm{v}^{\hat{\varphi}}}{r} & \frac{1}{r}\frac{\partial \mathrm{v}^{\hat{\varphi}}}{\partial \varphi} + \frac{\mathrm{v}^{\hat{r}}}{r} & \frac{\partial \mathrm{v}^{\hat{\varphi}}}{\partial z} \\ \frac{\partial \mathrm{v}^{\hat{z}}}{\partial r} & \frac{1}{r}\frac{\partial \mathrm{v}^{\hat{z}}}{\partial \varphi} & \frac{\partial \mathrm{v}^{\hat{z}}}{\partial z} \end{bmatrix}.$$

You are now in the position to solve some interesting problems. The end of the year is approaching, hence it makes sense to ask the question "how to extrude a cork from the neck of a bottle efficiently". Using continuum mechanics one can answer the question, see De Pascalis et al. (2007).

De Pascalis, R., M. Destrade, and G. Saccomandi (2007). The stress field in a pulled cork and some subtle points in the semi-inverse method of nonlinear elasticity. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences 463*(2087), 2945–2959.