1. Let $\mathbb{A} \in \mathbb{R}^{3\times 3}$ be a skew-symmetric matrix

$$\mathbb{A} =_{\text{def}} \begin{bmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{bmatrix},$$

such that $A_{12}^2 + A_{13}^2 + A_{23}^2 = 1$, and let $\varphi \in [0, 2\pi)$ be an arbitrary number. Show that

$$e^{\varphi \mathbb{A}} = \mathbb{I} + (\sin\varphi)\, \mathbb{A} + (1 - \cos\varphi)\, \mathbb{A}^2.$$

(Cayley–Hamilton theorem might be useful.)

2. Let $\mathbb{A} \in \mathbb{R}^{3\times 3}$ a $\mathbb{B} \in \mathbb{R}^{3\times 3}$ be invertible matrices. Show that

$$\det(\mathbb{A} + \mathbb{B}) = \det \mathbb{A} + \operatorname{Tr}\left(\mathbb{A}^\top \operatorname{cof} \mathbb{B}\right) + \operatorname{Tr}\left(\mathbb{B}^\top \operatorname{cof} \mathbb{A}\right) + \det \mathbb{B},$$

where $\operatorname{cof} \mathbb{C} =_{\text{def}} (\det \mathbb{C})\, \mathbb{C}^{-\top}$ denotes the cofactor matrix of matrix $\mathbb{C}$.

3. Let $\mathbb{A} \in \mathbb{R}^{3\times 3}$ be an invertible matrix and let $\mathbf{u}$ and $\mathbf{v}$ be arbitrary fixed vectors in $\mathbb{R}^3$ such that $\mathbf{v} \bullet \mathbb{A}^{-1}\mathbf{u} \neq -1$. Show that

$$(\mathbb{A} + \mathbf{u} \otimes \mathbf{v})^{-1} = \mathbb{A}^{-1} - \frac{1}{1 + \mathbf{v} \bullet \mathbb{A}^{-1}\mathbf{u}} \left(\mathbb{A}^{-1}\mathbf{u}\right) \otimes \left(\mathbb{A}^{-\top}\mathbf{v}\right).$$

4. Let $\mathbb{A} \in \mathbb{R}^{3\times 3}$ be a symmetric matrix and let $\mathbb{B} \in \mathbb{R}^{3\times 3}$ be a skew-symmetric matrix. Show that

$$\mathbb{A} : \mathbb{B} = 0.$$

(Recall that $\mathbb{A} : \mathbb{B} =_{\text{def}} \operatorname{Tr}\left(\mathbb{A}\mathbb{B}^\top\right)$.)