1. Consider the deformation $\boldsymbol{\chi}$ of a rectangle $\boldsymbol{X} \in [0, L] \times [0, H]$ that is described by the function $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X})$ given by the formulae

$$x_1 = X_1 + kX_2^2,$$
$$x_2 = X_2,$$
$$x_3 = X_3.$$

Consider a point

$$\boldsymbol{B} = \begin{bmatrix} \frac{L}{2} \\ \frac{H}{2} \\ 0 \end{bmatrix}$$

in the reference configuration, and a curve $\boldsymbol{\Gamma} : s \in \mathbb{R} \mapsto \boldsymbol{X} \in \mathbb{R}^3$ in the reference configuration that is given by the formula

$$\boldsymbol{\Gamma} = \begin{bmatrix} \frac{L}{2} \\ s \\ 0 \end{bmatrix}.$$

1. Find the position of point $\boldsymbol{B}$ after the deformation. Denote this point as $\boldsymbol{b}$.
2. Find the image of the curve $\boldsymbol{\Gamma}$ in the current configuration. Denote this curve as $\boldsymbol{\gamma}$.
3. Find the tangent vector $\frac{\mathrm{d}\boldsymbol{\Gamma}}{\mathrm{d}s}$ to the curve $\boldsymbol{\Gamma}$ at point $\boldsymbol{B}$.
4. Find the tangent vector $\frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}s}$ to the curve $\boldsymbol{\gamma}$ at point $\boldsymbol{b}$.
5. Find the deformation gradient $\mathbb{F}$ and verify that

$$\frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}s} = \mathbb{F}\frac{\mathrm{d}\boldsymbol{\Gamma}}{\mathrm{d}s}.$$

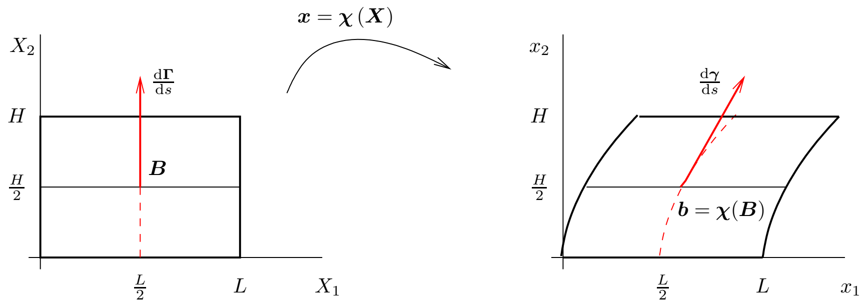

Figure 1: Sketch of the deformation studied in Question #1.

2. Find deformation gradient for the deformation $\boldsymbol{\chi}$ given by the following formulae

$$r = f(X, Y),$$
$$\varphi = g(X, Y),$$
$$z = Z,$$

where f and g are given functions. This means that the deformation $\boldsymbol{\chi}$ is given as a function that takes coordinates $[X, Y, Z]$ in the reference configuration (with respect tot the Cartesian coordinate system) and returns the position of that point in terms of cylindrical coordinates $[r, \varphi, z]$ in the current configuration, see Figure 2. (I recall that the components of the deformation $\boldsymbol{\chi}$ are usually denoted as $\chi^{\hat{x}}$, $\chi^{\hat{y}}$ and $\chi^{\hat{z}}$ if we are dealing with the Cartesian coordinates and $\chi^{\hat{r}}$, $\chi^{\hat{\varphi}}$ and $\chi^{\hat{z}}$ if we are dealing with the cylindrical coordinates.)

If you want, you can solve the problem using the guidelines provided below.

1. Recall that the relation between the Cartesian coordinates $[x, y, z]$ in the current configuration and the cylindrical coordinates $[r, \varphi, z]$ in the current configuration is given by the formulae

$$x = r\cos\varphi,$$
$$y = r\sin\varphi,$$
$$z = z.$$

The base vectors $\boldsymbol{e}_{\hat{r}}$, $\boldsymbol{e}_{\hat{\varphi}}$ and $\boldsymbol{e}_{\hat{z}}$ in the cylindrical coordinate system are generated as the tangent vectors to the coordinate curves, that is

$$\boldsymbol{e}_{\hat{r}} =_{\mathrm{def}} \frac{\frac{\mathrm{d}\boldsymbol{\gamma}_r}{\mathrm{d}s}}{\left|\frac{\mathrm{d}\boldsymbol{\gamma}_r}{\mathrm{d}s}\right|},$$

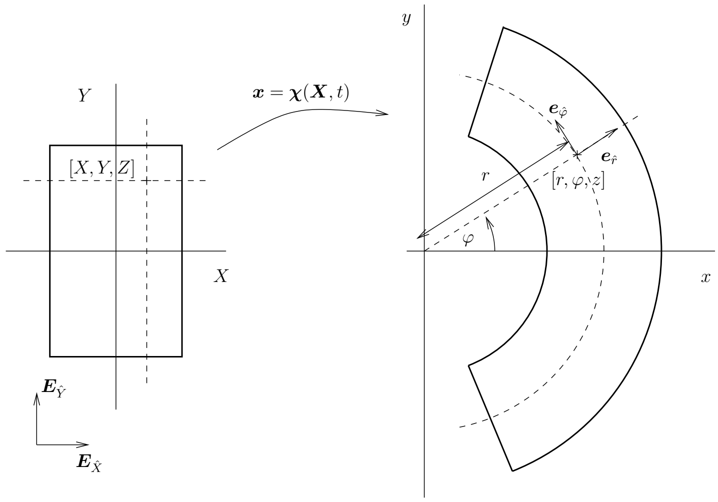

Figure 2: Problem geometry for Question #2.

where

$$\boldsymbol{\gamma}_r(s) =_{\text{def}} \begin{bmatrix} s\cos\varphi \\ s\sin\varphi \\ z \end{bmatrix},$$

and so forth. Show that the relations between the Cartesian base vectors $\boldsymbol{e}_{\hat{x}}$, $\boldsymbol{e}_{\hat{y}}$ and $\boldsymbol{e}_{\hat{z}}$ and the base vectors $\boldsymbol{e}_{\hat{r}}$, $\boldsymbol{e}_{\hat{\varphi}}$ and $\boldsymbol{e}_{\hat{z}}$ read

$$\begin{aligned}
\boldsymbol{e}_{\hat{r}} &= \cos\varphi \boldsymbol{e}_{\hat{x}} + \sin\varphi \boldsymbol{e}_{\hat{y}}, \\
\boldsymbol{e}_{\hat{\varphi}} &= -\sin\varphi \boldsymbol{e}_{\hat{x}} + \cos\varphi \boldsymbol{e}_{\hat{y}}, \\
\boldsymbol{e}_{\hat{z}} &= \boldsymbol{e}_{\hat{z}},
\end{aligned}$$

or, in the opposite direction,

$$\boldsymbol{e}_{\hat{x}} = \cos\varphi \boldsymbol{e}_{\hat{r}} - \sin\varphi \boldsymbol{e}_{\hat{\varphi}}, \tag{5a}$$

$$\boldsymbol{e}_{\hat{y}} = \sin\varphi \boldsymbol{e}_{\hat{r}} + \cos\varphi \boldsymbol{e}_{\hat{\varphi}}, \tag{5b}$$

$$\boldsymbol{e}_{\hat{z}} = \boldsymbol{e}_{\hat{z}}. \tag{5c}$$

2. If we want to use Cartesian coordinates both in the reference and the current configuration, we know that

$$\mathbb{F} = \frac{\partial \boldsymbol{\chi}(\boldsymbol{X}, t)}{\partial \boldsymbol{X}} = \frac{\partial \chi^i(\boldsymbol{X}, t)}{\partial X_j} \boldsymbol{e}_i \otimes \boldsymbol{E}^j,$$

or in full

$$\begin{aligned}
\mathbb{F} = \frac{\partial \chi^{\hat{x}}}{\partial X} \boldsymbol{e}_{\hat{x}} \otimes \boldsymbol{E}^{\hat{X}} + \frac{\partial \chi^{\hat{x}}}{\partial Y} \boldsymbol{e}_{\hat{x}} \otimes \boldsymbol{E}^{\hat{Y}} + \frac{\partial \chi^{\hat{x}}}{\partial Z} \boldsymbol{e}_{\hat{x}} \otimes \boldsymbol{E}^{\hat{Z}} + \frac{\partial \chi^{\hat{y}}}{\partial X} \boldsymbol{e}_{\hat{y}} \otimes \boldsymbol{E}^{\hat{X}} + \frac{\partial \chi^{\hat{y}}}{\partial Y} \boldsymbol{e}_{\hat{y}} \otimes \boldsymbol{E}^{\hat{Y}} + \frac{\partial \chi^{\hat{y}}}{\partial Z} \boldsymbol{e}_{\hat{y}} \otimes \boldsymbol{E}^{\hat{Z}} \\
+ \frac{\partial \chi^{\hat{z}}}{\partial X} \boldsymbol{e}_{\hat{z}} \otimes \boldsymbol{E}^{\hat{X}} + \frac{\partial \chi^{\hat{z}}}{\partial Y} \boldsymbol{e}_{\hat{z}} \otimes \boldsymbol{E}^{\hat{Y}} + \frac{\partial \chi^{\hat{z}}}{\partial Z} \boldsymbol{e}_{\hat{z}} \otimes \boldsymbol{E}^{\hat{Z}}.
\end{aligned} \tag{6}$$

This means that

$$\mathbb{F} = \begin{bmatrix} \mathrm{F}^{\hat{x}}{}_{\hat{X}} & \mathrm{F}^{\hat{x}}{}_{\hat{Y}} & \mathrm{F}^{\hat{x}}{}_{\hat{Z}} \\ \mathrm{F}^{\hat{y}}{}_{\hat{X}} & \mathrm{F}^{\hat{y}}{}_{\hat{Y}} & \mathrm{F}^{\hat{y}}{}_{\hat{Z}} \\ \mathrm{F}^{\hat{z}}{}_{\hat{X}} & \mathrm{F}^{\hat{z}}{}_{\hat{Y}} & \mathrm{F}^{\hat{z}}{}_{\hat{Z}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \chi^{\hat{x}}}{\partial X} & \frac{\partial \chi^{\hat{x}}}{\partial Y} & \frac{\partial \chi^{\hat{x}}}{\partial Z} \\ \frac{\partial \chi^{\hat{y}}}{\partial X} & \frac{\partial \chi^{\hat{y}}}{\partial Y} & \frac{\partial \chi^{\hat{y}}}{\partial Z} \\ \frac{\partial \chi^{\hat{z}}}{\partial X} & \frac{\partial \chi^{\hat{z}}}{\partial Y} & \frac{\partial \chi^{\hat{z}}}{\partial Z} \end{bmatrix}$$

or in other notation

$$\mathbb{F} = \begin{bmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{bmatrix}.$$

However, we want to express the deformation gradient in the form

$$\begin{aligned}
\mathbb{F} = \mathrm{F}^{\hat{r}}{}_{\hat{X}} \boldsymbol{e}_{\hat{r}} \otimes \boldsymbol{E}^{\hat{X}} + \mathrm{F}^{\hat{r}}{}_{\hat{Y}} \boldsymbol{e}_{\hat{r}} \otimes \boldsymbol{E}^{\hat{Y}} + \mathrm{F}^{\hat{r}}{}_{\hat{Z}} \boldsymbol{e}_{\hat{r}} \otimes \boldsymbol{E}^{\hat{Z}} + \mathrm{F}^{\hat{\varphi}}{}_{\hat{X}} \boldsymbol{e}_{\hat{\varphi}} \otimes \boldsymbol{E}^{\hat{X}} + \mathrm{F}^{\hat{\varphi}}{}_{\hat{Y}} \boldsymbol{e}_{\hat{\varphi}} \otimes \boldsymbol{E}^{\hat{Y}} + \mathrm{F}^{\hat{\varphi}}{}_{\hat{Z}} \boldsymbol{e}_{\hat{\varphi}} \otimes \boldsymbol{E}^{\hat{Z}} \\
+ \mathrm{F}^{\hat{z}}{}_{\hat{X}} \boldsymbol{e}_{\hat{z}} \otimes \boldsymbol{E}^{\hat{X}} + \mathrm{F}^{\hat{z}}{}_{\hat{Y}} \boldsymbol{e}_{\hat{z}} \otimes \boldsymbol{E}^{\hat{Y}} + \mathrm{F}^{\hat{z}}{}_{\hat{Z}} \boldsymbol{e}_{\hat{z}} \otimes \boldsymbol{E}^{\hat{Z}},
\end{aligned} \tag{7}$$

and we want to identify the coefficients $\mathrm{F}^{\hat{r}}{}_{\hat{X}}$ and so forth in terms of the derivatives of f and g. Unfortunately, we only know how to calculate the deformation gradient in the Cartesian coordinates, see (6), hence we need to convert (6) to the form (7). In order to do so, we need to:

(a) Convert the coefficients such as $\frac{\partial \chi^{\hat{x}}}{\partial X}$ using the chain rule. For example, for the first coefficient we get[1]

$$\frac{\partial \chi^{\hat{x}}}{\partial X} = \frac{\partial}{\partial X}\left(\chi^{\hat{r}} \cos \chi^{\hat{\varphi}}\right) = \frac{\partial \chi^{\hat{r}}}{\partial X} \cos \varphi - r \sin \varphi \frac{\partial \chi^{\hat{\varphi}}}{\partial X} = \frac{\partial f}{\partial X} \cos g - f \sin g \frac{\partial g}{\partial X}.$$

This must be repeated for all coefficients in (6).

(b) We need to rewrite the Cartesian base vectors $\boldsymbol{e}_{\hat{x}}$, $\boldsymbol{e}_{\hat{y}}$ and $\boldsymbol{e}_{\hat{z}}$ in terms of the base vectors $\boldsymbol{e}_{\hat{r}}$, $\boldsymbol{e}_{\hat{\varphi}}$ and $\boldsymbol{e}_{\hat{z}}$ in the cylindrical coordinate system. In other words, we need to exploit (5), and rewrite all tensor products of the type $\boldsymbol{e}_{\hat{x}} \otimes \boldsymbol{E}^{\hat{X}}$ in terms of $\boldsymbol{e}_{\hat{r}} \otimes \boldsymbol{E}^{\hat{X}}$, $\boldsymbol{e}_{\hat{\varphi}} \otimes \boldsymbol{E}^{\hat{X}}$ and $\boldsymbol{e}_{\hat{z}} \otimes \boldsymbol{E}^{\hat{X}}$. For example, we know that $\boldsymbol{e}_{\hat{x}} = \cos \varphi \boldsymbol{e}_{\hat{r}} - \sin \varphi \boldsymbol{e}_{\hat{\varphi}}$, hence

$$\boldsymbol{e}_{\hat{x}} \otimes \boldsymbol{E}^{\hat{X}} = (\cos \varphi \boldsymbol{e}_{\hat{r}} - \sin \varphi \boldsymbol{e}_{\hat{\varphi}}) \otimes \boldsymbol{E}^{\hat{X}} = \cos \varphi \boldsymbol{e}_{\hat{r}} \otimes \boldsymbol{E}^{\hat{X}} - \sin \varphi \boldsymbol{e}_{\hat{\varphi}} \otimes \boldsymbol{E}^{\hat{X}}.$$

This must be repeated for all tensor products in (6).

(c) Once we have converted all the terms in (6), we can rearrange the result into the form

$$\begin{aligned}\mathbb{F} = [\cdots]\boldsymbol{e}_{\hat{r}} \otimes \boldsymbol{E}^{\hat{X}} + [\cdots]\boldsymbol{e}_{\hat{r}} \otimes \boldsymbol{E}^{\hat{Y}} + [\cdots]\boldsymbol{e}_{\hat{r}} \otimes \boldsymbol{E}^{\hat{Z}} + [\cdots]\boldsymbol{e}_{\hat{\varphi}} \otimes \boldsymbol{E}^{\hat{X}} + [\cdots]\boldsymbol{e}_{\hat{\varphi}} \otimes \boldsymbol{E}^{\hat{Y}} + [\cdots]\boldsymbol{e}_{\hat{\varphi}} \otimes \boldsymbol{E}^{\hat{Z}} \\ + [\cdots]\boldsymbol{e}_{\hat{z}} \otimes \boldsymbol{E}^{\hat{X}} + [\cdots]\boldsymbol{e}_{\hat{z}} \otimes \boldsymbol{E}^{\hat{Y}} + [\cdots]\boldsymbol{e}_{\hat{z}} \otimes \boldsymbol{E}^{\hat{Z}},\end{aligned}$$

which can be compared to (7). This finally leads to the identification of the coefficients $\mathrm{F}^{\hat{r}}{}_{\hat{X}}$ and so forth in terms of functions f and g and their derivatives.

If the functions f and g are chosen in such a way that they indeed correspond to Figure 2 (bending of a rectangular block), then the deformation gradient can be in fact obtained in a straightforward way by a geometric insight. (Try it.) There is no need to bother yourself with such a lengthy computation. However, the procedure outlined above can be clearly applied in more complex setting, where the geometric intuition can fail.

We are—in a naive and intuitive way—introducing concepts that are well known in differential geometry and analysis on manifolds. If you want to see how to apply this mathematical formalism in continuum mechanics, please see the review paper by Stumpf and Hoppe (1997) or the book by Marsden and Hughes (1994).

Marsden, J. E. and T. J. R. Hughes (1994). *Mathematical foundations of elasticity.* New York: Dover Publications Inc. Corrected reprint of the 1983 original.

Stumpf, H. and U. Hoppe (1997). The application of tensor algebra on manifolds to nonlinear continuum mechanics—invited survey article. *Z. angew. Math. Mech.* *77*(5), 327–339.

[1] Recall that r (the coordinate in the current configuration) is given by $r = \chi^{\hat{r}}(\cdots)$ and similarly for the other variables. This means that x (the Cartesian coordinate) in the current configuration is given by $x = r \cos \varphi = \chi^{\hat{r}} \cos \chi^{\hat{\varphi}}$, that is $\chi^{\hat{x}}(X, Y, Z) = \chi^{\hat{r}}(X, Y, Z) \cos\left(\chi^{\hat{\varphi}}(X, Y, Z)\right)$.