Let $\mathbb{A}$ be a sufficiently smooth tensor field in $\mathbb{R}^3$, and let $\boldsymbol{v} \in \mathbb{R}^3$ be an arbitrary, but fixed vector field. Then the tensor $\operatorname{rot} \mathbb{A}$ that satisfies

$$(\operatorname{rot} \mathbb{A})^\top \boldsymbol{v} = \operatorname{rot} \left(\mathbb{A}^\top \boldsymbol{v}\right) \tag{1}$$

for all $\boldsymbol{v}$ is called the curl of the tensor field $\mathbb{A}$. If we want to work with components of $\operatorname{rot} \mathbb{A}$, then it is easy to see that (1) implies in Cartesian coordinate system

$$[\operatorname{rot} \mathbb{A}]_{ij} = \epsilon_j{}^{kl} \frac{\partial \mathrm{A}_{il}}{\partial x_k}. \tag{2}$$

1. Show that the following identities hold

$$\operatorname{rot} (\nabla \boldsymbol{u}) = \mathbb{0},$$
$$\operatorname{div} (\operatorname{rot} \mathbb{A}) = \mathbf{0}$$

   for any smooth vector field $\boldsymbol{u}$ and tensor field $\mathbb{A}$.

Let us now try to answer the following question. What is the condition that guarantees that a given tensor field $\mathbb{e}$ is generated as a symmetric part of a gradient of a vector field? That is whether there exists a vector field $\boldsymbol{U}$ such that

$$\mathbb{e} = \frac{1}{2} \left(\nabla \boldsymbol{U} + (\nabla \boldsymbol{U})^\top\right).$$

Recall that we are already able to answer the question whether a given tensor field $\mathbb{F}$ is generated as a gradient of some vector function. If the domain is simply connected, the necessary and sufficient condition reads

$$\operatorname{rot} \mathbb{F} = \mathbb{0}.$$

Show that in the present case, the necessary and sufficient condition for $\mathbb{e}$ being generated as a symmetric part of the gradient of a vector field reads

$$\operatorname{rot} \left((\operatorname{rot} \mathbb{e})^\top\right) = \mathbb{0}. \tag{3}$$

(We again assume that the domain of interest is simply connected.) You can proceed as follows.

1. (Necessary condition) Assume that there exists a vector field $\boldsymbol{U}$ such that $\nabla \boldsymbol{U} = \mathbb{e} + \mathbb{w}$, where $\mathbb{e}$ is the symmetric part of the gradient and $\mathbb{w}$ is the skew symmetric part of the gradient. Show that in such a case we have

$$\operatorname{rot} \mathbb{e} = \frac{1}{2} \left(\nabla \left(\operatorname{rot} \boldsymbol{U}\right)\right)^\top.$$

   and condition (3) follows immediately.

2. (Sufficient condition) Fulfillment of (3) and the fact that the domain is simply connected implies that there exists a vector field $\boldsymbol{a}$ such that $(\operatorname{rot} \mathbb{e})^\top = \nabla \boldsymbol{a}$. Let $\mathbb{A}_{\boldsymbol{a}}$ denotes the skew-symmetric matrix associated to vector $\boldsymbol{a}$. (Identity $\mathbb{A}_{\boldsymbol{a}} \boldsymbol{w} = \boldsymbol{a} \times \boldsymbol{w}$ holds for any $\boldsymbol{w}$.) Show that

$$\operatorname{rot} \mathbb{A}_{\boldsymbol{a}} = (\operatorname{div} \boldsymbol{a})\, \mathbb{I} - (\nabla \boldsymbol{a})^\top \tag{4a}$$

   and that

$$\operatorname{div} \boldsymbol{a} = 0. \tag{4b}$$

   Now construct the tensor field $\mathbb{g}$ as

$$\mathbb{g} =_{\mathrm{def}} \mathbb{e} + \mathbb{A}_{\boldsymbol{a}},$$

   and show that this tensor field has a potential, that is there exists a vector field $\boldsymbol{U}$ such that $\nabla \boldsymbol{U} = \mathbb{e} + \mathbb{A}_{\boldsymbol{a}}$ which completes the proof. (You may find formulae (4) useful in the course of the proof.)

3. (**You do not need to answer this question.**) Given a tensor field $\mathbb{e}$ that satisfies the compatibility condition $\operatorname{rot} \left((\operatorname{rot} \mathbb{e})^\top\right) = \mathbb{0}$ in a simply connected domain, is it possible to *uniquely* determine $\boldsymbol{U}$ such that $\mathbb{e} = \frac{1}{2} \left(\nabla \boldsymbol{U} + (\nabla \boldsymbol{U})^\top\right)$? If not, is it possible to fully characterize the arising ambiguity in the specification of $\boldsymbol{U}$? (In other words, is it possible to say that two different $\boldsymbol{U}$ generating the same $\mathbb{e}$ differ at most by a certain class of motions?)