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Anisotropic function spaces and maximal regularity for parabolic problems

Part 1: Function spaces

Herbert Amann

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Abstract. In this first part of a treatise on maximal regularity for linear parabolic systems we provide an in-depth investigation of anisotropic function spaces. Special attention is given to trace and extension theorems on hyperplanes and corners. Building on these results, parabolic evolution equations in strong and weak settings will be studied in Part 2.

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Contents

Introduction

The topic of principal interest in this treatise is the optimal solvability of linear parabolic initial boundary value problems in strong and weak L_p-settings. More precisely, we fix any positive real number T and set $J := [0, T]$. Then we consider systems of the form

$$\left.\begin{aligned} \partial_t u + \mathcal{A}u &= f \quad \text{on} \quad M \times J, \\ \mathcal{B}u &= g \quad \text{on} \ \partial M \times J, \\ u(\cdot, 0) &= u^0 \quad \text{on} \quad M, \end{aligned}\right\} \tag{P}$$

where, in general, M is an oriented Riemannian manifold with (possibly empty) boundary ∂M, $\partial_t + \mathcal{A}$ is a Petrowskii parabolic differential operator, $\mathcal{B}$ is a system of boundary operators satisfying the Lopatinskii-Shapiro conditions, and f, g, and u^0 are given sections in vector bundles over $M \times J$, $\partial M \times J$, and M, respectively.

These problems have already been intensely studied, in the strong setting, by several authors, most notably by V.A. Solonnikov [62] (see also the book by O.A. Ladyzhenskaya, V.A. Solonnikov, and N.N. Ural'ceva [45]), by M.S. Agranovich and M.I. Vishik [1] and G. Grubb and V.A. Solonnikov [33] in the L_2-setting, and by G. Grubb [30]. In the latter two papers even pseudo-differential boundary value problems are considered. The results of Solonnikov in [62] and of Grubb are optimal in the strong L_p-setting.

More recently, R. Denk, M. Hieber, and J. Prüss [20], [21] established a maximal L_p-L_q-regularity theory for (P) for differential and boundary operators with operator-valued coefficients using recent Fourier multiplier theorems for operator-valued symbols. Their results extend an earlier (scalar) maximal L_p-L_q-regularity result of P. Weidemaier [68], [69]. This author was the first to discover that Triebel-Lizorkin spaces occur as trace spaces for anisotropic mixed L_p-L_q-Sobolev spaces. For further optimal regularity results we refer also to D. Guidetti [34], [36].

Thus, since everything concerning optimal solvability in an L_p-setting is known, why do we come back to the study of these problems? There are two motivations for this. First, the present work is the initial and basic step of our program to study linear and quasilinear parabolic problems on non-smooth manifolds, that is, on manifolds with edges, corners, conical singularities, etc. For our approach to such problems it is of utmost importance to have complete and precise control of the dependence of a priori estimates on all data. This information is needed since in the presence of singularities we are led to study non-compact non-complete manifolds with non-compact boundaries. This is in stark contrast to practical all work mentioned above. In fact, except for V.A. Solonnikov's early work in the classical Sobolev space setting, in all papers known to the author it is either assumed that M is compact, or $M = \mathbb{R}^n$, or $\mathring{M}$, the interior of M, is an open subset of $\mathbb{R}^n$ with a compact boundary.[1] In addition, to be able to deal efficiently with quasilinear problems it is mandatory to study linear problems with low regularity for the coefficients $(\mathcal{A}, \mathcal{B})$.

In principle, it should be possible to go through the various proofs in the literature and extract the needed information from there. However, this seems to be more difficult than to start from scratch and derive the optimal a priori estimates by taking care of the various dependencies. The latter we do in this work. Although the problem is classical, we believe that there are some points of interest for experts also.

The second, equally important, motivation comes from the desire to possess an optimal existence theory for (P) in weak and very weak L_p-settings. This is an open problem since the pioneering work of J.-L. Lions and E. Magenes [47] who gave a partial solution in the L_2-setting. It is well-known that weak theories are of great importance in the qualitative theory of (quasilinear) reaction-diffusion systems, in problems of mathematical physics, the mathematical theory of incompressible fluids in particular, and in control theory. Thus the second main objective of this treatise is to provide a completion of the L_2-case as well as an extension to the L_p-setting of the Lions–Magenes theory. This aspect is explained in more detail in the following section.

Our approach necessitates — among other things — a thorough knowledge of anisotropic function spaces, more precisely, anisotropic Bessel potential and Besov spaces of distributions. In the classical case, that is for scalar distributions, anisotropic spaces of positive order have been extensively investigated, starting with the fundamental contributions of S.M. Nikol′skiĭ and his school (e.g., [51]). However, for our purposes anisotropic Bessel potential and Besov spaces of negative order of Banach space valued distributions, and their duality theory, are of paramount importance. Furthermore, anisotropic trace and extension theorems on manifolds with corners are a fundamental tool for the investigation of (P) in weak settings. Except for a single extension theorem due to P. Grisvard [27] for anisotropic Sobolev spaces we could not find an in-depth study of these questions. For this reason we develop in this work the necessary machinery in order to get a firm basis for our study of parabolic initial boundary value problems in weak settings.

This treatise consists of two parts. Part 1, the part which is presented here, is concerned with the theory of anisotropic vector-valued Bessel potential and Besov spaces. It also contains Fourier multiplier estimates for certain classes of symbols which are basic for establishing maximal regularity results for constant coefficient boundary value problems to be given in Part 2.

In Part 1 we restrict ourselves to spaces on model domains, namely on the full space, on closed half-spaces, and on corners. The extension of our results to manifolds is postponed to Part 2.

[1] Solonnikov allows uniformly regular unbounded boundaries.

Although most of the theory of anisotropic spaces of distributions on all of $\mathbb{R}^d$ developed here does not cause surprises, it is hoped that even specialists of this subject will find something of interest.

Once again I had the great fortune to get help from Pavol Quittner and Gieri Simonett who read preliminary drafts of this paper. Besides of pointing out numerous lapses and inconsistencies they also contributed valuable hints which led to significant improvements of earlier versions. Their painstaking, unselfish, and invaluable work is greatly appreciated.

Some notation and conventions We use standard notation for function spaces. In particular, $\mathcal{D}$, $\mathcal{S}$, $\mathcal{O}_M$, $\mathcal{D}'$, and $\mathcal{S}'$ are the locally convex topological vector spaces (*LCS* for short) of test functions, rapidly decreasing smooth functions, slowly increasing smooth functions, distributions, and tempered distributions, respectively, and are given their usual topology. Furthermore, BC, BUC, and C_0 are the Banach spaces of bounded and continuous, bounded and uniformly continuous, and continuous functions vanishing at infinity, respectively. The norm in these spaces is denoted by $\|\cdot\|_\infty$ if no confusion seems likely, and $\|\cdot\|_p$ stands for the usual norm in L_p, $1 \leq p \leq \infty$. In general, these spaces consist of functions defined on suitable domains X and have values in a complex Banach space E so that we usually write $\mathfrak{F}(X, E)$ if $\mathfrak{F}$ denotes any one of the preceding spaces. In the important scalar case $E = \mathbb{C}$ we abbreviate $\mathfrak{F}(X, \mathbb{C})$ to $\mathfrak{F}(X)$.

The norm in an abstract Banach space is generally denoted by $|\cdot|$. However, $|\cdot|$ stands also for the Euclidean norm in $\mathbb{R}^n$ (or $\mathbb{C}^n$) and for the usual modulus of multi-indices. The reader will have no difficulty with the correct interpretation in a given frame.

As usual, we denote by c, or $c(\alpha, \beta, \ldots)$, constants which may depend on otherwise specified quantities $\alpha, \beta \ldots$ and, generally, have different values in different formulas, but are always independent of the free variables in a given setting. If M is a nonempty subset of some vector space, then $\dot{M} = M \backslash \{0\}$. Moreover, $\doteq$ means: equal except for equivalent norms.

Given LCSs E and F, we write $\mathcal{L}(E, F)$ for the space of continuous linear maps from E into F and endow it with the topology of uniform convergence on bounded sets. Thus it is a Banach space if E and F are Banach spaces. We set $\mathcal{L}(E) = \mathcal{L}(E, E)$, and $\mathcal{L}\mathrm{is}(E, F)$ is the subset of all isomorphisms in $\mathcal{L}(E, F)$, the set of toplinear isomorphisms, and $\mathcal{L}\mathrm{aut}(E) = \mathcal{L}\mathrm{is}(E, E)$. We denote continuous injection by $\hookrightarrow$, and $E \stackrel{d}{\hookrightarrow} F$ means that E is continuously and densely injected in F. For further (standard) notation we refer to Section 5 of the Introduction in H. Amann [4].

We make free use of interpolation theory and refer to Section I.2 in [4] for a summary of the basic definitions, results, and notation. In particular, given $s_0, s_1 \in \mathbb{R}$, we always set $s_\theta := (1-\theta)s_0 + \theta s_1$ for $0 \leq \theta \leq 1$. Furthermore, $[\cdot,\cdot]_\theta$, $(\cdot,\cdot)_{\theta,q}$, and $(\cdot,\cdot)^0_{\theta,\infty}$ denote the complex, the real (for $1 \leq q \leq \infty$), and the continuous interpolation functor of exponent $\theta \in (0,1)$, respectively. Thus, if E_0 and E_1 are Banach spaces with $E_1 \stackrel{d}{\hookrightarrow} E_0$, then $(E_0, E_1)^0_{\theta,\infty}$ is the closure of E_1 in $(E_0, E_1)_{\theta,\infty}$. Since in this situation E_1 is always dense in $(E_0, E_1)_{\theta,q}$ for $1 \leq q < \infty$, for the sake of a unified presentation we put $(\cdot,\cdot)^0_{\theta,q} := (\cdot,\cdot)_{\theta,q}$ for $1 \leq q < \infty$.

Finally, the reader is reminded that a retraction from an LCS E onto an LCS F is a continuous linear map from E onto F possessing a continuous right inverse, a coretraction.

Parabolic equations

In this introductory section we explain the results for problem (P) which will be proved in detail and much greater generality in Part 2. To avoid technical complications we restrict ourselves to second order scalar equations on compact manifolds and omit all lower order terms.

To be precise: we assume that M is an oriented n-dimensional compact Riemannian C^2 manifold. We write

$$\Gamma := \partial M, \quad \Omega := \mathring{M} = M \backslash \Gamma$$

and set[2]

$$Q := \Omega \times \mathring{J}, \qquad \Sigma := \Gamma \times \mathring{J},$$

so that $\overline{Q} = M \times J$ and $\overline{\Sigma} = \Gamma \times J$. We denote by $\boldsymbol{n}$ the outward pointing unit normal vector field on Γ. Furthermore, $\mathrm{grad} = \mathrm{grad}_M$ and $\mathrm{div} = \mathrm{div}_M$ are the gradient and the divergence operator on M. We also use $\boldsymbol{n}$ to denote the outward pointing unit normal vector field on Σ, that is, in this case we simply write $\boldsymbol{n}$ for $(\boldsymbol{n}, 0)$ without fearing confusion. Thus $\partial_{\boldsymbol{n}}$ is either the normal derivative on Γ or on Σ, according to the context.

We assume (using obvious identifications)

$$a \in C^{(1,0)}\big(\overline{Q}, (0,\infty)\big) := C\big(J, C^1\big(M, (0,\infty)\big)\big)$$

and put

$$\mathcal{A} := -\,\mathrm{div}(a \,\mathrm{grad}\,\cdot).$$

Then we consider the parabolic Dirichlet problem

$$\left.\begin{aligned} \partial u + \mathcal{A}u &= f \quad && \text{on } Q, \\ u &= g && \text{on } \Sigma, \\ u(\cdot,0) &= u^0 && \text{on } \Omega, \end{aligned}\right\} \tag{0.1}$$

where $\partial := \partial_t$, and also the Neumann problem

$$\left.\begin{aligned} \partial u + \mathcal{A}u &= f \quad && \text{on } Q, \\ a\partial_{\boldsymbol{n}} u &= g && \text{on } \Sigma, \\ u(\cdot,0) &= u^0 && \text{on } \Omega. \end{aligned}\right\} \tag{0.2}$$

For a more precise and concise presentation we denote by $\gamma = \gamma_\Sigma = |_\Sigma$ the trace operator for Σ, that is 'the restriction from Q to Σ', and by $\gamma_\tau = |_{t=\tau}$ the one from Q to $\Omega_\tau = \Omega \times \{\tau\}$ for $\tau \in J$, whenever they exist. We also identify Ω_0 with Ω. Lastly, we fix $\chi \in \{0,1\}$ and put

$$\mathcal{B} := \chi a \partial_{\boldsymbol{n}} + (1-\chi)\gamma.$$

[2] Readers not comfortable with manifolds may consider, at a first perusal, the case where Ω is a bounded C^2 domain in $\mathbb{R}^n$.

Then

$$\left.\begin{aligned} \partial u + \mathcal{A}u &= f \quad \text{on } Q, \\ \mathcal{B}u &= g \quad \text{on } \Sigma, \\ \gamma_0 u &= u^0 \quad \text{on } \Omega, \end{aligned}\right\} \tag{0.3}$$

coincides with (0.1) if $\chi = 0$, and it is the Neumann problem (0.2) if $\chi = 1$. Of course, there are no boundary conditions if $\Gamma = \emptyset$. In this case all explicit or implicit references to Γ have to be neglected in what follows.

For $1 < p < \infty$ and $r \in \{0, 1, 2\}$ we denote by $W_p^r(\mathring{X})$ the usual Sobolev spaces on $\mathring{X}$ for $X \in \{M, \Gamma\}$, so that $W_p^0(\mathring{X}) = L_p(\mathring{X})$. We identify $L_p(\mathring{X})$ and $L_p(X)$ in the obvious way. (Note $\mathring{\Gamma} = \Gamma$.) Consequently, $W_p^r(X) = W_p^r(\mathring{X})$. If $r \in (0, 2)\backslash\{1\}$, then $W_p^r(X) = W_p^r(\mathring{X})$ are the Slobodeckii (or fractional order Sobolev) spaces which can be characterized (locally, for example) by the standard Slobodeckii norm, or by

$$W_p^s(X) \doteq \big(L_p(X), W_p^2(X)\big)_{s/2,p}, \qquad 0 < s < 2, \quad s \neq 1.$$

Let E be a Banach space. Then

$$W_p^s(J, E) = W_p^s(\mathring{J}, E), \qquad 0 \le s \le 1,$$

denotes the standard Sobolev space if $s = 0$ or 1, respectively Slobodeckii space if $0 < s < 1$, of E-valued functions on J with the usual Sobolev–Slobodeckii norm. Equivalently,

$$W_p^s(J, E) = \big(L_p(J, E), W_p^1(J, E)\big)_{s,p}, \qquad 0 < s < 1.$$

Now we can define anisotropic Sobolev–Slobodeckii spaces on $X \times J$ by

$$W_p^{(s,s/2)}(X \times J) := L_p\big(J, W_p^s(X)\big) \cap W_p^{s/2}\big(J, L_p(X)\big), \qquad 0 \le s \le 2,$$

so that $W_p^{(0,0/2)}(X \times J) = L_p(X \times J)$. We also denote by $\Pi := \Sigma \cup \overline{\Omega}$ the parabolic boundary of Q and set

$$\partial_\chi W_p^{(2,1)}(\Pi) := W_p^{(2-\chi-1/p)(1,1/2)}(\Sigma) \times W_p^{2-2/p}(\Omega).$$

It follows that

$$\mathcal{P} := \big(\partial + \mathcal{A}, (\mathcal{B}, \gamma_0)\big) \in \mathcal{L}\big(W_p^{(2,1)}(Q), L_p(Q) \times \partial_\chi W_p^{(2,1)}(\Pi)\big).$$

However, $\mathcal{P}$ is not surjective, in general. In fact, suppose $p > 3/(2 - \chi)$. Denote by $\gamma_{0\Sigma}$ the restriction of γ_0 to Σ. Hence $\gamma_{0\Sigma}$ is the trace operator from Σ onto $\Gamma = \Gamma \times \{0\}$. It is well-defined; in fact,

$$\gamma_{0\Sigma} \in \mathcal{L}\big(W_p^{(2-\chi-1/p)(1,1/2)}(\Sigma), W_p^{2-\chi-3/p}(\Gamma)\big).$$

Similarly, let $\mathcal{B}_0 = \mathcal{B}(\cdot, 0)$ be the restriction of $\mathcal{B}$ to the initial hypersurface Ω. Then

$$\mathcal{B}_0 \in \mathcal{L}\big(W_p^{2-2/p}(\Omega), W_p^{2-\chi-3/p}(\Gamma)\big).$$

Furthermore,

$$\mathcal{B}_0\gamma_0 = \gamma_{0\Sigma}\mathcal{B}. \tag{0.4}$$

Thus, if $\mathcal{P}u = \big(f, (g, u^0)\big)$, we see that the compatibility condition

$$\mathcal{B}(\cdot, 0)u^0 = g|_{t=0}$$

has to be satisfied.

Motivated by this we define a closed linear subspace of $\partial_\chi W_p^{(2,1)}(\Pi)$ by

$$\partial_\chi^{cc} W_p^{(2,1)}(\Pi) := \begin{cases} \partial_\chi W_p^{(2,1)}(\Pi), & 1 < p \le 3/(2-\chi), \\ \{\, (g,u^0) \in \partial_\chi W_p^{(2,1)}(\Pi) \ ; \ \mathcal{B}_0 u^0 = \gamma_{0\Sigma} g \,\}, & p > 3/(2-\chi). \end{cases}$$

Now we can formulate the following unique solvability result for (0.3) in the ‘strong’ setting.

0.1 Theorem *Suppose $p \ne 3/(2-\chi)$ if $\Gamma \ne \emptyset$. Then problem (0.3) possesses a unique solution $u \in W_p^{(2,1)}(Q)$, a **strong** $W_p^{(2,1)}$ **solution**, iff*

$$\big(f,(g,u^0)\big) \in L_p(Q) \times \partial_\chi^{cc} W_p^{(2,1)}(\Pi).$$

More precisely,

$$\mathcal{P} \in \mathcal{L}\mathrm{is}\big(W_p^{(2,1)}(Q), L_p(Q) \times \partial_\chi^{cc} W_p^{(2,1)}(\Pi)\big)$$

and it depends analytically on $a \in C^{(1,0)}\big(\overline{Q},(0,\infty)\big)$.

Note that this is an optimal result, a *maximal regularity theorem.* It can be derived, for example, from the more general results of V.A. Solonnikov (who considers the ‘singular value’ $p = 3/(2-\chi)$ also) or of R. Denk, M. Hieber, and J. Prüss, referred to in the Introduction. If M is smooth, that is a C^∞ manifold, then it also follows from G. Grubb’s results in [30]. A complete proof covering general parabolic systems on not necessarily compact manifolds will be given in Part 2.

For $p > 3/(2-\chi)$ we can, due to (0.4), define the trace operator $\gamma_{0\mathcal{B}}$ from Q onto the corner manifold $\Gamma = \Gamma \times \{0\}$ by

$$\gamma_{0\mathcal{B}} := (\gamma_{0\Sigma}\mathcal{B} + \mathcal{B}_0\gamma_0)/2 \in \mathcal{L}\big(W_p^{(2,1)}(Q), W_p^{2-\chi-3/p}(\Gamma)\big).$$

Then, if

$$\big(f,(g,u^0)\big) \in L_p(Q) \times \partial_\chi^{cc} W_p^{(2,1)}(\Pi),$$

u is a strong $W_p^{(2,1)}$ solution of (0.3) iff

$$\left.\begin{aligned} (\partial + \mathcal{A})u &= f && \text{on } Q, \\ \mathcal{B}u &= g && \text{on } \Sigma, \\ \gamma_0 u &= u^0 && \text{on } \Omega, \\ \gamma_{0\mathcal{B}} u &= h && \text{on } \Gamma \quad \text{if } p > 3/(2-\chi), \end{aligned}\right\} \tag{0.5}$$

where $h := \mathcal{B}_0 u^0 = \gamma_{0\Sigma} g$. The reason for writing (0.3) in this form will become clear below.

Now we address ourselves to the investigation of the weak solvability of (0.3). For this we need some preparation.

Let N be a compact Riemannian C^2 manifold, which may have corners, as is the case for $\overline{Q}$ or $\overline{\Sigma}$. Denoting its volume measure by $dV = dV_N$ we write

$$\langle u, v\rangle_N := \int_N uv\, dV, \qquad u, v : N \to \mathbb{C},$$

whenever this integral exists. In particular,

$$\langle\cdot,\cdot\rangle_N : L_{p'}(N) \times L_p(N) \to \mathbb{C}$$

is a separating continuous bilinear form for $1/p + 1/p' = 1$, the $L_p(N)$ **duality pairing**, by which we identify, as usual, the dual $L_p(N)'$ of $L_p(N)$ with $L_{p'}(N)$. We write $\mathcal{D}(\mathring{N})$ for the space of all test functions on $\mathring{N}$, that is, of all C^2 functions having compact support in $\mathring{N}$, endowed with the usual inductive limit topology. Then $\mathcal{D}'(N)$, the space of distributions on N, is the dual of $\mathcal{D}(\mathring{N})$. We identify $u \in L_1(N)$ with the 'regular' distribution

$$\varphi \mapsto \langle u, \varphi \rangle_N, \qquad \varphi \in \mathcal{D}(\mathring{N}).$$

Then, given any Banach space $F(N)$ of functions on N satisfying

$$\mathcal{D}(\mathring{N}) \overset{d}{\hookrightarrow} F(N) \overset{d}{\hookrightarrow} L_1(N),$$

it follows $F(N)' \hookrightarrow \mathcal{D}'(N)$, that is, $F(N)'$ is, via the $L_p(N)$ duality pairing, naturally identified with a space of distributions on N. In particular, we obtain

$$\mathcal{D}(\mathring{N}) \overset{d}{\hookrightarrow} F(N) \overset{d}{\hookrightarrow} L_{p'}(N) \quad \Rightarrow \quad L_p(N) \hookrightarrow F(N)' \overset{d}{\hookrightarrow} \mathcal{D}'(N).$$

The first injection on the right-hand side is also dense if $F(N)$ is reflexive.

Now suppose $N \in \{X, X \times J\}$ and set $\{s\} := s$ if $N = X$, and $\{s\} := (s, s/2)$ otherwise. Denote by $\mathring{W}_p^{\{s\}}(N)$ the closure of $\mathcal{D}(\mathring{N})$ in $W_p^{\{s\}}(N)$. Then

$$\mathring{W}_p^{\{s\}}(N) \hookrightarrow W_p^{\{s\}}(N) \overset{d}{\hookrightarrow} L_p(N), \qquad 0 < s \le 2. \tag{0.6}$$

Hence, setting

$$W_p^{-\{s\}}(N) := \big(\mathring{W}_{p'}^{\{s\}}(N)\big)' \tag{0.7}$$

with respect to the $L_p(N)$ duality pairing $\langle \cdot, \cdot \rangle_N$,

$$L_p(N) \overset{d}{\hookrightarrow} W_p^{-\{s\}}(N) \overset{d}{\hookrightarrow} \mathcal{D}'(N), \qquad 0 < s \le 2,$$

since $W_p^{\{s\}}(N)$ is reflexive. Thus $W_p^{-\{s\}}(N)$ is a space of distributions on N.

We write $\widetilde{W}_p^{-\{s\}}(N)$ for the dual of $W_{p'}^{\{s\}}(N)$ with respect to $\langle \cdot, \cdot \rangle_N$. Then (0.6) and reflexivity imply

$$L_p(N) \overset{d}{\hookrightarrow} \widetilde{W}_p^{-\{s\}}(N), \qquad 0 < s \le 2.$$

However, $\widetilde{W}_p^{-\{s\}}(N)$ is *not a space of distributions on* N, in general, since $\mathcal{D}(\mathring{N})$ is not dense in $W_p^{\{s\}}(N)$ if s is big enough.

Suppose E is a reflexive Banach space, E' is its dual, and $\langle \cdot, \cdot \rangle_E : E' \times E \to \mathbb{C}$ the duality pairing. Then

$$\langle v, u \rangle_{J,E} := \int_J \big\langle v(t), u(t) \big\rangle_E \, dt$$

defines a continuous bilinear form

$$\langle \cdot, \cdot \rangle_{J,E} : L_{p'}(J, E') \times L_p(J, E) \to \mathbb{C}$$

by which we identify $L_p(J, E)'$ with $L_{p'}(J, E')$. From this,

$$W_{p'}^s\big(J, L_{p'}(X)\big) \overset{d}{\hookrightarrow} L_{p'}\big(J, L_{p'}(X)\big),$$

and reflexivity we infer that

$$W_p^{-s}\big(J, L_p(X)\big) := \Big(W_{p'}^s\big(J, L_{p'}(X)\big)\Big)', \qquad 0 < s \le 1,$$

is well-defined with respect to $\langle \cdot, \cdot \rangle_{J,E}$ and

$$L_p\big(J, L_p(X)\big) \overset{d}{\hookrightarrow} W_p^{-s}\big(J, L_p(X)\big), \qquad 0 < s \leq 1.$$

Using these facts it will be shown that the negative order anisotropic Sobolev–Slobodeckii space $W_p^{-(s/s/2)}(X \times J)$, defined in (0.7), can also be characterized by

$$W_p^{-(s/s/2)}(X \times J) \doteq L_p\big(J, W_p^{-s}(X)\big) + W_p^{-s/2}\big(J, L_p(X)\big), \qquad 0 < s \leq 2.$$

Negative order anisotropic Sobolev spaces occur naturally in the study of distributional solutions of

$$\partial u + \mathcal{A} u = f \quad \text{on } Q. \tag{0.8}$$

Indeed, a distributional L_p solution of (0.8) is a function $u \in L_p(Q)$ satisfying

$$\big\langle (-\partial + \mathcal{A})\varphi, u \big\rangle_Q = \langle \varphi, f \rangle_Q, \qquad \varphi \in \mathcal{D}(Q).$$

If $u \in L_p(Q)$, then we see

$$f := \partial u + \mathcal{A} u \in W_p^{-1}\big(J, L_p(M)\big) + L_p\big(J, W_p^{-2}(M)\big).$$

This suggests that $W_p^{-(2,1)}(Q)$ might be the largest space for which (0.8) has a distributional $L_p(Q)$ solution. Indeed, this is true if $\Gamma = \emptyset$. However, in the presence of a nonempty boundary the situation is more complicated.

To find distributional solutions for problem (0.3) if $\Gamma \neq \emptyset$ we have to use a space of test functions, $\Phi(Q)$, larger than $\mathcal{D}(Q)$, since the latter space 'does not see the boundary'. The correct choice turns out to be

$$\Phi(Q) := \big\{ \varphi \in C^{(2,1)}(\overline{Q}) \ ; \ \mathcal{B}\varphi = 0, \ \gamma_T \varphi = 0 \big\},$$

where

$$C^{(2,1)}(\overline{Q}) := C\big(J, C^2(M)\big) \cap C^1\big(J, C(M)\big).$$

This follows from Green's formula. Indeed, set

$$\mathcal{C} := -(1-\chi) a \partial_{\boldsymbol{n}} + \chi \gamma.$$

Then, given $u \in C^{(2,1)}(\overline{Q})$ and $\varphi \in \Phi(Q)$,

$$\begin{aligned} &\int_0^T \int_M u(-\partial + \mathcal{A})\varphi \, dV_M \, dt \\ &\qquad = \int_0^T \int_M \varphi(\partial + \mathcal{A}) u \, dV_M \, dt + \int_0^T \int_\Gamma \mathcal{C}\varphi \mathcal{B} u \, dV_\Gamma \, dt + \int_M \gamma_0 \varphi \gamma_0 u \, dV_M, \end{aligned}$$

that is, using $dV_Q = dV_M \otimes dt$ and $dV_\Sigma = dV_\Gamma \otimes dt$,

$$\big\langle (-\partial + \mathcal{A})\varphi, u \big\rangle_Q = \big\langle \varphi, (\partial + \mathcal{A}) u \big\rangle_Q + \langle \mathcal{C}\varphi, \mathcal{B} u \rangle_\Sigma + \langle \gamma_0 \varphi, \gamma_0 u \rangle_M. \tag{0.9}$$

Thus, if $u \in W_p^{(2,1)}(Q)$ is a solution of (0.3), it follows

$$\big\langle (-\partial + \mathcal{A})\varphi, u \big\rangle_Q = \langle \varphi, f \rangle_Q + \langle \mathcal{C}\varphi, g \rangle_\Sigma + \langle \gamma_0 \varphi, u^0 \rangle_\Omega, \qquad \varphi \in \Phi(Q), \tag{0.10}$$

where $\big(f, (g, u^0)\big) = \mathcal{P} u$.

We denote by $\Phi_{p'}^{(2,1)}(Q)$ the closure of $\Phi(Q)$ in $W_{p'}^{(2,1)}(Q)$. Then

$$\mathcal{D}(Q) \hookrightarrow \Phi_{p'}^{(2,1)}(Q) \overset{d}{\hookrightarrow} L_{p'}(Q)$$

implies, due to reflexivity,

$$L_p(Q) \overset{d}{\hookrightarrow} \Phi_p^{-(2,1)}(Q) := \big(\Phi_{p'}^{(2,1)}(Q)\big)'$$

with respect to $\langle\cdot,\cdot\rangle_Q$. However, $\mathcal{D}(Q)$ is not dense in $\Phi_{p'}^{(2,1)}(Q)$. Consequently, $\Phi_p^{-(2,1)}(Q)$ *is not a space of distributions on* Q.

Put

$$\langle\!\langle \varphi, \mathcal{P}u\rangle\!\rangle := \big\langle \varphi, (\partial+\mathcal{A})u\big\rangle_Q + \langle \mathcal{C}\varphi, \mathcal{B}u\rangle_\Sigma + \langle \gamma_0\varphi, \gamma_0 u\rangle_M .$$

Then, as we shall see,

$$\Phi_{p'}^{(2,1)}(Q) \times W_p^{(2,1)}(Q) \to \mathbb{C}, \quad (\varphi, u) \mapsto \langle\!\langle \varphi, \mathcal{P}u\rangle\!\rangle \tag{0.11}$$

is a separating continuous bilinear form, and (0.9) implies

$$\langle\!\langle \varphi, \mathcal{P}u\rangle\!\rangle = \big\langle (-\partial+\mathcal{A})\varphi, u\big\rangle_Q, \qquad (\varphi,u) \in \Phi_{p'}^{(2,1)}(Q) \times W_p^{(2,1)}(Q). \tag{0.12}$$

Suppose $p \neq 3/(2-\chi)$. It is a consequence of Theorem 0.1 that $-\partial+\mathcal{A}$ is a toplinear isomorphism from $\Phi_{p'}^{(2,1)}(Q)$ onto $L_{p'}(Q)$. Hence we infer from (0.12) that (0.11) has a unique continuous bilinear extension

$$\Phi_{p'}^{(2,1)}(Q) \times L_p(Q) \to \mathbb{C}, \quad (\varphi, u) \mapsto \langle\!\langle \varphi, \overline{\mathcal{P}}u\rangle\!\rangle,$$

where

$$\overline{\mathcal{P}} :\ L_p(Q) \to \Phi_p^{-(2,1)}(Q)$$

is the unique continuous extension of $\mathcal{P}$. This proves essentially the following general theorem where we use $\cong$ to denote toplinear isomorphisms.

0.2 Theorem *Suppose* $p \neq 3/(2-\chi)$ *if* $\Gamma \neq \emptyset$. *Given any* $F \in \Phi_p^{-(2,1)}(Q)$, *there exists a unique* $u \in L_p(Q)$, *an* ***ultra-weak solution*** *of* (0.3), *satisfying*

$$\big\langle (-\partial+\mathcal{A})\varphi, u\big\rangle_Q = \langle F, \varphi\rangle_Q, \qquad \varphi \in \Phi(Q).$$

Moreover, u *is the unique strong* $W_p^{(2,1)}$ *solution if* F *is given by*

$$\langle F, \varphi\rangle_Q = \langle \varphi, f\rangle_Q + \langle \mathcal{C}\varphi, g\rangle_\Sigma + \langle \gamma_0\varphi, u^0\rangle_M, \qquad \varphi \in \Phi(Q),$$

with $\big(f,(g,u^0)\big) \in L_p(Q) \times \partial_\chi^{cc} W_p^{(2,1)}(\Pi)$. *More precisely: there exist* $\overline{\mathcal{P}}$ *and* i_χ *such that the following diagram is commuting:*[3]

$$\begin{tikzcd}
W_p^{(2,1)}(Q) \arrow[r, "\mathcal{P}", "\cong"'] \arrow[d, hook, "d"] & L_p(Q) \times \partial_\chi^{cc} W_p^{(2,1)}(\Pi) \arrow[d, hook, "d", "i_\chi"'] \\
L_p(Q) \arrow[r, "\overline{\mathcal{P}}", "\cong"'] & \Phi_p^{-(2,1)}(Q)
\end{tikzcd}$$

and $\overline{\mathcal{P}}u = F$. *Furthermore,* $\overline{\mathcal{P}}$ *depends analytically on* $a \in C^{(1,0)}\big(\overline{Q}, (0,\infty)\big)$.

[3] $E \overset{d}{\underset{j}{\hookrightarrow}} F$ means that $j :\ E \to F$ is injective with dense image.

It remains to understand the structure of $\Phi_p^{-(2,1)}(Q)$ and to represent its elements by distributions on Q and its parabolic boundary Π, if possible.

For this we set

$$\partial_\chi W_p^{-(2,1)}(\Pi) := W_p^{-(\chi+1/p)(1,1/2)}(\Sigma) \times W_p^{-2/p}(\Omega)$$

and

$$\partial_\chi^{cc} W_p^{-(2,1)}(\Pi) := \begin{cases} \partial_\chi W_p^{-(2,1)}(\Pi), & p \geq 3/(2-\chi), \\ \partial_\chi W_p^{-(2,1)}(\Pi) \times W_p^{2-\chi-3/p}(\Gamma), & p < 3/(2-\chi). \end{cases}$$

It will be a consequence of our general extension results for manifolds with corners that the following isomorphism theorem is true.

0.3 Theorem *Suppose $p \neq 3/(2-\chi)$ if $\Gamma \neq \emptyset$. Then*

$$\Phi_p^{-(2,1)}(Q) \cong W_p^{-(2,1)}(Q) \times \partial_\chi^{cc} W_p^{-(2,1)}(\Pi).$$

Using this fact we can now represent ultra-weak solutions in terms of distributions on Q and Π.

0.4 Theorem

(i) *Suppose $p \neq 3/(2-\chi)$ if $\Gamma \neq \emptyset$. Fix*

$$\mathcal{T} \in \mathcal{L}\mathrm{is}\big(\Phi_p^{-(2,1)}(Q), W_p^{-(2,1)}(Q) \times \partial_\chi^{cc} W_p^{-(2,1)}(\Pi)\big) \tag{0.13}$$

and set $\mathcal{P}_{-2} := \mathcal{T}\overline{\mathcal{P}}$. Then

$$\mathcal{P}_{-2} \in \mathcal{L}\mathrm{is}\big(L_p(Q), W_p^{-(2,1)}(Q) \times \partial_\chi^{cc} W_p^{-(2,1)}(\Pi)\big)$$

and it depends analytically on $a \in C^{(1,0)}\big(\overline{Q}, (0,\infty)\big)$. Furthermore, there exists j_χ such that the diagram

$$\begin{tikzcd} W_p^{(2,1)}(Q) \arrow[r, "\mathcal{P}", "\cong"'] \arrow[d, hook, "d"] & L_p(Q) \times \partial_\chi^{cc} W_p^{(2,1)}(\Pi) \arrow[d, hook, "d", "j_\chi"'] \\ L_p(Q) \arrow[r, "\mathcal{P}_{-2}", "\cong"'] & W_p^{-(2,1)}(Q) \times \partial_\chi^{cc} W_p^{-(2,1)}(\Pi) \end{tikzcd} \tag{0.14}$$

is commuting.

(ii) *The dual $\mathcal{T}'$ of $\mathcal{T}$ belongs to*

$$\mathcal{L}\mathrm{is}\big(\mathring{W}_{p'}^{(2,1)}(Q) \times \partial_{1-\chi}^{cc} W_{p'}^{(2,1)}(\Pi), \Phi_{p'}^{(2,1)}(Q)\big).$$

(iii) *If $p > 3/(2-\chi)$ provided $\Gamma \neq \emptyset$, then problem (0.3) has for each*

$$(f, g, u^0) \in W_p^{-(2,1)}(Q) \times W_p^{-(\chi+1/p)(1,1/2)}(\Sigma) \times W_p^{-2/p}(\Omega) \tag{0.15}$$

a unique ultra-weak L_p solution, namely $u = \mathcal{P}_{-2}^{-1}(f, g, u^0)$. It is the unique $u \in L_p(Q)$ satisfying

$$\left.\begin{aligned} \big\langle(-\partial + \mathcal{A})\varphi, u\big\rangle_Q &= \langle f, \xi\rangle_Q + \langle g, \eta\rangle_\Sigma + \langle u^0, \zeta\rangle_\Omega \\ \varphi &= \mathcal{T}'(\xi, \eta, \zeta) \end{aligned}\right\} \tag{0.16}$$

for all $(\xi, \eta, \zeta) \in \mathcal{D}(Q) \times \mathcal{D}(\Sigma) \times \mathcal{D}(\Omega)$. Furthermore, it solves $(\partial + \mathcal{A})u = f$ on Q in the distributional sense.

(iv) *Suppose $\Gamma \neq \emptyset$ and $p < 3/(2-\chi)$. Then there exists for each*

$$(f, g, u^0, h) \text{ in } W_p^{-(2,1)}(Q) \times W_p^{-(\chi+1/p)(1,1/2)}(\Sigma) \times W_p^{-2/p}(\Omega) \times W_p^{2-\chi-3/p}(\Gamma) \tag{0.17}$$

a unique ultra-weak L_p solution of (0.3), *namely $u = \mathcal{P}_{-2}^{-1}(f, g, u^0, h)$. It is the unique $u \in L_p(Q)$ satisfying*

$$\left.\begin{aligned} \big\langle(-\partial + \mathcal{A})\varphi, u\big\rangle_Q &= \langle f, \xi\rangle_Q + \langle g, \eta\rangle_\Sigma + \langle u^0, \zeta\rangle_\Omega + \langle h, \vartheta\rangle_\Gamma \\ \varphi &= \mathcal{T}'(\xi, \eta, \zeta, \vartheta) \end{aligned}\right\} \tag{0.18}$$

for all $(\xi, \eta, \zeta, \vartheta) \in \mathcal{D}(Q) \times \mathcal{D}(\Sigma) \times \mathcal{D}(\Omega) \times \mathcal{D}(\Gamma)$. It solves $(\partial + \mathcal{A})u = f$ on Q in the sense of distributions.

PROOF. (i) follows from Theorems 0.2 and 0.3.
As for (ii): it suffices to observe that

$$\big(\partial_\chi^{cc} W_p^{-(2,1)}(\Pi)\big)' = \partial_{1-\chi}^{cc} W_{p'}^{(2,1)}(\Pi)$$

with respect to the duality pairing $\langle\cdot,\cdot\rangle_\Sigma + \langle\cdot,\cdot\rangle_\Omega$ if $p > 3/(2-\chi)$, and with respect to $\langle\cdot,\cdot\rangle_\Sigma + \langle\cdot,\cdot\rangle_\Omega + \langle\cdot,\cdot\rangle_\Gamma$ otherwise. Assertions (iii) and (iv) are now evident except for the respective last statement. For this we refer to Part 2. $\square$

Clearly, the 'extended initial boundary value problem' $\mathcal{P}_{-2}$ depends on the choice of $\mathcal{T}$. Thus the representation of the ultra-weak L_p solution of (0.3) depends on $\mathcal{T}$ also, as is indicated by (0.16) and (0.18). However, for each choice of $\mathcal{T}$ the 'generalized' parabolic boundary value problem $\mathcal{P}_{-2} = \mathcal{T}\overline{\mathcal{P}}$ establishes an isomorphism between $L_p(Q)$ and $W_p^{-(2,1)}(Q) \times \partial_\chi^{cc} W_p^{-(2,1)}(\Pi)$. Thus *the ultra-weak solvability properties of problem* (0.3) *are independent of the particular choice of $\mathcal{T}$.* We fix now — once and for all — an isomorphism $\mathcal{T}$. (The proof of Theorem 0.3 will provide us with rather detailed information on the structure of $\mathcal{T}$.)

Suppose either $\Gamma = \emptyset$ or $p > 3/(2-\chi)$. Then, given (f, g, u^0) satisfying (0.15), we call u **ultra-weak L_p solution** of (0.3) iff $u \in L_p(Q)$ and (0.16) is satisfied. If $\Gamma \neq \emptyset$ and $p < 3/(2-\chi)$, then, given (f, g, u^0, h) satisfying (0.17), u is said to be an **ultra-weak L_p solution** of

$$\left.\begin{aligned} (\partial + \mathcal{A})u &= f && \text{on } Q, \\ \mathcal{B}u &= g && \text{on } \Sigma, \\ \gamma_0 u &= u^0 && \text{on } \Omega, \\ \gamma_{0\mathcal{B}} u &= h && \text{on } \Gamma, \end{aligned}\right\} \tag{0.19}$$

iff $u \in L_p(Q)$ and (0.18) is satisfied. Clearly, (0.3) and (0.19) are *formal notation only.*

For the reader's convenience we restate Theorem 0.4 using this intuitive formulation.

0.5 Theorem

(i) *Suppose either* $\Gamma = \emptyset$ *or* $p > 3/(2-\chi)$. *Then*

$$\left.\begin{aligned} (\partial + \mathcal{A})u &= f \quad \text{on } Q,\\ \mathcal{B}u &= g \quad \text{on } \Sigma,\\ \gamma_0 u &= u^0 \quad \text{on } \Omega \end{aligned}\right\} \tag{0.20}$$

has a unique ultra-weak solution $u \in L_p(Q)$ *iff*

$$(f, g, u^0) \in W_p^{-(2,1)}(Q) \times W_p^{-(\chi+1/p)(1,1/2)}(\Sigma) \times W_p^{-2/p}(\Omega).$$

(ii) *Suppose* $\Gamma \neq \emptyset$ *and* $p < 3/(2-\chi)$. *Then*

$$\begin{aligned} (\partial + \mathcal{A})u &= f \quad \text{on } Q,\\ \mathcal{B}u &= g \quad \text{on } \Sigma,\\ \gamma_0 u &= u^0 \quad \text{on } \Omega,\\ \gamma_{0\mathcal{B}} u &= h \quad \text{on } \Gamma \end{aligned}$$

has an ultra-weak solution $u \in L_p(Q)$ *iff* (f, g, u^0, h) *belongs to*

$$W_p^{-(2,1)}(Q) \times W_p^{-(\chi+1/p)(1,1/2)}(\Sigma) \times W_p^{-2/p}(\Omega) \times W_p^{2-\chi-3/p}(\Gamma).$$

(iii) *The mapping* $(f, g, u^0) \mapsto u$, *respectively* $(f, g, u^0, h) \mapsto u$, *is a toplinear isomorphism onto* $L_p(Q)$ *depending analytically on* $a \in C^{(1,0)}\big(\overline{Q}, (0,\infty)\big)$.

It should be observed that the regularity assumptions for the data are natural in comparison with the ones of Theorem 0.1: the order of all spaces containing u, f, g, and u^0 is 2 less than the one of the corresponding spaces of the strong version.

Now, specializing to $p = 2$, we can compare our results with those of J.-L. Lions and E. Magenes [47]. As usual, we write H^s for W_2^s.

First we consider the *Dirichlet problem*. Theorem 0.5 guarantees that

$$\left.\begin{aligned} (\partial + \mathcal{A})u &= f \quad \text{on } Q,\\ \gamma u &= g \quad \text{on } \Sigma,\\ \gamma_0 u &= u^0 \quad \text{on } \Omega \end{aligned}\right\} \tag{0.21}$$

possesses a unique ultra-weak solution $u \in L_2(Q)$ iff

$$(f, g, u^0) \in H^{-(2,1)}(Q) \times H^{-(1/2,1/4)}(\Sigma) \times H^{-1}(\Omega).$$

In Volume 2 of [47] the authors assume that Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary and $a \in \mathcal{D}(\overline{Q})$. They put

$$D^0(Q) := \left\{\, v \in L_2(Q) \ ;\ (\partial + \mathcal{A})v \in L_2(Q) \,\right\}$$

equipped with the graph norm of the maximal restriction of $\partial + \mathcal{A} \in \mathcal{L}\big(\mathcal{D}'(Q)\big)$ to $L_2(Q)$. They also introduce a certain space $\Xi^{-(2,1)}(Q)$ satisfying

$$L_2(Q) \overset{d}{\hookrightarrow} \Xi^{-(2,1)}(Q) \underset{\neq}{\hookrightarrow} H^{-(2,1)}(Q).$$

Then they prove (cf. Remark[4] IV.12.3 in [47]) that, given

$$(f, g, u^0) \in \Xi^{-(2,1)}(Q) \times H^{-(1/2,1/4)}(\Sigma) \times H^{-1}(\Omega),$$

there exists a unique solution $u \in D^0(Q)$ such that the differential equation on Q is satisfied in the distributional sense and the boundary conditions in the sense of continuous extensions of the standard trace operators. Furthermore, the map $(f, g, u^0) \mapsto u$ is a toplinear isomorphism onto $D^0(Q)$. In addition, u is the strong $H^{(2,1)}$ solution if the data satisfy the conditions of Theorem 0.5 with $p = 2$.

Note that J.-L. Lions and E. Magenes provide a maximal regularity theorem in this case. However, by our results we can solve the Dirichlet problem for a larger class of distributions f on Q than can be done by the Lions–Magenes theory. Moreover, in concrete situations the space $D^0(Q)$ is not easy to handle.

Next we consider the *Neumann problem.* Theorem 0.5 guarantees that

$$\left.\begin{aligned} (\partial + \mathcal{A})u &= f && \text{on } Q, \\ a\partial_{\boldsymbol{n}} u &= g && \text{on } \Sigma, \\ \gamma_0 u &= u^0 && \text{on } \Omega, \\ \gamma_{0\mathcal{B}} u &= h && \text{on } \Gamma \end{aligned}\right\} \tag{0.22}$$

possesses a unique ultra-weak solution $u \in L_2(\Omega)$ iff

$$(f, g, u^0, h) \in H^{-(2,1)}(Q) \times H^{-(3/2,3/4)}(\Sigma) \times H^{-1}(\Omega) \times H^{-1/2}(\Gamma). \tag{0.23}$$

J.-L. Lions and E. Magenes introduce intermediate spaces

$$L_2(\Sigma) \overset{d}{\hookrightarrow} H^{-3/2}\Xi^{-3/4}(\Sigma) \underset{\neq}{\hookrightarrow} H^{-(3/2,3/4)}(\Sigma)$$

and

$$L_2(\Omega) \overset{d}{\hookrightarrow} \Xi^{-1}(\Omega) \underset{\neq}{\hookrightarrow} H^{-1}(\Omega)$$

and establish the following result (see Theorem IV.12.1 in [47]):

There exists a continuous linear map

$$\Xi^{-(2,1)}(Q) \times H^{-3/2}\Xi^{-3/4}(\Sigma) \times \Xi^{-1}(\Omega) \to D^0(Q), \quad (f, g, u^0) \mapsto u$$

such that

$$(\partial + \mathcal{A})u = f \quad \text{in } \mathcal{D}'(Q)$$

and the boundary and initial conditions

$$a\partial_{\boldsymbol{n}} u = g \quad \text{on } \Sigma, \qquad \gamma_0 u = u^0 \quad \text{on } \Omega$$

are satisfied in the sense of continuous linear extension. Thus u is a generalized solution of the Neumann problem

$$\begin{aligned} (\partial + \mathcal{A})u &= f && \text{on } Q, \\ a\partial_{\boldsymbol{n}} u &= g && \text{on } \Sigma, \\ \gamma_0 u &= u^0 && \text{on } \Omega. \end{aligned}$$

However, as observed in [47], *this is not a maximal regularity result*: the data (f, g, u^0) belong to a space smaller than the ‘optimal space’ and $(f, g, u^0) \mapsto u$ *is*

[4] Roman numbers indicate chapters.

not an isomorphism. In fact, not even uniqueness of the generalized solution is guaranteed.

The fact that in the case of the Neumann problem there may occur a space of distributions on Γ has already been observed (if $p = 2$) by C. Baiocchi [15] (also see Section IV.12.3 in [47]). This author established an isomorphism theorem for the Neumann problem, however between $D^0(Q)$ and

$$L_2(Q) \times H^{-(3/2,3/4)}(\Sigma) \times H^{-1}(\Omega) \times H^{-1/2}(\Gamma).$$

This is a much weaker result than (0.22), (0.23).

We repeat that the Lions–Magenes theory as well as C. Baiocchi's result are restricted to $p = 2$, whereas $1 < p < \infty$ in our work.

In H. Amann [3] we introduced the concept of very weak L_p solutions for $1 < p < \infty$. More precisely, a **very weak L_p solution** of (0.3) is a function $u \in C\big(J, L_p(\Omega)\big)$ satisfying

$$\big\langle(-\partial + \mathcal{A})\varphi, u\big\rangle_Q = \langle\varphi, f\rangle_Q + \langle\mathcal{C}\varphi, g\rangle_\tau + \langle\gamma_0\varphi, u^0\rangle_\Omega, \qquad \varphi \in \Phi.$$

Theorem 11.4 of [3] guarantees that, given $\sigma > 0$, problem (0.3) has for each

$$(f, g, u^0) \in C^\sigma\big(J, W^{-2}_{p,\mathcal{B}}(\Omega)\big) \times C^\sigma\big(J, W_p^{-\chi-1/p}(\Gamma)\big) \times L_p(\Omega), \tag{0.24}$$

where

$$W^{-2}_{p,\mathcal{B}}(\Omega) := \big\{\, v \in W^2_{p'}(\Omega) \ ;\ \mathcal{B}v = 0 \,\big\}',$$

a unique very weak solution satisfying, in addition,

$$u \in C\big(J, L_p(\Omega)\big) \cap C^1\big(J, W^{-2}_{p,\mathcal{B}}(\Omega)\big).$$

Note that $W^{-2}_{p,\mathcal{B}}(\Omega)$ is not a space of distributions on Ω. Hence there is some ambiguity in (0.24): namely, g could be omitted since it can be identified with an element of $W^{-2}_{p,\mathcal{B}}(\Omega)$ (cf. Section 11 in [3]). In fact, in Theorem 10 of H. Amann [6] it is shown that

$$W^{-2}_{p,\mathcal{B}}(\Omega) \cong W_p^{-2}(\Omega) \times W_p^{-\chi-1/p}(\Gamma).$$

To compare very weak and ultra-weak solutions we recall

$$W_p^{-(\chi+1/p)(1,1/2)}(\Sigma) = L_p\big(J, W_p^{-\chi-1/p}(\Gamma)\big) + W_p^{-(\chi+1/p)/2}\big(J, L_p(\Gamma)\big).$$

Consequently,

$$C^\sigma\big(J, W_p^{-\chi-1/p}(\Gamma)\big) \hookrightarrow W_p^{-(\chi+1/p)(1,1/2)}(\Sigma).$$

This implies that every very weak solution is an ultra-weak one. The main difference between very weak and ultra-weak solutions is the fact that the latter possess a distributional time derivative belonging to $W_p^{-1}\big(J, L_p(\Omega)\big)$, whereas such information is not available for very weak solutions.

The concept of ultra-weak solutions, introduced in the present work, is a vast generalization of the one of very weak solutions. In addition to allowing for a more general class of data, ultra-weak solutions lead to maximal regularity results, that is, isomorphism theorems. It is not possible to derive such a theorem within the theory of very weak solutions.

The idea of very weak solutions has been adapted to the Navier–Stokes equations in H. Amann [7] (also see [8], [9]). Subsequently, this work has been extended by a number of authors and applied to derive new regularity results (see, for example, G.P. Galdi, C.G. Simader, and H. Sohr [26], R. Farwig, G.P. Galdi, and H. Sohr [24], R. Farwig, H. Kozono, and H. Sohr [25], K. Schumacher [58]). Following the lines of H. Amann [7], [8], [9], the theory of ultra-weak solutions can also be extended to the Navier–Stokes equations. This will be done elsewhere.

One of the most important fields of research in which parabolic equations with distributional data on the boundary are unalterable is control theory. Starting with the early work of J.-L. Lions and his school (cf. Chapter VI in [47]), most of the mathematical control theory for systems governed by partial differential equations has been developed in the L_2 framework (cf., for example, I. Lasiecka and R. Triggiani [46] for more recent developments). This setting imposes severe restrictions on the possible choice of controls on the boundary. For instance, it does not allow the use of point controls on Σ. The situation is very different with the L_p theory, since for p sufficiently close to 1 we can consider arbitrary Radon measures as boundary data. In fact, using the theory of very weak solutions, this has already been shown in H. Amann [6] and, in the nonlinear framework, in H. Amann and P. Quittner [12]. For further applications of the theory of very weak solutions to control theory we refer to H. Amann and P. Quittner [13], [14]. Using the theory of ultra-weak solutions we can improve on those results as is indicated now.

For a separable locally compact space X we denote by $\mathcal{M}(X)$ the Banach space of all bounded complex-valued Radon measures on X. Then $\mathcal{M}(X) = C_0(X)'$ with respect to the identification

$$\langle \mu, \varphi \rangle_{C_0(X)} = \int_X \varphi \, d\mu, \qquad (\varphi, \mu) \in C_0(X) \times \mathcal{M}(X).$$

Using this characterization and Sobolev embedding theorems for anisotropic spaces we find

$$\mathcal{M}(Q) \times \mathcal{M}(\Omega) \hookrightarrow W_p^{-(2,1)}(Q) \times W_p^{-2/p}(\Omega), \qquad p < (n+2)/n,$$

and

$$\mathcal{M}(\Sigma) \times \mathcal{M}(\Gamma) \hookrightarrow W_p^{-(\chi+1/p)(1,1/2)}(\Sigma) \times W_p^{2-\chi-3/p}(\Gamma), \qquad p < (n+2)/(n+1-\chi).$$

The following theorem is an easy consequence of these embeddings and Theorem 0.5.

0.6 Theorem *Suppose $\Gamma \neq \emptyset$ and $p < (n+2)/(n+1-\chi)$. Then, given*

$$(\mu, \nu, \rho, \sigma) \in \mathcal{M}(Q) \times \mathcal{M}(\Sigma) \times \mathcal{M}(\Omega) \times \mathcal{M}(\Gamma),$$

there exists a unique ultra-weak solution $u \in L_p(Q)$ of

$$\begin{aligned} (\partial + \mathcal{A})u &= \mu \quad \text{on } Q, \\ \mathcal{B}u &= \nu \quad \text{on } \Sigma, \\ \gamma_0 u &= \rho \quad \text{on } \Omega, \\ \gamma_{0\mathcal{B}} u &= \sigma \quad \text{on } \Gamma. \end{aligned}$$

In other words, there exists a unique $u \in L_p(Q)$ satisfying

$$\begin{aligned} \big\langle (-\partial + \mathcal{A})\varphi, u \big\rangle_Q &= \int_Q \xi \, d\mu + \int_\Sigma \eta \, d\nu + \int_\Omega \zeta \, d\rho + \int_\Gamma \vartheta \, d\sigma \\ \varphi &= \mathcal{T}'(\xi, \eta, \zeta, \vartheta) \end{aligned}$$

for all $(\xi, \eta, \zeta, \vartheta) \in \mathcal{D}(Q) \times \mathcal{D}(\Sigma) \times \mathcal{D}(\Omega) \times \mathcal{D}(\Gamma)$. The map $(\mu, \nu, \rho, \sigma) \mapsto u$ is linear and continuous and depends analytically on $a \in C^{(1,0)}\big(\overline{Q}, (0, \infty)\big)$.

Note that $p < 2$ unless $\chi = 1$ and $n \le 2$.

As an extreme case we see, for example, that the Dirichlet problem

$$\begin{aligned} (\partial_t + \mathcal{A})u &= 0 && \text{on } Q, \\ \gamma u &= 0 && \text{on } \Sigma, \\ \gamma_0 u &= 0 && \text{on } \Omega, \\ \gamma_{0\mathcal{B}} u &= \delta_{x_0} && \text{on } \Gamma \end{aligned}$$

has a unique ultra-weak L_p solution if $p < (n+2)/(n+1)$ and δ_{x_0} is the Dirac measure with support $\{x_0\}$.

Theorems 0.1 and 0.2 concern the border cases $s = 2$ and $s = 0$, respectively, of the solution space $W_p^{s(1,1/2)}(Q)$. It is to be expected that, by interpolation, one can derive maximal regularity theorems for (0.3) in intermediate spaces, that is, for $0 < s < 2$. By and large this is correct. We do not give details here but refer the reader to Part 2. The difficulty resides in concrete characterizations of the pertinent interpolation spaces. For this we have to have a thorough understanding of interpolation properties of anisotropic Besov and Bessel potential spaces in the presence of boundary conditions, questions which are addressed in Part 1 of this treatise.

CHAPTER 1

Multiplier estimates

This chapter is of preparatory nature. We discuss the concept of anisotropic dilations and derive multiplier estimates for parameter-dependent symbols. These concepts are of basic importance for the whole treatise.

Although we are mostly interested in the case of ‘parabolic weight vectors’ we consider the general anisotropic case. This can be done without additional complications. In fact, the general setting and consequent use of appropriate notation clarify many results which would appear mysterious otherwise. In addition, general anisotropic function spaces are of interest in their own right.

1.1 Anisotropic dilations

A systematic study of anisotropic Banach spaces, more specifically, anisotropic Bessel potential and Besov spaces, is based on weighted dilations of the underlying space $\mathbb{R}^d$ and corresponding linear representations on suitable spaces of distributions. In this section we introduce these dilations and fix the setting for the whole part.

Let G be the multiplicative group $(\dot{\mathbb{R}}^+, \cdot)$ and V an LCS. A linear representation of G on V is a map

$$G \to \mathcal{L}(V), \quad t \mapsto T_t$$

satisfying

$$T_s T_t = T_{st}, \quad T_1 = 1_V.$$

It follows that $\{\, T_t \ ; \ t \in G \,\}$ is a commutative subgroup of $\mathcal{L}\mathrm{aut}(V)$, and

$$(T_t)^{-1} = T_{t^{-1}} = T_{1/t}$$

for $t \in G$. The representation is strongly continuous if $(t \mapsto T_t v) \in C(G, V)$ holds for $v \in V$.

Throughout this part
$d \in \dot{\mathbb{N}}$ and $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_d) \in \dot{\mathbb{N}}^d$ is a fixed ***weight vector****.*

We denote by

$$\omega \text{ the least common multiple of } \{\omega_1, \ldots, \omega_d\}$$

and set[1]

$$|\boldsymbol{\omega}| := \omega_1 + \cdots + \omega_d, \quad \alpha \cdot \boldsymbol{\omega} := \alpha^1\omega_1 + \cdots + \alpha^d\omega_d$$

for $\alpha = (\alpha^1, \ldots, \alpha^d) \in \mathbb{N}^d$.

We define an action of G on $\mathbb{R}^d$ by

$$t \bullet x = (t^{\omega_1}x^1, \ldots, t^{\omega_d}x^d), \qquad t > 0, \quad x \in \mathbb{R}^d, \tag{1.1.1}$$

the (anisotropic) **dilation** with weight $\boldsymbol{\omega}$.

Let E be a Banach space. The (anisotropic) **dilation** with weight $\boldsymbol{\omega}$ is defined on $\mathcal{S}(\mathbb{R}^d, E)$ by

$$\sigma_t u(x) := u(t \bullet x), \qquad u \in \mathcal{S}(\mathbb{R}^d, E), \quad x \in \mathbb{R}^d, \tag{1.1.2}$$

for $t > 0$. It is extended to $\mathcal{S}'(\mathbb{R}^d, E)$ by setting

$$\sigma_t u(\varphi) := t^{-|\boldsymbol{\omega}|} u(\sigma_{1/t}\varphi), \qquad u \in \mathcal{S}'(\mathbb{R}^d, E), \quad \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{1.1.3}$$

The most important examples in our context are the **trivial weight vector** $(1, 1, \ldots, 1)$ and the $2m$-**parabolic weight vector** $(1, \ldots, 1, 2m)$, where $m \in \dot{\mathbb{N}}$. In the first case, (1.1.1) and (1.1.2) are the standard (isotropic) dilations on $\mathbb{R}^d$.

Now we collect the basic properties of σ_t in a proposition and leave its simple proof to the reader.

1.1.1 Proposition *The map $t \mapsto \sigma_t$ is a strongly continuous linear representation of $(\dot{\mathbb{R}}^+, \cdot)$ on $\mathcal{S}(\mathbb{R}^d, E)$ and on $\mathcal{S}'(\mathbb{R}^d, E)$. It restricts from $\mathcal{S}'(\mathbb{R}^d, E)$ to a representation on $L_q(\mathbb{R}^d, E)$ for $1 \le q \le \infty$, which is strongly continuous for $q < \infty$, and possesses the following properties*:

(i) $\partial^\alpha \circ \sigma_t = t^{\alpha \cdot \boldsymbol{\omega}} \sigma_t \circ \partial^\alpha$, $\alpha \in \mathbb{N}^d$;

(ii) $\|\sigma_t u\|_q = t^{-|\boldsymbol{\omega}|/q} \|u\|_q$, $u \in L_q(\mathbb{R}^d, E)$, $1 \le q \le \infty$;

(iii) $\mathcal{F} \circ \sigma_t = t^{-|\boldsymbol{\omega}|} \sigma_{1/t} \circ \mathcal{F}$, $t > 0$.

1.2 Homogeneity

In the investigation of linear elliptic and parabolic boundary value problems, Fourier analysis plays a predominant rôle. In this connection one has to study model problems on $\mathbb{R}^d$ and half-spaces thereof which lead to Fourier multiplier operators being (anisotropically) homogeneous and parameter-dependent. In this section we introduce spaces of parameter-dependent functions which are homogeneous with respect to dilations with weight $\boldsymbol{\omega}$ and collect some elementary properties. The systematic use of parameter-dependent homogeneous functions will greatly simplify our calculations and allow the control of the parameter-dependence in various estimates, which is crucial for our approach.

Throughout this part we also assume[2]
H *is a closed cone in* $\mathbb{C}$ *containing* $\mathbb{R}^+ = \mathbb{R}^+ + i0$.

[1]The simultaneous use of boldface and standard letters for vectors in $\mathbb{N}^d$ is inconsistent, of course. However, it is justified by the particular rôle of the weight vector and allows us to denote by ω the least common multiple of the components of $\boldsymbol{\omega}$.

We put

$$\mathsf{Z} := \mathbb{R}^d \times \mathsf{H},$$

denote its general point by $\zeta = (\xi, \eta)$ with $\xi = (\xi^1, \ldots, \xi^d) \in \mathbb{R}^d$ and $\eta \in \mathsf{H}$, and write

$$a_\eta := a(\cdot, \eta) : \mathbb{R}^d \to E, \qquad \eta \in \mathsf{H},$$

for $a : \mathsf{Z} \to E$. We also extend the anisotropic dilation with weight $\boldsymbol{\omega}$ to Z by setting

$$t \bullet \zeta := (t \bullet \xi, t\eta), \qquad t > 0, \quad \zeta = (\xi, \eta) \in \mathsf{Z}, \tag{1.2.1}$$

and

$$\sigma_t a(\zeta) := a(t \bullet \zeta) \tag{1.2.2}$$

for $a : \mathsf{Z} \to E$.

Suppose $z \in \mathbb{C}$. A map $a : \dot{\mathsf{Z}} \to E$ is **positively z-homogeneous** (with respect to the action (1.2.1)) if

$$\sigma_t a = t^z a, \qquad t > 0. \tag{1.2.3}$$

Let S be a nonempty set. We define an equivalence relation $\sim$ on $(0,\infty)^S$ by setting

$$f \sim g \text{ for } f, g : S \to (0,\infty) \Longleftrightarrow (1/\kappa) f \le g \le \kappa f \text{ for some } \kappa \ge 1.$$

In particular,

$$\|\cdot\|_1 \sim \|\cdot\|_2$$

means that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms on a given vector space.

In the following lemma we present some elementary properties of positively homogeneous maps. They are basic for the computations to follow below.

1.2.1 Lemma *Suppose $a \in C(\dot{\mathsf{Z}}, E)$ and $a_\eta \in C^k\big((\mathbb{R}^d)^\bullet, E\big)$ for some $k \in \mathbb{N}$ and all $\eta \in \dot{\mathsf{H}}$, and a is positively z-homogeneous. Also suppose $\mathsf{M} \in C\big(\dot{\mathsf{Z}}, (0,\infty)\big)$ is positively 1-homogeneous. Then*

(i) $\sigma_t \partial_\xi^\alpha a = t^{z - \alpha\cdot\boldsymbol{\omega}} \partial_\xi^\alpha a, \quad |\alpha| \le k, \quad t > 0.$

(ii) *Set*

$$\zeta_{\mathsf{M}}^* := \big(1/\mathsf{M}(\zeta)\big) \bullet \zeta, \qquad \zeta \in \dot{\mathsf{Z}}.$$

Then $\mathsf{M}(\zeta_{\mathsf{M}}^*) = 1$ and

$$\partial_\xi^\alpha a(\zeta) = \mathsf{M}^{z-\alpha\cdot\boldsymbol{\omega}}(\zeta) \partial_\xi^\alpha a(\zeta_{\mathsf{M}}^*), \qquad \zeta \in \dot{\mathsf{Z}}. \tag{1.2.4}$$

(iii) $[\mathsf{M} = 1]$ *is compact.*

(iv) *If $\mathsf{N} \in C\big(\dot{\mathsf{Z}}, (0,\infty)\big)$ is positively 1-homogeneous, then $\mathsf{N} \sim \mathsf{M}$. Furthermore,*

$$(1/\kappa)\mathsf{M} \le \mathsf{N} \le \kappa \mathsf{M}$$

implies $[\mathsf{M} = 1] \subset [\kappa^{-1} \le \mathsf{N} \le \kappa]$.

---

[2] More generally, H can be a closed cone in $\mathbb{C}^N$ containing $\mathbb{R}^+ := (\mathbb{R}^+ + i0) \times \{0\}$ for some $N \ge 2$. However, such generality is not needed in this work.

PROOF. (i) follows by differentiating (1.2.3) and using Proposition 1.1.1(i).

(ii) Since $\zeta = \mathsf{M}(\zeta) \bullet \zeta^*_{\mathsf{M}}$, the positive 1-homogeneity implies $\mathsf{M}(\zeta) = \mathsf{M}(\zeta)\mathsf{M}(\zeta^*_{\mathsf{M}})$. Hence $\mathsf{M}(\zeta^*_{\mathsf{M}}) = 1$. Furthermore, by (i),

$$\partial^\alpha_\xi a(\zeta) = \sigma_{\mathsf{M}(\zeta)} \partial^\alpha_\xi a(\zeta^*_{\mathsf{M}}) = \mathsf{M}(\zeta)^{z-\alpha\cdot\boldsymbol{\omega}} \partial^\alpha_\xi a(\zeta^*_{\mathsf{M}})$$

for $\zeta \in \dot{\mathsf{Z}}$.

(iii) is a consequence of the closedness of H and the continuity of M. Moreover, setting

$$\kappa := \|\mathsf{M}\|_{\infty,[\mathsf{N}=1]} \vee \|\mathsf{N}\|_{\infty,[\mathsf{M}=1]},$$

assertion (iv) is obvious. □

For $k \in \mathbb{N}$ and $z \in \mathbb{C}$ we write

$$\mathcal{H}^k_z(\mathsf{Z}, E)$$

for the vector space of all $a \in C(\dot{\mathsf{Z}}, E)$ which are positively z-homogeneous and satisfy

$$a_\eta \in C^k\big((\mathbb{R}^d)^\bullet, E\big), \qquad \eta \neq 0,$$

and

$$\|a\|^{\mathsf{M}}_{\mathcal{H}^k_z} := \max_{|\alpha|\le k} \|\partial^\alpha_\xi a\|_{\infty,[\mathsf{M}=1]} < \infty$$

for some positively 1-homogeneous $\mathsf{M} \in C\big(\dot{\mathsf{Z}}, (0,\infty)\big)$.

Clearly, $\|\cdot\|^{\mathsf{M}}_{\mathcal{H}^k_z}$ is a norm on $\mathcal{H}^k_z(\mathsf{Z}, E)$, and this space is independent of M in the sense that another choice of M leads to an equivalent norm. In fact, if N belongs to $C\big(\dot{\mathsf{Z}}, (0,\infty)\big)$ and is positively 1-homogeneous, then $\mathsf{N} \sim \mathsf{M}$ and, consequently,

$$\|\cdot\|^{\mathsf{N}}_{\mathcal{H}^k_z} \sim \|\cdot\|^{\mathsf{M}}_{\mathcal{H}^k_z}$$

by Lemma 1.2.1(iv).

It is easily verified that $\mathcal{H}^k_z(\mathsf{Z}, E)$ is a Banach space (with any one of its norms $\|\cdot\|^{\mathsf{M}}_{\mathcal{H}^k_z}$). Hence

$$\mathcal{H}^\infty_z(\mathsf{Z}, E) := \bigcap_{k\in\mathbb{N}} \mathcal{H}^k_z(\mathsf{Z}, E)$$

is a Fréchet space.

1.3 Quasi-norms

Of particular importance are positively 1-homogeneous scalar functions a such that

$$a_\eta \in C^\infty(\mathbb{R}^d), \qquad \eta \in \dot{\mathsf{H}}, \tag{1.3.1}$$

being positive, and satisfying a triangle inequality. We call them **quasi-norms**.[3] More precisely, we denote by

$$\mathfrak{Q} := \mathfrak{Q}(\mathsf{Z})$$

[3]Our terminology is different from the one used by other authors, for example by H. Triebel [66], and should not be confused with their concepts. In particular, a quasi-norm (in our sense) is not isotropically homogeneous, in general.

the set of all $\mathsf{Q} \in \mathcal{H}_1^\infty(\mathsf{Z})$ satisfying (1.3.1) and

$$0 < \mathsf{Q}(\zeta + \widetilde{\zeta}) \le \mathsf{Q}(\zeta) + \mathsf{Q}(\widetilde{\zeta}), \qquad \zeta, \widetilde{\zeta} \in \dot{\mathsf{Z}}. \tag{1.3.2}$$

Each $\mathsf{Q} \in \mathfrak{Q}$ is continuously extended over Z by setting $\mathsf{Q}(0) := 0$.

1.3.1 Remarks and examples (a) Suppose $\mathsf{Q} \in \mathfrak{Q}$ and set

$$d_{\mathsf{Q}}(\xi, \widetilde{\xi}) := \mathsf{Q}_0(\xi - \widetilde{\xi}), \qquad \xi, \widetilde{\xi} \in \mathbb{R}^d.$$

Then d_{Q} is a translation-invariant metric on $\mathbb{R}^d$. It follows from Lemma 1.2.1(iv) that $d_{\mathsf{Q}} \sim d_{\mathsf{M}}$ for $\mathsf{M} \in \mathfrak{Q}$.

(b) Put

$$\mathsf{N}(\zeta) := \Big(\sum_{j=1}^d |\xi^j|^{2\omega/\omega_j} + |\eta|^{2\omega} \Big)^{1/2\omega}, \qquad \zeta \in \mathsf{Z}.$$

Then $\mathsf{N} \in \mathfrak{Q}$ and it is called **natural $\boldsymbol{\omega}$-quasi-norm**. Note that d_{N} is the Euclidean distance on $\mathbb{R}^d$ iff $\omega = 1$.

Proof. It is obvious that N is smooth, positive, and positively 1-homogeneous and (1.3.2) holds. For $1 \le j \le d$,

$$\partial_j \mathsf{N}(\zeta) = (1/\omega_j)\, |\xi^j|^{2(\omega/\omega_j - 1)}\, \xi^j / \mathsf{N}(\zeta)^{2\omega - 1}.$$

From this and $\omega/\omega_j \in \dot{\mathbb{N}}$ we infer by induction

$$|\partial_\xi^\alpha \mathsf{N}(\zeta)| \le c(k, \boldsymbol{\omega}), \qquad |\alpha| \le k, \quad \zeta \in [\mathsf{N} = 1],$$

where $k \in \dot{\mathbb{N}}$ (cf. Lemma 1.4.2 below). Hence $\|\mathsf{N}\|_{\mathcal{H}_1^k}^{\mathsf{N}} \le c(k, \boldsymbol{\omega})$. □

(c) Suppose $\boldsymbol{\omega}$ is **clustered** in the following sense: there are positive integers $\ell,\ d_1, \dots, d_\ell,\ \nu_1, \dots, \nu_\ell$ satisfying

$$\boldsymbol{\omega} = (\underbrace{\nu_1, \dots, \nu_1}_{d_1}, \underbrace{\nu_2, \dots, \nu_2}_{d_2}, \dots, \underbrace{\nu_\ell, \dots, \nu_\ell}_{d_\ell}).$$

Set

$$\boldsymbol{d} := (d_1, \dots, d_\ell), \quad \boldsymbol{\nu} := (\nu_1, \dots, \nu_\ell).$$

Then $(\boldsymbol{d}, \boldsymbol{\nu})$ is a **reduced weight system associated with $\boldsymbol{\omega}$**, and

$$\mathbb{R}^d = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_\ell}$$

is the corresponding **$\boldsymbol{d}$-splitting** of $\mathbb{R}^d$. We then denote the general point of $\mathbb{R}^d$ by $x = (x_1, \dots, x_\ell)$ with $x_i \in \mathbb{R}^{d_i}$ and write $\xi = (\xi_1, \dots, \xi_\ell)$ for the dual variable.[4] We also set

$$\nu := \omega$$

so that ν is the least common multiple of $\nu_1, \dots, \nu_\ell$. Finally, Λ, defined by

$$\Lambda(\zeta) := \Big(\sum_{i=1}^\ell |\xi_i|^{2\nu/\nu_i} + |\eta|^{2\nu} \Big)^{1/2\nu}, \qquad \zeta \in \mathsf{Z},$$

belongs to $\mathfrak{Q}$ and is called **$(\boldsymbol{d}, \boldsymbol{\nu})$-quasi-norm**.

[4] The reader is advised to observe carefully the distinction between ξ^i, the i-th component of $\xi \in \mathbb{R}^d$, and the d_i-tuple ξ_i. Here and in similar situations we do not notationally distinguish between a d_i-tuple and a vector in $\mathbb{R}^{d_i}$.

Note that it is not assumed that the ν_i are pair-wise disjoint. Thus $\boldsymbol{\omega}$ is always clustered by the **trivial clustering** (or **non-reduced** weight system)

$$\ell = d, \quad \boldsymbol{d} = (1, \ldots, 1), \quad \boldsymbol{\nu} = (\omega_1, \ldots, \omega_d).$$

In this case $\Lambda = \mathsf{N}$.

(d) Let $\boldsymbol{\omega}$ be the $2m$-**parabolic weight vector** for some $m \in \dot{\mathbb{N}}$. Then

$$\boldsymbol{d} := (d-1, 1), \quad \boldsymbol{\nu} := (1, 2m)$$

form the (canonical) **reduced** $2m$-**parabolic weight system**. We then write $x = (x', t)$ for the general point of the corresponding $\boldsymbol{d}$-splitting, and $\xi = (\xi', \tau)$ for the dual variable. Then

$$\Lambda(\zeta) = (|\xi'|^{4m} + |\tau|^2 + |\eta|^{4m})^{1/4m}, \qquad \zeta \in \mathsf{Z}.$$

If $m = 1$, then the metric d_Λ is equivalent to the usual **parabolic metric**

$$(\xi, \widetilde{\xi}) \mapsto \sqrt{|\xi' - \widetilde{\xi}'|^2 + |\tau - \widetilde{\tau}|} \tag{1.3.3}$$

on $\mathbb{R}^d$.

(e) Put $\mathsf{E}(0) := 0$ and let $\mathsf{E}(\zeta)$ be, for $\zeta \in \dot{\mathsf{Z}}$, the smallest $t > 0$ satisfying

$$|t^{-1} \bullet \zeta|^2 = |t^{-1} \bullet \xi|^2 + |t^{-1}\eta|^2 = 1.$$

Then E belongs to $\mathfrak{Q}$ and is called **Euclidean $\boldsymbol{\omega}$-quasi-norm**.

Proof. From $s \bullet t \bullet \zeta = (st) \bullet \zeta$ we see that E is positively 1-homogeneous. To prove (1.3.2) we follow J. Johnsen and W. Sickel [41]. We have to show that, given $\zeta, \zeta' \in \dot{\mathsf{Z}}$,

$$a(\zeta, \zeta') := \sum_{j=1}^{d} \frac{|\xi^j + (\xi')^j|^2}{\big(\mathsf{E}(\zeta) + \mathsf{E}(\zeta')\big)^{2\omega_j}} + \frac{|\eta + \eta'|^2}{\big(\mathsf{E}(\zeta) + \mathsf{E}(\zeta')\big)^2} \le 1.$$

Note that $a(t \bullet \zeta, t \bullet \zeta') = a(\zeta, \zeta')$ for $t > 0$. Hence we can assume $\mathsf{E}(\zeta) + \mathsf{E}(\zeta') = 1$. Thus $\mathsf{E}(\zeta) \le 1$ implies

$$1 = \sum_{j=1}^{d} \frac{|\xi^j|^2}{\mathsf{E}(\zeta)^{2\omega_j}} + \frac{|\eta|^2}{\mathsf{E}(\zeta)^2} \ge \frac{|\zeta|^2}{\mathsf{E}(\zeta)^2}$$

and, consequently,

$$|\zeta + \zeta'| \le |\zeta| + |\zeta'| \le \mathsf{E}(\zeta) + \mathsf{E}(\zeta') = 1,$$

that is, $a(\zeta, \zeta') \le 1$.

By differentiating the identity

$$1 = \big|\big(1/\mathsf{E}(\zeta)\big) \bullet \zeta\big|^2$$

and solving for $\partial_j \mathsf{E}(\zeta)$ we obtain

$$\partial_j \mathsf{E}(\zeta) = \xi^j \mathsf{E}(\zeta)^{1-2\omega_j} \Big(\sum_{k=1}^{d} \omega_k \big(\xi^k\big)^2 / \mathsf{E}(\zeta)^{2\omega_k} + |\eta|^2 / \mathsf{E}(\zeta)^2\Big)^{-1}$$

for $1 \le j \le d$. Hence

$$|\partial_j \mathsf{E}(\zeta)| \le |\xi^j| \le 1 \text{ for } \mathsf{E}(\zeta) = 1.$$

From these formulas we see by induction that $\mathsf{E}_\eta \in C^\infty(\mathbb{R}^d)$ for $\eta \in \dot{\mathsf{H}}$ and

$$\|\partial_\xi^\alpha \mathsf{E}\|_{\infty,[\mathsf{E}=1]} \le c(k), \qquad |\alpha| \le k, \quad k \in \mathbb{N}.$$

This proves the claim. □

1.4 Products and inverses

The present section is essentially an exercise on calculus. Its purpose is to establish simple sufficient criteria for some functions to belong to $\mathcal{H}_z^k$ for suitable $k \in \mathbb{N}$ and $z \in \mathbb{C}$. For later applications it is important to control the dependence of norm estimates on all involved quantities.

Throughout the remainder of this part • $\boldsymbol{\omega}$ *is clustered;* • $(\boldsymbol{d}, \boldsymbol{\nu})$ *is a reduced weight system for it.*

In accordance with the $\boldsymbol{d}$-splitting of $\mathbb{R}^d$ we write

$$\alpha = (\alpha_1, \dots, \alpha_\ell) \in \mathbb{N}^{d_1} \times \dots \times \mathbb{N}^{d_\ell} = \mathbb{N}^d$$

and note $\alpha \cdot \boldsymbol{\omega} = |\alpha_1|\, \nu_1 + \dots + |\alpha_\ell|\, \nu_\ell$. We also put

$$\|\cdot\|_{\mathcal{H}_z^k} := \|\cdot\|_{\mathcal{H}_z^k}^\Lambda, \quad \zeta^* := \zeta_\Lambda^*.$$

In the following, we denote by

$$\bar{\mathcal{H}}_z^\infty(\mathsf{Z}, E)$$

the closed linear subspace of $\mathcal{H}_z^\infty(\mathsf{Z}, E)$ consisting of all a therein such that a_η belongs to $C^\infty(\mathbb{R}^d, E)$ for $\eta \ne 0$. Note that $\mathfrak{Q}(\mathsf{Z}) \subset \bar{\mathcal{H}}_z^\infty(\mathsf{Z})$.

1.4.1 Lemma

(i) *Let $E_1 \times E_2 \to E$ be a multiplication of Banach spaces.*[5] *Then its point-wise extension satisfies*

$$\mathcal{H}_{z_1}^k(\mathsf{Z}, E_1) \times \mathcal{H}_{z_2}^k(\mathsf{Z}, E_2) \to \mathcal{H}_{z_1+z_2}^k(\mathsf{Z}, E)$$

for $k \in \bar{\mathbb{N}}$ and $z_1, z_2 \in \mathbb{C}$. It is a multiplication as well.

(ii) *For $\eta \in \dot{\mathsf{H}}$,*

$$(a \mapsto a_\eta) \in \mathcal{L}\big(\bar{\mathcal{H}}_z^\infty(\mathsf{Z}, E), \mathcal{O}_M(\mathbb{R}^d, E)\big).$$

Proof. (i) Since

$$\sigma_t(ab) = (\sigma_t a)\sigma_t b, \qquad (a, b) \in \mathcal{H}_{z_1}^k(\mathsf{Z}, E_1) \times \mathcal{H}_{z_2}^k(\mathsf{Z}, E_2)$$

the statement follows from Leibniz' rule.

(ii) By Lemma 1.2.1(ii)

$$|\partial^\alpha a_\eta(\xi)| \le \Lambda_\eta^{\operatorname{Re} z - \alpha\cdot\boldsymbol{\omega}}(\xi)\, \|a\|_{\mathcal{H}_z^k}, \qquad \xi \in \mathbb{R}^d, \quad \eta \in \dot{\mathsf{H}},$$

for $|\alpha| \le k$ and $k \in \mathbb{N}$. Now the claim is obvious. □

[5] A **multiplication** $\mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}$ of LCSs is a continuous bilinear map.

Now we consider invertible elements of $\mathcal{H}_z^k$. For this we first establish a semi-explicit formula for derivatives of inverses

1.4.2 Lemma *Suppose $m \in \dot{\mathbb{N}}$, X is open in $\mathbb{R}^d$, and $a \in C^m\big(X, \mathcal{L}\mathrm{aut}(E)\big)$. Set*

$$a^{-1}(x) := a(x)^{-1}, \qquad x \in X.$$

Then $a^{-1} \in C^m\big(X, \mathcal{L}\mathrm{aut}(E)\big)$ and, given $\alpha \in \mathbb{N}^d$ with $|\alpha| = m$,

$$\partial^\alpha a^{-1} = \sum_{j=1}^m \sum_{\beta \in \mathsf{B}_{j,\alpha}} \varepsilon_\beta a^{-1}(\partial^{\beta_1} a) a^{-1} \cdots (\partial^{\beta_j} a) a^{-1}, \tag{1.4.1}$$

where

$$\mathsf{B}_{j,\alpha} := \big\{\, \beta := \{\beta_1, \dots, \beta_j\} \ ; \ \beta_i \in \mathbb{N}^d,\ |\beta_i| > 0,\ \textstyle\sum_{i=1}^j \beta_i = \alpha \,\big\}, \qquad \varepsilon_\beta \in \mathbb{Z}.$$

Proof. Recall that $\mathcal{L}\mathrm{aut}(E)$ is open in $\mathcal{L}(E)$ and

$$\mathrm{inv} : \ \mathcal{L}\mathrm{aut}(E) \to \mathcal{L}\mathrm{aut}(E), \quad b \mapsto b^{-1}$$

is smooth with

$$\big(\partial\, \mathrm{inv}(b)\big)c = -b^{-1} c b^{-1}, \qquad b \in \mathcal{L}\mathrm{aut}(E), \quad c \in \mathcal{L}(E). \tag{1.4.2}$$

(For this and further results on calculus in Banach spaces used below we refer to H. Amann and J. Escher [11, Chapter VII].) Hence $a^{-1} \in C^m\big(X, \mathcal{L}\mathrm{aut}(E)\big)$.

From (1.4.2) and the chain rule we infer

$$\partial_j a^{-1} = -a^{-1}(\partial_j a) a^{-1}, \quad 1 \le j \le d.$$

Now the claim follows by induction. □

1.4.3 Lemma *Suppose $a \in \mathcal{H}_z^\infty\big(\mathsf{Z}, \mathcal{L}(E)\big)$ with $a(\zeta^*) \in \mathcal{L}\mathrm{aut}(E)$ for $\zeta \in \dot{\mathsf{Z}}$. Then*

$$a^{-1} \in \mathcal{H}_{-z}^\infty\big(\mathsf{Z}, \mathcal{L}(E)\big), \quad \|a^{-1}\|_{\mathcal{H}_{-z}^k} \le c\big(\|a\|_{\mathcal{H}_z^k}, \|a^{-1}\|_{\infty,[\Lambda=1]}, k\big)$$

for $k \in \mathbb{N}$.

Proof. (1) Setting $\alpha = 0$ and $\mathsf{M} = \Lambda$ in Lemma 1.2.1(ii) we find that $a(\zeta)$ belongs to $\mathcal{L}\mathrm{aut}(E)$ for $\zeta \in \dot{\mathsf{Z}}$. Hence $a(\zeta)a^{-1}(\zeta) = 1_E$ implies

$$1_E = \sigma_t(aa^{-1}) = (\sigma_t a)\sigma_t a^{-1} = t^z a \sigma_t a^{-1}$$

and, similarly, $1_E = \sigma_t(a^{-1}a) = t^z \sigma_t(a^{-1})a$. Thus $t^z \sigma_t a^{-1} = a^{-1}$, that is, a^{-1} is positively $(-z)$-homogeneous.

(2) Suppose $\alpha \in \mathbb{N}^d$ satisfies $0 < |\alpha| =: m \le k$. Let $j \in \{1, \dots, m\}$ and let $\{\beta_1, \dots, \beta_j\}$ belong to $\mathsf{B}_{j,\alpha}$. Then we infer from Lemma 1.2.1(ii) and (1)

$$(\partial_\xi^{\beta_i} a) a^{-1}(\zeta) = \Lambda^{-\beta_i \cdot \boldsymbol{\omega}}(\zeta) \partial_\xi^{\beta_i} a(\zeta^*) a^{-1}(\zeta^*). \tag{1.4.3}$$

Consequently, due to $\beta_1 + \cdots + \beta_j = \alpha$,

$$|a^{-1}(\partial^{\beta_1} a) a^{-1} \cdots (\partial^{\beta_j} a) a^{-1}(\zeta)| \le c \Lambda^{-\mathrm{Re}\, z - \alpha \cdot \boldsymbol{\omega}}(\zeta), \qquad \zeta \in \dot{\mathsf{Z}}, \tag{1.4.4}$$

where

$$c = c(\|a\|_{\mathcal{H}_z^k}, \|a^{-1}\|_{\infty,[\Lambda=1]}).$$

Now the claim follows from Lemma 1.4.2. □

1.5 Resolvent estimates for symbols

In this section we assume

- *F is a finite-dimensional Banach space.*

We denote for $s \in \mathbb{R}$ by

$$\mathfrak{P}_s\big(\mathsf{Z}, \mathcal{L}(F)\big)$$

the set of all $a \in \mathcal{H}_s^\infty\big(\mathsf{Z}, \mathcal{L}(F)\big)$ which are 'positive' in the sense that

$$\sigma\big(a(\zeta^*)\big) \subset [\operatorname{Re} z > 0], \qquad \zeta \in \dot{\mathsf{Z}}.$$

Note

$$\mathfrak{Q}(\mathsf{Z}) \subset \mathfrak{P}_1(\mathsf{Z}). \tag{1.5.1}$$

Since $[\Lambda = 1]$ is compact, a is continuous on $\dot{\mathsf{Z}}$, and the spectrum is upper semi continuous, there exists $\kappa \geq 1$ such that

$$\sigma\big(a(\zeta^*)\big) \subset [\operatorname{Re} z \geq 1/\kappa], \quad |a(\zeta^*)| \leq \kappa, \qquad \zeta \in \dot{\mathsf{Z}}. \tag{1.5.2}$$

Given any $\kappa \geq 1$, we write

$$\mathfrak{P}_s(\kappa) := \mathfrak{P}_s\big(\mathsf{Z}, \mathcal{L}(F); \kappa\big)$$

for the set of all $a \in \mathfrak{P}_s\big(\mathsf{Z}, \mathcal{L}(F)\big)$ satisfying (1.5.2). Hence

$$\mathfrak{P}_s\big(\mathsf{Z}, \mathcal{L}(F)\big) = \bigcup_{\kappa \geq 1} \mathfrak{P}_s\big(\mathsf{Z}, \mathcal{L}(F); \kappa\big).$$

Set

$$\varphi(\kappa) := \arccos(1/\kappa^2). \tag{1.5.3}$$

From $\sigma\big(a(\zeta^*)\big) \subset \big[|z| \leq |a(\zeta^*)|\,\big]$ it follows

$$\sigma\big(a(\zeta^*)\big) \subset [\operatorname{Re} z \geq 1/\kappa] \cap [\,|z| \leq \kappa] \subset \mathsf{S}_{\varphi(\kappa)}, \qquad \zeta \in \dot{\mathsf{Z}}, \tag{1.5.4}$$

for $a \in \mathfrak{P}_s(\kappa)$, where[6]

$$\mathsf{S}_\varphi := [\,|\arg z| \leq \varphi] \cup \{0\}, \qquad 0 \leq \varphi < \pi.$$

Hence $a(\zeta) = \Lambda^s(\zeta) a(\zeta^*)$ implies

$$\sigma\big(a(\zeta)\big) \subset \mathsf{S}_{\varphi(\kappa)}, \qquad \zeta \in \dot{\mathsf{Z}}, \quad a \in \mathfrak{P}_s(\kappa). \tag{1.5.5}$$

The next lemma establishes a resolvent estimate which is uniform with respect to $a \in \mathfrak{P}_s(\kappa)$.

1.5.1 Lemma *Suppose $\kappa \geq 1$. Then*

$$\big|\big(\lambda + a(\zeta)\big)^{-1}\big| \leq \frac{c(\kappa)}{\Lambda^s(\zeta) + |\lambda|}, \qquad \zeta \in \dot{\mathsf{Z}}, \quad \lambda \in \mathsf{S}_{\pi - \varphi(2\kappa)},$$

for all $s \in \mathbb{R}$ and $a \in \mathfrak{P}_s\big(\mathsf{Z}, \mathcal{L}(F); \kappa\big)$.

[6] We denote by $\arg z \in (-\pi, \pi]$ the principle value of $z \in \dot{\mathbb{C}}$.

PROOF. By introducing a basis, we can suppose that $F = \mathbb{C}^N$ so that $\mathcal{L}(F)$ is identified with $\mathbb{C}^{N\times N}$.

(1) Assume $b \in \mathbb{C}^{N\times N}$ is invertible. Let $b^\natural$ be the algebraic adjoint of b, defined by

$$b^\natural = \det(b)b^{-1} \tag{1.5.6}$$

and given explicitly by

$$b^\natural = \left[(-1)^{j+k}B_k^j\right]^\top \in \mathbb{C}^{N\times N}, \tag{1.5.7}$$

where B_k^j is the determinant of the matrix obtained by deleting the j^{th} row and the k^{th} column of b.

Assume $\rho \geq 1$ and $\sigma(b) \subset [\,|z| \geq 1/\rho]$. Then $|\det b| \geq \rho^{-N}$ and (1.5.6), (1.5.7) imply

$$|b^{-1}| \leq c(N)\rho^N\,|b|^{N-1}.$$

(2) From (1.5.4) we infer

$$\operatorname{dist}\Big(\sigma\big(a(\zeta^*)\big), \big[|\arg(z)| \geq \varphi(2\kappa)\big] \cup \{0\}\Big) \geq \kappa\sin\big(\varphi(2\kappa) - \varphi(\kappa)\big) =: \frac{1}{\rho(\kappa)}.$$

Since $\big[|\arg(z)| \geq \varphi(2\kappa)\big] \cup \{0\} = -\mathsf{S}_{\pi-\varphi(2\kappa)}$ it follows

$$\sigma\big(\lambda + a(\zeta^*)\big) \subset \big[|z| \geq 1/\rho(\kappa)\big], \qquad \lambda \in \mathsf{S}_{\pi-\varphi(2\kappa)}, \quad \zeta \in \dot{\mathsf{Z}}.$$

If $|\lambda| \geq 2\kappa \geq 2\,|a(\zeta^*)|$, then

$$\left|\big(\lambda + a(\zeta^*)\big)^{-1}\right| \leq |\lambda|^{-1}\left|\big(1 + a(\zeta^*)/\lambda\big)^{-1}\right| \leq \frac{|\lambda|^{-1}}{1 - |\lambda|^{-1}\,|a(\zeta^*)|} \leq \frac{2}{|\lambda|}.$$

From this and step (1) we deduce

$$\left|\big(\lambda + a(\zeta^*)\big)^{-1}\right| \leq \frac{c(\kappa)}{1+|\lambda|}, \qquad \lambda \in \mathsf{S}_{\pi-\varphi(2\kappa)}, \quad \zeta \in \dot{\mathsf{Z}},$$

for $a \in \mathfrak{P}_s(\kappa)$. Now the assertion follows from

$$\left|\big(\lambda + a(\zeta)\big)^{-1}\right| = \Lambda^{-s}(\zeta)\left|\big(\Lambda^{-s}(\zeta)\lambda + a(\zeta^*)\big)^{-1}\right| \leq \frac{c}{\Lambda^s(\zeta) + |\lambda|}$$

since $\Lambda^{-s}(\zeta)\mathsf{S}_{\pi-\varphi(2\kappa)} \subset \mathsf{S}_{\pi-\varphi(2\kappa)}$. □

Now we extend this resolvent estimate by including derivatives.

1.5.2 Proposition *Suppose $\kappa \geq 1$. Then*

$$\max_{|\alpha|\leq k} |\Lambda_\eta^{\alpha\cdot\boldsymbol{\omega}}(\xi)\partial^\alpha(\lambda + a_\eta)^{-1}(\xi)| \leq \frac{c(\kappa, \|a\|_{\mathcal{H}_s^k}, k)}{\Lambda^s(\zeta) + |\lambda|}$$

for $\xi \in \mathbb{R}^d$, $\eta \in \dot{\mathsf{H}}$, $\lambda \in \mathsf{S}_{\pi-\varphi(2\kappa)}$, $k \in \mathbb{N}$, $s \in \mathbb{R}$, and $a \in \mathfrak{P}_s\big(\mathsf{Z}, \mathcal{L}(F);\kappa\big)$. If $s \geq 0$, then

$$\max_{|\alpha|\leq k} \|\Lambda_\eta^{\alpha\cdot\boldsymbol{\omega}}\partial^\alpha(\lambda + a_\eta)^{-1}\|_\infty \leq \frac{c(\kappa, \|a\|_{\mathcal{H}_s^k}, k)}{|\eta|^s + |\lambda|}, \qquad \eta \in \dot{\mathsf{H}}, \quad \lambda \in \mathsf{S}_{\pi-\varphi(2\kappa)},$$

for $k \in \mathbb{N}$ and $a \in \mathfrak{P}_s\big(\mathsf{Z}, \mathcal{L}(F);\kappa\big)$.

PROOF. Let $\beta \in \mathbb{N}^d$ satisfy $0 < |\beta| \le k$. Then we deduce from Lemma 1.5.1 and (1.2.4)

$$\begin{aligned}|\partial_\xi^\beta(\lambda + a)(\lambda + a)^{-1}| &= |(\partial_\xi^\beta a)(\lambda + a)^{-1}| \\ &\le \frac{c(\kappa)\Lambda^{s-\beta\cdot\boldsymbol{\omega}}\,\|a\|_{\mathcal{H}_s^k}}{\Lambda^s + |\lambda|} \le c(\kappa)\Lambda^{-\beta\cdot\boldsymbol{\omega}}\,\|a\|_{\mathcal{H}_s^k}\end{aligned}$$

for $\lambda \in \mathsf{S}_{\pi-\varphi(2\kappa)}$ and $a \in \mathfrak{P}_s(\kappa)$. Now the assertion follows from Lemma 1.4.2 (cf. step (2) of the proof of Lemma 1.4.3), the last estimate being implied by $\Lambda^s(\zeta) \ge |\eta|^s$ for $s \ge 0$. □

Suppose $a \in \mathfrak{P}_s\big(\mathsf{Z}, \mathcal{L}(F)\big)$ for some $s \in \mathbb{R}$. Denote by $\psi(a)$ the **spectral angle** of a, that is, the minimum of all $\psi \in [0, \pi/2)$ such that $\sigma\big(a(\zeta)\big) \subset \mathsf{S}_\psi$ for all $\zeta \in \dot{\mathsf{Z}}$. It follows from (1.5.2) and (1.5.5) that $\psi(a)$ is well-defined. Fix any $\psi \in \big(\psi(a), \pi/2\big)$ and suppose $h : \mathring{\mathsf{S}}_\psi \to \mathbb{C}$ is holomorphic. Let Γ be any positively oriented contour contained in $\mathring{\mathsf{S}}_\psi$ and containing $\sigma\big(a(\zeta^*)\big)$ for all $\zeta \in \dot{\mathsf{Z}}$ in its interior. It follows also from (1.5.2) (see (1.5.4)) that such a Γ exists. Denote by $\tau\Gamma$ the image of Γ under the dilation $(z \to \tau z) : \mathbb{C} \to \mathbb{C}$ for $\tau \in \dot{\mathbb{C}}$. Then $\Lambda^s(\zeta)\Gamma$ contains $\sigma\big(a(\zeta)\big)$ in its interior and is contained in $\mathring{\mathsf{S}}_\psi$. Hence

$$h\big(a(\zeta)\big) := \frac{1}{2\pi i}\int_{\Lambda^s(\zeta)\Gamma} h(\lambda)\big(\lambda - a(\zeta)\big)^{-1}\,d\lambda$$

is well-defined for $\zeta \in \dot{\mathsf{Z}}$. It follows from Cauchy's theorem that $h\big(a(\zeta)\big)$ is independent of the particular contour Γ and angle ψ. In fact, the well-known Dunford calculus (cf. [22, Section VII.1]) shows that $h\big(a(\zeta)\big)$ depends only on the values of h on $\sigma\big(a(\zeta)\big)$.

By means of Proposition 1.5.2 we now establish estimates for derivatives of $h(a)$. They are of importance in connection with Fourier multipliers as will be apparent in the following chapters.

1.5.3 Lemma *Suppose $s \in \mathbb{R}$ and $\kappa \ge 1$. Set*

$$\Omega_{2\kappa} := [\operatorname{Re} z \ge 1/2\kappa] \cap [\,|z| \le 2\kappa] \subset \mathsf{S}_{\varphi(2\kappa)}.$$

Then

$$\max_{|\alpha|\le k}\big|\Lambda^{\alpha\cdot\boldsymbol{\omega}}(\zeta)\partial_\xi^\alpha h\big(a(\zeta)\big)\big| \le c(\kappa, \|a\|_{\mathcal{H}_s^k}, k)\,\|h\|_{\infty,\Lambda^s(\zeta)\Omega_{2\kappa}}, \qquad \zeta \in \dot{\mathsf{Z}}, \quad k \in \mathbb{N},$$

for all $a \in \mathfrak{P}_s\big(\mathsf{Z}, \mathcal{L}(F); \kappa\big)$ and each holomorphic function h on $\mathring{\mathsf{S}}_{\varphi(3\kappa)}$.

PROOF. Let $h : \mathring{\mathsf{S}}_{\varphi(3\kappa)} \to \mathbb{C}$ be holomorphic. Denote by Γ the positively oriented boundary of $\Omega_{2\kappa}$. Then

$$2\pi i\, h\big(a(\zeta)\big) = \int_{\Lambda^s(\zeta)\Gamma} h(\lambda)\big(\lambda - a(\zeta)\big)^{-1}\,d\lambda = \int_{-\Lambda^s(\zeta)\Gamma} g(\mu)\big(\mu + a(\zeta)\big)^{-1}\,d\mu$$

for $\zeta \in \dot{\mathsf{Z}}$, where

$$g : \mathbb{C}\backslash\mathsf{S}_{\pi-\varphi(3\kappa)} \to \mathbb{C}, \quad \mu \mapsto h(-\mu)$$

is holomorphic, $-\Lambda^s(\zeta)\Gamma \subset \mathbb{C}\backslash\mathsf{S}_{\pi-\varphi(3\kappa)}$ for $\zeta \in \dot{\mathsf{Z}}$, and $\|g\|_{\infty,-\Lambda^s(\zeta)\Gamma} = \|h\|_{\infty,\Lambda^s(\zeta)\Gamma}$.

Fix $\zeta_0 \in \dot{\mathsf{Z}}$. There exists a neighborhood U of ζ_0 in $\dot{\mathsf{Z}}$ such that $\sigma\bigl(-a(U)\bigr)$ is contained in the interior of $-\Lambda^s(\zeta_0)\Gamma$. Thus, by Cauchy's theorem,

$$h(a)(\zeta) := h\bigl(a(\zeta)\bigr) = \frac{1}{2\pi i}\int_{-\Lambda^s(\zeta_0)\Gamma} g(\mu)\bigl(\mu + a(\zeta)\bigr)^{-1}\,d\mu, \qquad \zeta \in U.$$

Consequently,

$$\partial_\xi^\alpha h(a)(\zeta) = \frac{1}{2\pi i}\int_{-\Lambda^s(\zeta_0)\Gamma} g(\mu)\partial_\xi^\alpha\bigl(\mu + a(\zeta)\bigr)^{-1}\,d\mu, \qquad \zeta \in U,$$

for $\alpha \in \mathbb{N}^d$. Since this holds, in particular, for $\zeta = \zeta_0$ and ζ_0 is arbitrary in $\dot{\mathsf{Z}}$ we find

$$\partial_\xi^\alpha h(a)(\zeta) = \frac{1}{2\pi i}\int_{-\Lambda^s(\zeta)\Gamma} g(\mu)\partial_\xi^\alpha\bigl(\mu + a(\zeta)\bigr)^{-1}\,d\mu, \qquad \zeta \in \dot{\mathsf{Z}}.$$

Now Proposition 1.5.2 implies

$$\begin{aligned}|\Lambda^{\alpha\cdot\boldsymbol{\omega}}(\zeta)\partial_\xi^\alpha h(a)(\zeta)| &\le c\int_{-\Lambda^s(\zeta)\Gamma} |g(\mu)|\,\frac{|d\mu|}{\Lambda^s(\zeta)+|\mu|}\\ &\le c\int_{-\Gamma}\bigl|g\bigl(\Lambda^s(\zeta)z\bigr)\bigr|\,|dz|/|z| \le c\sup_{z\in\Gamma}\bigl|h\bigl(\Lambda^s(\zeta)z\bigr)\bigr|\end{aligned}$$

for $|\alpha| \le k$ and $\zeta \in \dot{\mathsf{Z}}$, where c depends on κ, $\|a\|_{\mathcal{H}_s^k}$, and k only. □

Now we specialize to particularly important cases, namely power functions and exponentials. Recall that $\log z = \log|z| + i\arg z$ and

$$h_z(\lambda) := \lambda^z = e^{z\log\lambda}, \qquad \lambda \in \dot{\mathbb{C}},$$

is the principal value of the logarithm and the power function, respectively. Since h_z is holomorphic,

$$a^z := h_z(a) : \dot{\mathsf{Z}} \to \mathcal{L}(F)$$

is well-defined for $a \in \mathfrak{P}_s\bigl(\mathsf{Z}, \mathcal{L}(F)\bigr)$. Furthermore,

$$a^0 = 1_F, \quad a^1 = a, \quad a^{z_1+z_2} = a^{z_1}a^{z_2}, \qquad z_1, z_2 \in \mathbb{C},$$

by the Dunford calculus. Note $(a_\eta)^z = (a^z)_\eta =: a_\eta^z$ for $\eta \in \dot{\mathsf{H}}$.

1.5.4 Proposition *Suppose $s \ge 0$ and $\kappa \ge 1$. Then*

$$\max_{|\alpha|\le k}\|\Lambda_\eta^{\alpha\cdot\boldsymbol{\omega}}\partial^\alpha a_\eta^z\|_\infty \le c(\kappa, \|a\|_{\mathcal{H}_s^k}, k)\,(|\eta|^s/2\kappa)^{\operatorname{Re} z}\,e^{|\operatorname{Im} z|\,\varphi(2\kappa)}$$

for $a \in \mathfrak{P}_s\bigl(\mathsf{Z}, \mathcal{L}(F), \kappa\bigr)$, $\eta \in \dot{\mathsf{H}}$, and $\operatorname{Re} z \le 0$.

Proof. Since

$$|h_z(\lambda)| \le |\lambda|^{\operatorname{Re} z}\,e^{|\operatorname{Im} z|\,\varphi(2\kappa)}, \qquad \lambda \in \dot{\mathsf{S}}_{\varphi(2\kappa)},$$

the assertion follows from Lemma 1.5.3. □

1.5.5 Remark Suppose $a \in \mathfrak{P}_s\bigl(\mathsf{Z}, \mathcal{L}(F); \kappa\bigr)$ for some $s \ge 0$ and $\kappa \ge 1$. Fix any $\psi \in \bigl(\psi(a), \pi/2\bigr)$. Then an easy modification of the proof of Lemmas 1.5.1 and 1.5.3 shows that the term $e^{|\operatorname{Im} z|\,\varphi(2\kappa)}$ in the estimate of Proposition 1.5.4 can be replaced by $e^{|\operatorname{Im} z|\,\psi}$. In this case c depends also on $\psi(a)$ and $1/\bigl(\psi - \psi(a)\bigr)$. □

Next we turn to exponentials. For $t \in \mathbb{R}$ we denote by $g_t : \mathbb{C} \to \mathbb{C}$ the entire function $\lambda \mapsto e^{-t\lambda}$. Then

$$e^{-ta} := g_t(a) : \dot{\mathsf{Z}} \to \mathcal{L}(F)$$

is well-defined for $a \in \mathfrak{P}_s\big(\mathsf{Z}, \mathcal{L}(F)\big)$. Furthermore, the Dunford-calculus shows that $\{\, e^{-ta(\zeta)} \ ; \ t \in \mathbb{R} \,\}$ is a continuous subgroup of $\mathcal{L}(F)$. It is well-known (eg., Section 12 in H. Amann [2]) that $t \mapsto e^{-ta(\zeta)}$ is the unique solution in $\mathcal{L}(F)$ of

$$\dot{u} + a(\zeta)u = 0 \text{ on } \mathbb{R}, \quad u(0) = 1_F,$$

for $\zeta \in \dot{\mathsf{Z}}$.

1.5.6 Proposition *Suppose $s \geq 0$ and $\kappa \geq 1$. Then*

$$\max_{|\alpha| \leq k} \|\Lambda_\eta^{\alpha \cdot \boldsymbol{\omega}} \partial^\alpha e^{-ta_\eta}\|_\infty \leq c(\kappa, \|a\|_{\mathcal{H}_s^k}, k)\, e^{-t\,|\eta|^s/2\kappa}$$

for $t \geq 0$, $\eta \in \dot{\mathsf{H}}$, $k \in \mathbb{N}$, and $a \in \mathfrak{P}_s\big(\mathsf{Z}, \mathcal{L}(F), \kappa\big)$.

Proof. Due to $|e^{-t\lambda}| = e^{-t\,\mathrm{Re}\,\lambda}$ this is immediate from Lemma 1.5.3. $\square$

1.6 Multiplier spaces

We denote by

$$\mathcal{M}(\mathbb{R}^d, E) := \mathcal{M}_{(\boldsymbol{d},\boldsymbol{\nu})}(\mathbb{R}^d, E)$$

the set of all $a \in C^{d+\ell}\big((\mathbb{R}^d)^\bullet, E\big)$ satisfying

$$\|a\|_{\mathcal{M}} := \max_{|\alpha| \leq d+\ell} \|\Lambda_1^{\alpha \cdot \boldsymbol{\omega}} \partial^\alpha a\|_\infty < \infty.$$

It is a Banach space with the norm $\|\cdot\|_{\mathcal{M}}$.

In later chapters we shall show that the elements of $\mathcal{M}(\mathbb{R}^d, E)$ are Fourier multipliers for various function spaces. This will explain the choice of $d + \ell$ for the order of smoothness. Moreover, the importance of the following simple technical lemma will then be obvious.

1.6.1 Lemma

(i) *Let $E_1 \times E_2 \to E$ be a multiplication of Banach spaces. Then its point-wise extension satisfies*

$$\mathcal{M}(\mathbb{R}^d, E_1) \times \mathcal{M}(\mathbb{R}^d, E_2) \to \mathcal{M}(\mathbb{R}^d, E)$$

and it is a multiplication.

(ii) *Assume $k \in \mathbb{N}$ and $a \in C(\dot{\mathsf{Z}}, E)$ satisfies $a_\eta \in C^k\big((\mathbb{R}^d)^\bullet, E\big)$ for $\eta \in \dot{\mathsf{H}}$. Then*

$$\|\Lambda_1^{\alpha \cdot \boldsymbol{\omega}} \partial^\alpha (\sigma_{|\eta|} a_\eta)\|_\infty = \|\Lambda_\eta^{\alpha \cdot \boldsymbol{\omega}} \partial^\alpha a_\eta\|_\infty$$

for $\eta \in \dot{\mathsf{H}}$ and $|\alpha| \leq k$.

(iii) *Suppose* $\mathrm{Re}\, z \geq 0$. *Then*

$$(a \mapsto \sigma_{|\eta|} a_\eta) \in \mathcal{L}\big(\mathcal{H}_{-z}^{d+\ell}(\mathsf{Z}, E), \mathcal{M}(\mathbb{R}^d, E)\big)$$

and

$$\|\sigma_{|\eta|} a_\eta\|_{\mathcal{M}} \leq |\eta|^{-\mathrm{Re}\, z}\, \|a\|_{\mathcal{H}_{-z}^{d+\ell}}, \qquad \eta \in \dot{\mathsf{H}}.$$

Proof. (i) follows from Leibniz' rule.

(ii) Note

$$\sigma_{1/|\eta|}\Lambda_1^s = |\eta|^{-s}\,\Lambda_\eta^s, \qquad s \in \mathbb{R}. \tag{1.6.1}$$

Thus, due to Proposition 1.1.1(i),

$$\begin{aligned}\Lambda_1^{\alpha\cdot\boldsymbol{\omega}}\partial^\alpha(\sigma_{|\eta|}a_\eta) &= |\eta|^{\alpha\cdot\boldsymbol{\omega}}\Lambda_1^{\alpha\cdot\boldsymbol{\omega}}\sigma_{|\eta|}(\partial^\alpha a_\eta)\\ &= |\eta|^{\alpha\cdot\boldsymbol{\omega}}\sigma_{|\eta|}\big((\sigma_{1/|\eta|}\Lambda_1^{\alpha\cdot\boldsymbol{\omega}})\partial^\alpha a_\eta\big)\\ &= \sigma_{|\eta|}(\Lambda_\eta^{\alpha\cdot\boldsymbol{\omega}}\partial^\alpha a_\eta).\end{aligned}$$

Now the assertion follows from Proposition 1.1.1(ii).

(iii) By (1.2.4),

$$|\partial^\alpha a_\eta(\xi)| \le \Lambda^{-\operatorname{Re} z-\alpha\cdot\boldsymbol{\omega}}(\zeta)\,\|a\|_{\mathcal{H}_{-z}^{d+\ell}} \le |\eta|^{-\operatorname{Re} z}\,\Lambda^{-\alpha\cdot\boldsymbol{\omega}}(\zeta)\,\|a\|_{\mathcal{H}_{-z}^{d+\ell}}$$

for $a \in \mathcal{H}_{-z}^{d+\ell}$ and $\eta \in \dot{\mathsf{H}}$. Thus (ii) implies the statement. □

CHAPTER 2

Anisotropic Banach scales

It turns out that many properties of anisotropic Banach spaces can be obtained by Fourier multiplier theorems, irrespective of the underlying concrete realization. For this reason we now introduce a class of $\mathcal{M}$-admissible Banach spaces by requiring that they are Banach spaces of tempered distributions on which Fourier multiplier operators with (scalar) symbols in $\mathcal{M}(\mathbb{R}^d)$ act continuously.

2.1 Admissible Banach spaces

Let $E_1 \times E_2 \to E_0$ be a multiplication of Banach spaces. For $m \in \mathcal{S}'(\mathbb{R}^d, E_1)$ we put

$$\mathrm{dom}\big(m(D)\big) := \big\{\, u \in \mathcal{S}'(\mathbb{R}^d, E_2) \ ; \ m\widehat{u} \in \mathcal{S}'(\mathbb{R}^d, E_0) \,\big\}$$

and

$$m(D)u := \mathcal{F}^{-1} m \mathcal{F} u = \mathcal{F}^{-1}(m\widehat{u}),$$

denoting by $\mathcal{F}$ the Fourier transform.[1] Then $m(D)$ is a linear map from its domain in $\mathcal{S}'(\mathbb{R}^d, E_2)$ into $\mathcal{S}'(\mathbb{R}^d, E_0)$, a **Fourier multiplier operator** with **symbol** m. Note that

$$\mathcal{S}(\mathbb{R}^d, E_2) \subset \mathcal{O}_M(\mathbb{R}^d, E_2) \subset \mathrm{dom}\big(m(D)\big).$$

Furthermore, $m \in \mathcal{O}_M(\mathbb{R}^d, E_1)$ implies

$$m(D) \in \mathcal{L}\big(\mathcal{S}(\mathbb{R}^d, E_2), \mathcal{S}(\mathbb{R}^d, E_0)\big) \cap \mathcal{L}\big(\mathcal{S}'(\mathbb{R}^d, E_2), \mathcal{S}'(\mathbb{R}^d, E_0)\big). \tag{2.1.1}$$

Now we fix a Banach space E and set

$$\mathcal{S} = \mathcal{S}_E := \mathcal{S}(\mathbb{R}^d, E), \quad \mathcal{S}' = \mathcal{S}'_E := \mathcal{S}'(\mathbb{R}^d, E).$$

We write $\mathfrak{F}$, or $\mathfrak{F}_E$, for $\mathfrak{F}(\mathbb{R}^d, E)$ if the latter is a Banach space of tempered E-valued distributions on $\mathbb{R}^d$ containing $\mathcal{S}$.

We say $\mathfrak{F}$ is an ($\mathcal{M}$-)**admissible Banach space** (of distributions), provided

$$\left.\begin{aligned}
&\text{(i)} && \mathcal{S} \overset{d}{\hookrightarrow} \mathfrak{F} \overset{d}{\hookrightarrow} \mathcal{S}';\\
&\text{(ii)} && m(D)u \in \mathfrak{F} \text{ and}\\
& && \qquad \|m(D)u\|_{\mathfrak{F}} \le c\,\|m\|_{\mathcal{M}}\,\|u\|_{\mathfrak{F}}\\
& && \text{for } (m,u) \in \mathcal{M}(\mathbb{R}^d) \times \mathcal{S}.
\end{aligned}\right\} \tag{2.1.2}$$

Since $\mathcal{S}$ is dense in $\mathcal{S}'$, condition (i) is equivalent to $\mathcal{S} \overset{d}{\hookrightarrow} \mathfrak{F} \hookrightarrow \mathcal{S}'$.

[1] See Section III.4.1 in H. Amann [4] and [10] for notation and facts from the theory of vector-valued distributions of which we make free use.

If $\mathfrak{F}$ is $\mathcal{M}$-admissible, then there exists a unique extension in $\mathcal{L}(\mathfrak{F})$ of the linear map $m(D) : \mathcal{S} \to \mathfrak{F}$, which shows $\mathfrak{F} \subset \mathrm{dom}\big(m(D)\big)$.

The following simple observation is of fundamental importance for what follows. Here F, F_0, F_1, and F_2 are finite-dimensional Banach spaces.

2.1.1 Proposition *Let $\mathfrak{F}_E$ be admissible. Then $\mathfrak{F}_{E\otimes F}$ is also admissible and*

$$\mathcal{M}\big(\mathbb{R}^d, \mathcal{L}(F_1, F)\big) \to \mathcal{L}(\mathfrak{F}_{E\otimes F_1}, \mathfrak{F}_{E\otimes F}), \quad m \mapsto m(D)$$

is linear and continuous. Furthermore, the map

$$\mathcal{M}\big(\mathbb{R}^d, \mathcal{L}(F_2, F_0)\big) \times \mathcal{M}\big(\mathbb{R}^d, \mathcal{L}(F_1, F_2)\big) \to \mathcal{L}(\mathfrak{F}_{E\otimes F_1}, \mathfrak{F}_{E\otimes F_0}), \quad (a,b) \mapsto (ab)(D)$$

is a multiplication. In particular,

$$\mathcal{M}\big(\mathbb{R}^d, \mathcal{L}(F)\big) \to \mathcal{L}(\mathfrak{F}_{E\otimes F}), \quad m \mapsto m(D) \tag{2.1.3}$$

is a continuous algebra homomorphism.

Proof. (1) Suppose $F_i = \mathbb{C}$ for $i = 0,1,2$ so that $E \otimes F_i$ is canonically identified with E by identifying $e \otimes 1$ with e for $e \in E$. Suppose $m_1, m_2 \in \mathcal{M}(\mathbb{R}^d)$ and $u \in \mathcal{S}$. Then[2] $\langle\xi\rangle^k m_1 \widehat{u}$ and $\langle\xi\rangle^k m_2 m_1 \widehat{u}$ belong to $L_\infty(\mathbb{R}^d, E)$ for each $k \in \mathbb{N}$. Consequently, $m_1\widehat{u}, m_2 m_1 \widehat{u} \in \mathcal{O}'_C := \mathcal{O}'_C(\mathbb{R}^d, E)$. Hence $m_1(D)u = \mathcal{F}^{-1}(m_1\widehat{u}) \in \mathcal{O}_M$ and $m_2\mathcal{F}\big(m_1(D)u\big) = m_2 m_1 \widehat{u} \in \mathcal{O}'_C$. This implies

$$\begin{aligned} m_2(D)m_1(D)u &= \mathcal{F}^{-1}\big(m_2\mathcal{F}(m_1(D)u)\big) = \mathcal{F}^{-1}(m_2 m_1 \widehat{u}) \\ &= (m_2 m_1)(D)\widehat{u} \in \mathcal{O}_M \end{aligned}$$

for $u \in \mathcal{S}$. Since $m_1, m_2 \in \mathcal{M}(\mathbb{R}^d)$ it follows $m_1(D)u \in \mathfrak{F}$ and $m_2(D)\big(m_1(D)u\big) \in \mathfrak{F}$ for $u \in \mathcal{S}$. Now we obtain

$$m_2(D)m_1(D)u = (m_2 m_1)(D)u, \qquad u \in \mathfrak{F},$$

by density and continuity. Since it is obvious from (2.1.2) that

$$\mathcal{M}(\mathbb{R}^d) \to \mathcal{L}(\mathfrak{F}), \quad m \mapsto m(D)$$

is linear and continuous, the assertions follow in this case.

(2) By introducing a basis in F_i we can assume $F_i = \mathbb{C}^{N_i}$. Consequently, $\mathcal{L}(F_i, F_j) = \mathbb{C}^{N_j \times N_i}$ and $\mathfrak{F}_{E\otimes F_i}$ is canonically identified with $\mathfrak{F}_E^{N_i}$, the N_i-fold product of $\mathfrak{F}_E$. Now the statement follows by applying the result of step (1) componentwise in the obvious way and by taking Lemma 1.6.1(i) into consideration. $\square$

2.1.2 Corollary *Let $\mathfrak{F}_E$ be admissible and* $\mathrm{Re}\, z \geq 0$. *Then*

$$\big(a \mapsto (\sigma_{|\eta|} a_\eta)(D)\big) \in \mathcal{L}\big(\mathcal{H}^{d+\ell}_{-z}\big(\mathsf{Z}, \mathcal{L}(F_1, F)\big), \mathcal{L}(\mathfrak{F}_{E\otimes F_1}, \mathfrak{F}_{E\otimes F})\big) \tag{2.1.4}$$

and

$$\|(\sigma_{|\eta|} a_\eta)(D)\|_{\mathcal{L}(\mathfrak{F}_{E\otimes F_1}, \mathfrak{F}_{E\otimes F})} \leq c\, |\eta|^{-\mathrm{Re}\, z}\, \|a\|_{\mathcal{H}^{d+\ell}_{-z}}, \qquad \eta \in \dot{\mathsf{H}}.$$

Proof. This follows from Lemma 1.6.1(iii) and Proposition 2.1.1. $\square$

[2] Recall $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$.

The next lemma explains, to some extent, why we consider smooth homogeneous functions although derivatives of order at most $d+1$ are needed in connection with Fourier multiplier operators. Recall $m^{-1}(\xi) = m(\xi)^{-1}$ for $\xi \in \mathbb{R}^d$ and $m : \mathbb{R}^d \to \mathcal{L}\mathrm{aut}(F)$.

2.1.3 Lemma *Let $\mathfrak{F}$ be admissible. Suppose*

$$m \in \mathcal{O}_M\big(\mathbb{R}^d, \mathcal{L}\mathrm{aut}(F)\big) \quad \textit{and} \quad m^{-1} \in \mathcal{M}\big(\mathbb{R}^d, \mathcal{L}(F)\big).$$

Denote by M the $\mathfrak{F}_{E\otimes F}$-realization of $m(D)$. Then M is closed and densely defined in $\mathfrak{F}_{E\otimes F}$, $0 \in \rho(M)$, and $M^{-1} = m^{-1}(D)$.

Proof. Since $m^{-1} \in \mathcal{M}\big(\mathbb{R}^d, \mathcal{L}(F)\big)$ implies $m^{-1} \in L_\infty\big(\mathbb{R}^d, \mathcal{L}(F)\big)$ it follows from $m \in \mathcal{O}_M\big(\mathbb{R}^d, \mathcal{L}(F)\big)$ and Lemma 1.4.2 that $m^{-1} \in \mathcal{O}_M\big(\mathbb{R}^d, \mathcal{L}(F)\big)$. Hence, by (2.1.1),

$$m(D) \in \mathcal{L}\mathrm{aut}(\mathcal{S}_{E\otimes F}) \cap \mathcal{L}\mathrm{aut}(\mathcal{S}'_{E\otimes F}), \quad m(D)^{-1} = m^{-1}(D). \tag{2.1.5}$$

From this we see that M is well-defined and closed, and, since

$$m(D)u \in \mathcal{S} \subset \mathfrak{F}_{E\otimes F}, \qquad u \in \mathcal{S}_{E\otimes F},$$

it is densely defined. From (2.1.5) we also infer that M is bijective. Proposition 2.1.1 implies

$$m^{-1}(D) \in \mathcal{L}(\mathfrak{F}_{E\otimes F}).$$

Now the statement is clear. □

The next lemma shows that, starting with one admissible Banach space $\mathfrak{F}$, we can construct a wide variety of admissible spaces related to $\mathfrak{F}$. This will be exploited in Section 2.3 below.

2.1.4 Lemma

(i) *Let $\mathfrak{F}$ be admissible. Suppose $\mathfrak{F}_1$ satisfies* (2.1.2)(i) *and $a \in \mathcal{O}_M(\mathbb{R}^d)$. If*

$$a(D) \in \mathcal{L}\mathrm{is}(\mathfrak{F}, \mathfrak{F}_1),$$

then $\mathfrak{F}_1$ is admissible.

(ii) *Let $(\mathfrak{F}_0, \mathfrak{F}_1)$ be a densely injected Banach couple such that $\mathfrak{F}_0$ and $\mathfrak{F}_1$ are admissible. Suppose $0 < \theta < 1$ and*

$$(\cdot,\cdot)_\theta \in \big\{\, [\cdot,\cdot]_\theta, (\cdot,\cdot)^0_{\theta,q} \ ;\ 1 \le q \le \infty \,\big\}.$$

Then $\mathfrak{F}_\theta := (\mathfrak{F}_0, \mathfrak{F}_1)_\theta$ is admissible.

Proof. (i) Note $am \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$ for $m \in \mathcal{M}(\mathbb{R}^d)$. Also

$$a(D) \in \mathcal{L}\mathrm{aut}(\mathcal{S}) \cap \mathcal{L}\mathrm{aut}(\mathcal{S}').$$

Hence

$$a(D)m(D)u = (am)(D)u = m(D)a(D)u, \qquad u \in \mathcal{S},$$

and, consequently,

$$m(D)v = a(D)m(D)a(D)^{-1}v, \qquad v \in \mathcal{S}.$$

By the $\mathcal{M}$-admissibility of $\mathfrak{F}$,

$$\|m(D)v\|_{\mathfrak{F}_1} \le \|a(D)\|_{\mathcal{L}(\mathfrak{F},\mathfrak{F}_1)} \, \|m(D)\|_{\mathcal{L}(\mathfrak{F})} \, \|a(D)^{-1}v\|_{\mathfrak{F}} \le c\, \|v\|_{\mathfrak{F}_1}$$

for $v \in \mathcal{S}$. Hence $\mathfrak{F}_1$ satisfies (2.1.2)(ii).

(ii) Since

$$\mathcal{S} \overset{d}{\hookrightarrow} \mathfrak{F}_1 \overset{d}{\hookrightarrow} \mathfrak{F}_\theta,$$

condition (2.1.2)(i) holds for $\mathfrak{F}_\theta$. By interpolation we obtain also the validity of condition (2.1.2)(ii). □

2.1.5 Corollary *Suppose $m \in \mathcal{O}_M(\mathbb{R}^d)$ and $1/m \in \mathcal{M}(\mathbb{R}^d)$. Let $\mathfrak{F}$ be admissible and denote by M the $\mathfrak{F}$-realization of $m(D)$. Then*[3] *$D(M) \overset{d}{\hookrightarrow} \mathfrak{F}$ and $D(M)$ is $\mathcal{M}$-admissible.*

Proof. From Lemma 2.1.3 we infer $M \in \mathcal{L}\mathrm{is}\big(D(M), \mathfrak{F}\big)$. Hence we obtain the claim from step (i) of the preceding proof. □

2.2 Parameter-dependence and resolvent estimates

We now introduce parameter-dependent admissible Banach spaces and consider Fourier multiplier operators in such spaces. In particular, we derive resolvent estimates and semigroup representation theorems in such spaces. For the reader's convenience we begin by recalling some simple general facts.

Let $\mathcal{X}$ and $\mathcal{Y}$ be LCSs and suppose $\varphi \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Then the **image space** $\varphi\mathcal{X}$ is the image of $\mathcal{X}$ in $\mathcal{Y}$ under φ endowed with the unique locally convex Hausdorff topology for which $\widehat{\varphi}$, defined by the commutativity of the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "\varphi"] \arrow[dr] & & \varphi\mathcal{X} \subset \mathcal{Y} \\ & \mathcal{X}/\ker(\varphi) \arrow[ur, "\widehat{\varphi}"'] & \end{tikzcd} \tag{2.2.1}$$

is a toplinear isomorphism. Of course, the non labeled arrow represents the canonical projection.

2.2.1 Remarks (a) $\varphi\mathcal{X}$ is an LCS such that $\varphi\mathcal{X} \hookrightarrow \mathcal{Y}$ and φ is a continuous surjection onto $\varphi\mathcal{X}$. If $\mathcal{P}$ is a generating family of seminorms for $\mathcal{X}$, then, setting

$$\widehat{p}(y) := \inf\big\{\, p(x) \,;\ x \in \varphi^{-1}(y) \,\big\}, \qquad y \in \varphi\mathcal{X},$$

the family $\{\,\widehat{p} \,;\ p \in \mathcal{P}\,\}$ generates the topology of $\varphi\mathcal{X}$. If $\mathcal{X} = (\mathcal{X}, \|\cdot\|)$ is a Banach space, then $\varphi\mathcal{X}$ is one also with the ‘quotient norm’

$$y \mapsto \|y\|_{\varphi\mathcal{X}} := \inf\big\{\, \|x\| \,;\ x \in \varphi^{-1}(y) \,\big\}.$$

In particular, if φ is injective, then

$$\|y\|_{\varphi\mathcal{X}} = \|\varphi^{-1}y\|_{\mathcal{X}}, \qquad y \in \varphi\mathcal{X},$$

and φ is an isometric isomorphism from $\mathcal{X}$ onto $\varphi\mathcal{X}$.

Proof. This is a consequence of the closedness of $\ker(\varphi)$ and standard properties of quotient spaces. □

[3]Recall that $D(M)$ is the domain of M endowed with its graph norm.

(b) Let $\mathcal{X}_0$ and $\mathcal{Z}$ be LCSs such that $\mathcal{X}_0 \overset{i}{\hookrightarrow} \mathcal{X}$ and $\mathcal{Y} \overset{j}{\hookrightarrow} \mathcal{Z}$. Writing $\varphi\mathcal{X}_0$ for $(\varphi \circ i)\mathcal{X}_0$ it follows

$$\varphi\mathcal{X}_0 \hookrightarrow \varphi\mathcal{X} \hookrightarrow \mathcal{Z}.$$

If i has dense image (resp. the images of φ and j are dense), then the first (resp. second) injection is dense.

Proof. The first assertion is obvious. The second one follows from (2.2.1) and the continuity of the canonical projection. □

(c) If $\mathcal{X}$ is a reflexive or a separable Banach space, then $\varphi\mathcal{X}$ is also reflexive or separable, respectively.

Proof. Quotients of Banach spaces modulo closed linear subspaces possess these properties. □

For $\gamma \in \mathbb{R}$ and $\eta \in \dot{\mathsf{H}}$ we put

$$\rho_t^\gamma := t^\gamma \sigma_t, \qquad t > 0.$$

Let $a_\eta \in C(\mathsf{Z})$ and $u \in \mathcal{S}'(\mathbb{R}^d, E')$ be such that $a_\eta(D)u$ is well-defined in $\mathcal{S}'(\mathbb{R}^d, E)$. Then we infer from Proposition 1.1.1(iii)

$$\begin{aligned} \rho_{1/|\eta|}^\gamma \circ a_\eta(D)u &= |\eta|^{-\gamma}\, \sigma_{1/|\eta|}\mathcal{F}^{-1}(a_\eta \widehat{u}) \\ &= |\eta|^{-\gamma+|\boldsymbol{\omega}|}\, \mathcal{F}^{-1}\sigma_{|\eta|}(a_\eta \widehat{u}) = |\eta|^{-\gamma+|\boldsymbol{\omega}|}\, \mathcal{F}^{-1}(\sigma_{|\eta|}a_\eta)\sigma_{|\eta|}\mathcal{F}u \qquad (2.2.2) \\ &= |\eta|^{-\gamma}\, \mathcal{F}^{-1}(\sigma_{|\eta|}a_\eta)\mathcal{F}\sigma_{1/|\eta|}u = (\sigma_{|\eta|}a_\eta)(D)\rho_{1/|\eta|}^\gamma u \end{aligned}$$

for all such $u \in \mathcal{S}'(\mathbb{R}^d, E)$.

Assume $\mathfrak{F} = \mathfrak{F}_E$ is admissible and F is finite-dimensional. Put $\mathfrak{G} := \mathfrak{F}_{E\otimes F}$ and, for $\eta \in \dot{\mathsf{H}}$,

$$\mathfrak{G}_{\gamma,\eta} := \rho_{|\eta|}^\gamma \mathfrak{G} = \{\, u \in \mathcal{S}'_{E\otimes F} \ ;\ \rho_{1/|\eta|}^\gamma u \in \mathfrak{F}_{E\otimes F} \,\}$$

endowed with the norm[4]

$$u \mapsto \|u\|_{\mathfrak{G}_{\gamma,\eta}} := \|\rho_{1/|\eta|}^\gamma u\|_{\mathfrak{G}}.$$

It follows from Remarks 2.2.1 that $\mathfrak{G}_{\gamma,\eta}$ is a Banach space satisfying

$$\mathcal{S}_{E\otimes F} \overset{d}{\hookrightarrow} \mathfrak{G}_{\gamma,\eta} \overset{d}{\hookrightarrow} \mathcal{S}'_{E\otimes F},$$

and $\rho_{|\eta|}^\gamma$ is an isometric isomorphism from $\mathfrak{G}$ onto $\mathfrak{G}_{\gamma,\eta}$ with $(\rho_{|\eta|}^\gamma)^{-1} = \rho_{1/|\eta|}^\gamma$.

2.2.2 Lemma *Suppose $a \in C\big(\mathsf{Z}, \mathcal{L}(F)\big)$ is such that either*

$$(\alpha) \qquad a_\eta \in \mathcal{O}_M\big(\mathbb{R}^d, \mathcal{L}(F)\big)$$

or

$$(\beta) \qquad \sigma_{|\eta|}a_\eta \in \mathcal{M}\big(\mathbb{R}^d, \mathcal{L}(F)\big)$$

for $\eta \in \dot{\mathsf{H}}$. In case (α) denote by D_η the domain of the $\mathfrak{G}$-realization of $(\sigma_{|\eta|}a_\eta)(D)$ and by $D_{\gamma,\eta}$ the one of the $\mathfrak{G}_{\gamma,\eta}$-realization of $a_\eta(D)$, endowed with the graph norm.

[4]The reason for considering γ-dependent norms will become clear in Section 4.14 below when we study parameter-dependent Besov spaces.

If (β) *holds, then put* $D_\eta := \mathfrak{G}$ *and* $D_{\gamma,\eta} := \mathfrak{G}_{\gamma,\eta}$. *Then the diagram of continuous linear maps*

$$\begin{tikzcd}
\mathfrak{G} \arrow[d, "\rho^\gamma_{|\eta|}"', "\cong"] & D_\eta \arrow[l, hook', "d"'] \arrow[r, "(\sigma_{|\eta|}a_\eta)(D)"] \arrow[d, "\rho^\gamma_{|\eta|}"', "\cong"] & \mathfrak{G} \arrow[d, "\rho^\gamma_{|\eta|}"', "\cong"] \\
\mathfrak{G}_{\gamma,\eta} & D_{\gamma,\eta} \arrow[l, hook', "d"'] \arrow[r, "a_\eta(D)"] & \mathfrak{G}_{\gamma,\eta}
\end{tikzcd}$$

is commuting, the vertical arrows representing isometric isomorphisms.

Proof. This follows easily from Proposition 2.1.1, (2.2.2), and the preceding remarks. □

2.2.3 Proposition

(i) *Suppose* $\sigma_{|\eta|}a_\eta \in \mathcal{M}\big(\mathbb{R}^d, \mathcal{L}(F)\big)$ *for* $\eta \in \dot{\mathsf{H}}$. *Then* $a_\eta(D) \in \mathcal{L}(\mathfrak{G}_{\gamma,\eta})$ *and*

$$\|a_\eta(D)\|_{\mathcal{L}(\mathfrak{G}_{\gamma,\eta})} = \|(\sigma_{|\eta|}a_\eta)(D)\|_{\mathcal{L}(\mathfrak{G})} \le c\,\|(\sigma_{|\eta|}a_\eta)\|_{\mathcal{M}}$$

for $\eta \in \dot{\mathsf{H}}$.

(ii) *If* $\operatorname{Re} z \ge 0$, *then*

$$\big(a \mapsto a_\eta(D)\big) \in \mathcal{L}\big(\mathcal{H}^{d+\ell}_{-z}(\mathsf{Z}, \mathcal{L}(F)), \mathcal{L}(\mathfrak{G}_{\gamma,\eta})\big)$$

and

$$\|a_\eta(D)\|_{\mathcal{L}(\mathfrak{G}_{\gamma,\eta})} \le c\,|\eta|^{-\operatorname{Re} z}\,\|a\|_{\mathcal{H}^{d+\ell}_{-z}}$$

for $\eta \in \dot{\mathsf{H}}$.

Proof. Assertion (i) is immediate by Proposition 2.1.1, the preceding lemma, and the isometry of $\rho^\gamma_{|\eta|}$.

(ii) is a consequence of (i) and Lemma 1.6.1(iii). □

Now we can prove the main results of this section. For this we assume

$$s \ge 0 \quad \text{and} \quad a \in \mathfrak{P}_s \cap \bar{\mathcal{H}}^\infty_s\big(\mathsf{Z}, \mathcal{L}(F)\big).$$

Then $a_\eta \in \mathcal{O}_M\big(\mathbb{R}^d, \mathcal{L}(F)\big)$ for $\eta \in \dot{\mathsf{H}}$ by Lemma 1.4.1(ii). Hence $a_\eta(D)$ belongs to $\mathcal{L}(\mathcal{S}_{E\otimes F}) \cap \mathcal{L}(\mathcal{S}'_{E\otimes F})$ and, consequently,

$$A_{\gamma,\eta}, \text{ the } \mathfrak{G}_{\gamma,\eta}\text{-realization of } a_\eta(D),$$

is well-defined. For a Banach space X we denote by $\mathcal{H}_-(X)$ the set of all negative generators of exponentially decaying analytic semigroups on X.

2.2.4 Theorem *Suppose* $s \ge 0$. *Then* $\mathsf{S}_{\pi-\varphi(2\kappa)} \subset \rho(-A_{\gamma,\eta})$ *and*

$$A_{\gamma,\eta} \in \mathcal{H}_- \cap \mathcal{BIP}(\mathfrak{G}_{\gamma,\eta}).$$

More precisely, assume $\kappa \ge 1$. *Then*

$$(|\eta|^s + |\lambda|)\,\|(\lambda + A_{\gamma,\eta})^{-1}\|_{\mathcal{L}(\mathfrak{G}_{\gamma,\eta})} + e^{-|t|\,\varphi(2\kappa)}\,\|(A_{\gamma,\eta})^{it}\|_{\mathcal{L}(\mathfrak{G}_{\gamma,\eta})} \le c(\kappa, \|a\|_{\mathcal{H}^{d+\ell}_s})$$

for $t, \gamma \in \mathbb{R}$, $\eta \in \dot{\mathsf{H}}$, $\lambda \in \mathsf{S}_{\pi-\varphi(2\kappa)}$, *and* $a \in \mathfrak{P}_s\big(\mathsf{Z}, \mathcal{L}(F); \kappa\big) \cap \bar{\mathcal{H}}^\infty_s\big(\mathsf{Z}, \mathcal{L}(F)\big)$.

PROOF. (1) Denote by B_η the $\mathfrak{G}$-realization of $\sigma_\eta a_\eta(D)$. It is a consequence of Proposition 1.5.2 and Lemmas 1.6.1(ii) and 2.1.3 that $\lambda + B_\eta$ is closed and densely defined in $\mathfrak{G}$, has zero in its resolvent set, satisfies $(\lambda + B_\eta)^{-1} = (\lambda + \sigma_\eta a_\eta)^{-1}(D)$, and

$$(|\eta|^s + |\lambda|)\,\|(\lambda + B_\eta)^{-1}\|_{\mathcal{L}(\mathfrak{G})} \le c(\kappa, \|a\|_{\mathcal{H}_s^{d+\ell}})$$

for $\lambda \in \mathsf{S}_{\pi-\varphi(2\kappa)}$ and $\eta \in \dot{\mathsf{H}}$. As in part (1) of the proof of Lemma 1.4.3 we find

$$(\lambda + \sigma_\eta a_\eta)^{-1} = \sigma_\eta(\lambda + a_\eta)^{-1}.$$

Now $\mathsf{S}_{\pi-\varphi(2\kappa)} \subset \rho(-A_{\gamma,\eta})$ and $A_{\gamma,\eta} \in \mathcal{H}(\mathfrak{G}_{\gamma,\eta})$ as well as the asserted estimates for $(\lambda + A_{\gamma,\eta})^{-1}$ follow from Lemma 2.2.2.

(2) The second part of the claim follows similarly from Proposition 1.5.4 and the last part of Proposition 2.1.1. □

2.2.5 Remark Assume $s \ge 0$, $\kappa \ge 1$, and $a \in \mathfrak{P}_s\big(\mathsf{Z}, \mathcal{L}(F);\kappa\big) \cap \bar{\mathcal{H}}_s^\infty\big(\mathsf{Z}, \mathcal{L}(F)\big)$. Let $\psi(a)$ be the spectral angle of a. Then it follows from Remark 1.5.5 that $\psi(A_{\gamma,\eta})$, the spectral angle of $A_{\gamma,\eta}$, is bounded above by $\psi(a)$ for $\eta \in \dot{\mathsf{H}}$ and $\gamma \in \mathbb{R}$. □

The next theorem shows, in particular, that the semigroup generated by $-A_{\gamma,\eta}$ is the Fourier multiplier semigroup with symbol e^{-ta_η}. It also gives an explicit bound exhibiting the η-dependence.

2.2.6 Theorem *Suppose $s \ge 0$, $\gamma \in \mathbb{R}$, and $\kappa \ge 1$. Then*

$$\|e^{-tA_{\gamma,\eta}}\|_{\mathcal{L}(\mathfrak{G}_{\gamma,\eta})} \le c(\kappa, \|a\|_{\mathcal{H}_s^{d+\ell}})\, e^{-t\,|\eta|^s/2\kappa}$$

and

$$e^{-tA_{\gamma,\eta}} = (e^{-ta_\eta})(D)$$

for $t \ge 0$, $\eta \in \dot{\mathsf{H}}$, and $a \in \mathfrak{P}_s\big(\mathsf{Z}, \mathcal{L}(F), \kappa\big) \cap \bar{\mathcal{H}}_s^\infty\big(\mathsf{Z}, \mathcal{L}(F)\big)$.

PROOF. (1) Proposition 1.5.6 and Lemma 1.6.1(ii) imply

$$\|\sigma_{|\eta|} e^{-ta_\eta}\|_{\mathcal{M}} \le c(\kappa, \|a\|_{\mathcal{H}_s^{d+\ell}}) e^{-t\,|\eta|^s/2\kappa}, \qquad t \ge 0,$$

for $\eta \in \dot{\mathsf{H}}$.

(2) Fix $\eta \in \dot{\mathsf{H}}$. Denote by by Γ_∞ the negatively oriented boundary of $\mathsf{S}_{\pi-\varphi(2\kappa)}$. Since

$$\sigma(-A_{\gamma,\eta}) \subset \mathbb{C}\backslash \mathsf{S}_{\pi-\varphi(2\kappa)}$$

by Theorem 2.2.4, semigroup theory implies

$$\begin{aligned} e^{-tA_{\gamma,\eta}} &= \frac{1}{2\pi i}\int_{\Gamma_\infty} e^{\lambda t}(\lambda + A_{\gamma,\eta})^{-1}\,d\lambda = \frac{1}{2\pi i}\int_{\Gamma_\infty} e^{\lambda t}(\lambda + a_\eta)^{-1}(D)\,d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_\infty} e^{\lambda t}(\lambda + a_\eta)^{-1}\,d\lambda(D) = e^{-ta_\eta}(D) \end{aligned}$$

for $t \ge 0$. Indeed, the next to the last equality follows from the last part of Proposition 2.1.1, and the last one from Cauchy's theorem. Taking into account (1) and Proposition 2.2.3(i), the assertions follow. □

2.3 Fractional power scales

Starting with an admissible Banach space $\mathfrak{F}$ we now introduce the (two-sided) fractional power scale generated by $\mathfrak{F}$ and

$$J := \Lambda_1(D)$$

in the sense of H. Amann [4, Section V.1]. In addition, we also consider parameter-dependent versions. As will be shown in the next chapters, our abstract approach unifies and simplifies the general theory of (parameter-dependent) concrete function spaces.

We put

$$\mathcal{S} := \mathcal{S}(\mathbb{R}^d, E), \quad \mathcal{S}' := \mathcal{S}'(\mathbb{R}^d, E).$$

Since $\Lambda_\eta^z \in \mathcal{O}_M$ for $z \in \mathbb{C}$ and $\eta \in \dot{\mathsf{H}}$ it follows that

$$J_\eta^z := \Lambda_\eta^z(D)$$

is well-defined and satisfies

$$J_\eta^z \in \mathcal{L}\mathrm{aut}(\mathcal{S}) \cap \mathcal{L}\mathrm{aut}(\mathcal{S}'), \qquad J_\eta^{z_1+z_2} = J_\eta^{z_1} J_\eta^{z_2} \tag{2.3.1}$$

for $z, z_1, z_2 \in \mathbb{C}$ and $\eta \neq 0$, where $J_\eta^0 = \mathrm{id}_{\mathcal{S}'}$. We set $J^z := J_1^z$, $J_\eta := J_\eta^1$, so that $J = J_1$.

Let $\mathfrak{F} := \mathfrak{F}(\mathbb{R}^d, E)$ be an admissible Banach space. We put

$$\mathfrak{F}^s = \mathfrak{F}^s(\mathbb{R}^d, E) := J^{-s}\mathfrak{F} = \big(\{\, u \in \mathcal{S}' \ ; \ J^s u \in \mathfrak{F} \,\}, \ \|\cdot\|_{\mathfrak{F}^s}\big)$$

where

$$\|\cdot\|_{\mathfrak{F}^s} := \|J^s \cdot\|_{\mathfrak{F}}$$

for $s \in \mathbb{R}$. For $\gamma \in \mathbb{R}$ and $\eta \in \dot{\mathsf{H}}$ we also put

$$\mathfrak{F}_{\gamma,\eta} = \mathfrak{F}_{\gamma,\eta}(\mathbb{R}^d, E) := \rho_{|\eta|}^\gamma \mathfrak{F}$$

and

$$\mathfrak{F}_{\gamma,\eta}^s := J_\eta^{-s} \mathfrak{F}_{\gamma,\eta}$$

for $s \in \mathbb{R}$ and endow these spaces with their natural norm.

2.3.1 Lemma *The diagram*

$$\begin{tikzcd}
\mathfrak{F} \arrow[r, "\rho_{|\eta|}^{\gamma}", "\cong"'] \arrow[d, "J^{-s}"', "\cong"] & \mathfrak{F}_{\gamma,\eta} \arrow[d, "J_\eta^{-s}"', "\cong"] \\
\mathfrak{F}^s \arrow[r, "\rho_{|\eta|}^{\gamma-s}", "\cong"'] & \mathfrak{F}_{\gamma,\eta}^s
\end{tikzcd}$$

is commuting, and all these isomorphisms are isometric.

Proof. It follows from (1.6.1) and (2.2.2) that

$$J_\eta^{-s} = \rho_{|\eta|}^\gamma \circ (\sigma_{|\eta|} \Lambda_{|\eta|}^{-s})(D) \circ \rho_{1/|\eta|}^\gamma = \rho_{|\eta|}^{\gamma-s} \circ J^{-s} \circ \rho_{1/|\eta|}^\gamma. \tag{2.3.2}$$

This shows that the claim is true. □

Henceforth, we use the symbol $\underset{\eta}{\doteq}$ to mean: equal except for equivalent norms, uniformly with respect to $\eta \in \dot{\mathsf{H}}$, that is, **$\eta$-uniformly**. We also write $\underset{\eta}{\sim}$ for **uniformly equivalent norms**.

In the following theorem we collect the basic properties of the spaces $\mathfrak{F}^s$ and their parameter-dependent versions $\mathfrak{F}^s_{\gamma,\eta}$.

2.3.2 Theorem *Let $\mathfrak{F}$ be admissible and suppose $\gamma \in \mathbb{R}$. Then:*

(i) $\mathfrak{F}^s_{\gamma,\eta}$ *is a Banach space and*

$$\mathcal{S} \overset{d}{\hookrightarrow} \mathfrak{F}^s_{\gamma,\eta} \overset{d}{\hookrightarrow} \mathfrak{F}^t_{\gamma,\eta} \hookrightarrow \mathcal{S}', \qquad s > t. \tag{2.3.3}$$

(ii) J^t_η *is an isometric isomorphism from* $\mathfrak{F}^{s+t}_{\gamma,\eta}$ *onto* $\mathfrak{F}^s_{\gamma,\eta}$ *for* $s, t \in \mathbb{R}$.

(iii) $\|\cdot\|_{\mathfrak{F}^s_{\gamma,\eta}} \le c\,|\eta|^{-t}\,\|\cdot\|_{\mathfrak{F}^{s+t}_{\gamma,\eta}}$, $s, t \in \mathbb{R}$, $\eta \in \dot{\mathsf{H}}$.

(iv) $\mathfrak{F}^s$ *is admissible.*

(v) *Given $s_0 < s_1$ and $\theta \in (0,1)$,*

$$\mathfrak{F}^{s_\theta}_{\gamma,\eta} \underset{\eta}{\doteq} [\mathfrak{F}^{s_0}_{\gamma,\eta}, \mathfrak{F}^{s_1}_{\gamma,\eta}]_\theta.$$

Proof. Let (u_j) be a Cauchy sequence in $\mathfrak{F}^s$. Then $(J^s u_j)$ is a Cauchy sequence in $\mathfrak{F}$. Thus $J^s u_j \to v$ in $\mathfrak{F}$, hence in $\mathcal{S}'$, for some $v \in \mathfrak{F}$. By (2.3.1), $u_j \to u := J^{-s} v$ in $\mathcal{S}'$. This shows $u \in \mathfrak{F}^s$ and $u_j \to u$ in $\mathfrak{F}^s$. Thus $\mathfrak{F}^s$ is a Banach space. Since $\mathfrak{F}^s_{\gamma,\eta}$ is isomorphic to $\mathfrak{F}^s$ it is one too.

From $\mathcal{S} \overset{d}{\hookrightarrow} \mathfrak{F} \overset{d}{\hookrightarrow} \mathcal{S}'$ and Lemma 2.3.1 it follows

$$\mathcal{S} \overset{d}{\hookrightarrow} \mathfrak{F}_{\gamma,\eta} \overset{d}{\hookrightarrow} \mathcal{S}'. \tag{2.3.4}$$

Corollary 2.1.2 with $\eta = 1$ implies $J^{-s} \in \mathcal{L}(\mathfrak{F})$ for $s \ge 0$. Hence, by Lemma 2.3.1,

$$\|J^{-s}_\eta\|_{\mathcal{L}(\mathfrak{F}_{\gamma,\eta})} = |\eta|^{-s}\,\|\rho^\gamma_{|\eta|} J^{-s} \rho^\gamma_{1/|\eta|}\|_{\mathcal{L}(\mathfrak{F}_{\gamma,\eta})} \le |\eta|^{-s}\,\|J^{-s}\|_{\mathcal{L}(\mathfrak{F})}$$

and, similarly,

$$\|J^{-s}\|_{\mathcal{L}(\mathfrak{F})} \le |\eta|^s\,\|J^{-s}_\eta\|_{\mathcal{L}(\mathfrak{F}_{\gamma,\eta})}.$$

Consequently,

$$\|J^{-s}_\eta\|_{\mathcal{L}(\mathfrak{F}_{\gamma,\eta})} = |\eta|^{-s}\,\|J^{-s}\|_{\mathcal{L}(\mathfrak{F})}, \qquad \eta \in \dot{\mathsf{H}}. \tag{2.3.5}$$

Suppose $t < s$. Then

$$\|u\|_{\mathfrak{F}^t_{\gamma,\eta}} = \|J^t_\eta u\|_{\mathfrak{F}_{\gamma,\eta}} \le \|J^{t-s}_\eta\|_{\mathcal{L}(\mathfrak{F}_{\gamma,\eta})}\,\|J^s_\eta u\|_{\mathfrak{F}_{\gamma,\eta}} \le c\,|\eta|^{t-s}\,\|u\|_{\mathfrak{F}^s_{\gamma,\eta}}. \tag{2.3.6}$$

From this and (2.3.4) we obtain $\mathfrak{F}^s_{\gamma,\eta} \hookrightarrow \mathfrak{F}^t_{\gamma,\eta}$. Thus (i) is proved. (ii) follows from Lemma 2.3.1 and (iii) from (2.3.6). We shall obtain (iv) from Lemma 2.1.4(ii) once (v) will be established.

To prove (v) we use (ii), guaranteeing that $J^{s_0}_\eta$ is an isometric isomorphism from $\mathfrak{F}^{s_0}_{\gamma,\eta}$ onto $\mathfrak{F}_{\gamma,\eta}$ and from $\mathfrak{F}^{s_1}_{\gamma,\eta}$ onto $\mathfrak{F}^{s_1-s_0}_{\gamma,\eta}$. Hence we can assume $s_0 = 0$. From Theorem 2.2.4 we infer that the $\mathfrak{F}_{\gamma,\eta}$-realization of J_η has bounded imaginary powers whose norms are bounded independently of $\eta \in \dot{\mathsf{H}}$. Now the assertion follows from Seeley's proof [60, Theorem 3]. (Also see H. Triebel [65, Theorem 1.15.3].) $\square$

2.3.3 Remark Suppose $s > 0$. Then we identify J_η^s with the $\mathfrak{F}_{\gamma,\eta}$-realization of $\Lambda_\eta^s(D)$. Hence J_η^s is a closed linear operator in $\mathfrak{F}_{\gamma,\eta}$ with dense domain $\mathfrak{F}_{\gamma,\eta}^s$. Consequently,

$$\mathfrak{F}_{\gamma,\eta}^s = \big(\operatorname{dom}(J_\eta^s), \|\cdot\|_{\mathfrak{F}_{\gamma,\eta}^s}\big),$$

where $\|\cdot\|_{\mathfrak{F}_{\gamma,\eta}^s} = \|J_\eta^s \cdot\|_{\mathfrak{F}_{\gamma,\eta}}$ is equivalent to the graph norm of $\operatorname{dom}(J_\eta^s)$, due to the fact that $0 \in \rho(J_\eta^s)$. The bounded inverse of J_η^s, also denoted by J_η^{-s}, is the $\mathfrak{F}_{\gamma,\eta}$-realization of $\Lambda_\eta^{-s}(D)$ and $\mathfrak{F}_{\gamma,\eta}^{-s}$ is easily seen to be the completion of $(\mathfrak{F}_{\gamma,\eta}, \|J_\eta^{-s} \cdot\|_{\mathfrak{F}_{\gamma,\eta}})$ in $\mathcal{S}'$. Thus

$$[\,\mathfrak{F}_{\gamma,\eta}^s\ ;\ s \in \mathbb{R}\,] \tag{2.3.7}$$

is the **fractional power scale generated by** $(\mathfrak{F}_{\gamma,\eta}, J_\eta)$ in the sense of H. Amann [4, Section V.1]. Theorem 2.3.2(v) and [4, Theorem V.1.5.4] imply that it is the interpolation extrapolation scale generated by $(\mathfrak{F}_{\gamma,\eta}, J_\eta)$ and $[\cdot,\cdot]_\theta$, $0 < \theta < 1$. □

The fact that we use the same symbol, namely J_η^s, for $\Lambda_\eta^s(D) \in \mathcal{L}(\mathcal{S}')$ and for its restriction to $\mathfrak{F}_{\gamma,\eta}^s$ is justified and cannot cause confusion since these operators coincide on $\mathcal{S}$ and the latter space is dense in $\mathfrak{F}_{\gamma,\eta}^s$.

2.3.4 Corollary *Fix $s_0 \in \mathbb{R}$ and let $[\,\mathfrak{G}_{\gamma,\eta}^s\ ;\ s \in \mathbb{R}\,]$ be the fractional power scale generated by $(\mathfrak{F}_{\gamma,\eta}^{s_0}, J_\eta)$. Then $\mathfrak{G}_{\gamma,\eta}^s = \mathfrak{F}_{\gamma,\eta}^{s+s_0}$.*

The next lemma will be of repeated use.

2.3.5 Lemma *Let $\mathfrak{F}$ be admissible and $\gamma, s \in \mathbb{R}$. If $a \in \mathfrak{P}_s(\mathsf{Z})$, then*

$$\|\cdot\|_{\mathfrak{F}_{\gamma,\eta}^s} \underset{\eta}{\sim} \|a_\eta(D)\cdot\|_{\mathfrak{F}_{\gamma,\eta}}.$$

Proof. Since, by Lemma 1.4.3, Λ^{-s} and a^{-1} belong to $\mathcal{H}_{-s}^\infty(\mathsf{Z})$, Lemma 1.4.1(i) implies $\Lambda^s a^{-1}, \Lambda^{-s} a \in \mathcal{H}_0^\infty(\mathsf{Z})$. Thus

$$J_\eta^s = (\Lambda^s a^{-1})_\eta(D) a_\eta(D)$$

and

$$a_\eta(D) = (a\Lambda^{-s})_\eta(D) J_\eta^s$$

imply the assertion, due to Proposition 2.2.3(ii) and Lemma 2.3.1. □

2.3.6 Corollary *Let Q be a quasi-norm and $\gamma \in \mathbb{R}$. For $s \in \mathbb{R}$ and $\eta \in \dot{\mathsf{H}}$ set*

$$\mathfrak{G}_{\gamma,\eta}^s := \big(\{\, u \in \mathcal{S}'\ ;\ \mathsf{Q}_\eta^s(D)u \in \mathfrak{F}_{\gamma,\eta}\,\},\ \|\cdot\|_{\mathfrak{G}_{\gamma,\eta}^s}\big)$$

where

$$\|\cdot\|_{\mathfrak{G}_{\gamma,\eta}^s} := \|\mathsf{Q}_\eta^s(D)\cdot\|_{\mathfrak{F}_{\gamma,\eta}}.$$

Then $\mathfrak{G}_{\gamma,\eta}^s \underset{\eta}{\doteq} \mathfrak{F}_{\gamma,\eta}^s$ for $s \in \mathbb{R}$.

Our next lemma clarifies the mapping properties of derivatives in fractional power scales.

2.3.7 Lemma *Let $\mathfrak{F}$ be admissible, $s, \gamma \in \mathbb{R}$, and $\alpha \in \mathbb{N}^d$. Then*

$$\partial^\alpha \in \mathcal{L}(\mathfrak{F}_{\gamma,\eta}^s, \mathfrak{F}_{\gamma,\eta}^{s-\alpha\cdot\boldsymbol{\omega}})$$

η-uniformly.

PROOF. Set
$$a(\zeta) := \xi^\alpha \Lambda^{-\alpha\cdot\boldsymbol{\omega}}(\zeta), \qquad \zeta = (\xi,\eta) \in \dot{\mathsf{Z}}.$$
Then $a \in \mathcal{H}_0^\infty(\mathsf{Z})$ by Lemmas 1.4.1(i) and 1.4.3, since $(\zeta \mapsto \xi^\alpha) \in \mathcal{H}_{\alpha\cdot\boldsymbol{\omega}}(\mathsf{Z})$. Hence
$$a_\eta(D) = D^\alpha J_\eta^{-\alpha\cdot\boldsymbol{\omega}} \in \mathcal{L}(\mathfrak{F}_{\gamma,\eta})$$
η-uniformly by Corollary 2.1.2. Thus Theorem 2.3.2 implies
$$D^\alpha = (D^\alpha J_\eta^{-\alpha\cdot\boldsymbol{\omega}})J_\eta^{\alpha\cdot\boldsymbol{\omega}} \in \mathcal{L}(\mathfrak{F}_{\gamma,\eta}^{\alpha\cdot\boldsymbol{\omega}}, \mathfrak{F}_{\gamma,\eta})$$
η-uniformly. Now the assertion follows from Theorem 2.3.2(ii) and the commutativity of D^α and J^s. □

We close this section by proving a renorming theorem whose importance will be clear when we consider concrete realizations of fractional power scales.

2.3.8 Theorem *Let $\mathfrak{F}$ be admissible, $\gamma \in \mathbb{R}$, and $m \in \dot{\mathbb{N}}$. Then the following are equivalent:*

(i) $u \in \mathfrak{F}_{\gamma,\eta}^{m\omega}$;

(ii) $\partial^\alpha u \in \mathfrak{F}_{\gamma,\eta}, \quad \alpha\cdot\boldsymbol{\omega} \le m\omega$;

(iii) $u, \partial_j^{m\omega/\omega_j} u \in \mathfrak{F}_{\gamma,\eta}, \quad 1 \le j \le d$.

Furthermore,
$$\|\cdot\|_{\mathfrak{F}_{\gamma,\eta}^{m\omega}} \underset{\eta}{\sim} \sum_{\alpha\cdot\boldsymbol{\omega}\le m\omega} |\eta|^{m\omega-\alpha\cdot\boldsymbol{\omega}} \|\partial^\alpha \cdot\|_{\mathfrak{F}_{\gamma,\eta}} \underset{\eta}{\sim} |\eta|^{m\omega} \|\cdot\|_{\mathfrak{F}_{\gamma,\eta}} + \sum_{j=1}^d \|\partial_j^{m\omega/\omega_j} \cdot\|_{\mathfrak{F}_{\gamma,\eta}}.$$

PROOF. (1) From Theorem 2.3.2(iii) and Lemma 2.3.7 we deduce
$$\|D^\alpha\|_{\mathcal{L}(\mathfrak{F}_{\gamma,\eta}^{m\omega},\mathfrak{F}_{\gamma,\eta})} \le c\,|\eta|^{\alpha\cdot\boldsymbol{\omega}-m\omega}, \qquad \eta \ne 0.$$
Consequently,
$$|\eta|^{m\omega} \|\cdot\|_{\mathfrak{F}_{\gamma,\eta}} + \sum_{j=1}^d \|\partial_j^{m\omega/\omega_j} \cdot\|_{\mathfrak{F}_{\gamma,\eta}} \le \sum_{\alpha\cdot\boldsymbol{\omega}\le m\omega} |\eta|^{m\omega-\alpha\cdot\boldsymbol{\omega}} \|\partial^\alpha \cdot\|_{\mathfrak{F}_{\gamma,\eta}} \le c\,\|\cdot\|_{\mathfrak{F}_{\gamma,\eta}^{m\omega}}$$
for $\eta \in \dot{\mathsf{H}}$.

(2) Put
$$b(\zeta) := \sum_{j=1}^d (\xi^j)^{2\omega/\omega_j} + |\eta|^{2\omega}, \qquad \zeta \in \dot{\mathsf{Z}}.$$
Then $b \in \mathfrak{P}_{2\omega}(\mathsf{Z})$. Hence $\|\cdot\|_{\mathfrak{F}_{\gamma,\eta}^{2\omega}} \underset{\eta}{\sim} \|b_\eta(D)\cdot\|_{\mathfrak{F}_{\gamma,\eta}}$ by Lemma 2.3.5. Now we infer from Theorem 2.3.2(ii)
$$\|\cdot\|_{\mathfrak{F}_{\gamma,\eta}^{\omega}} = \|J_\eta^{-\omega}\cdot\|_{\mathfrak{F}_{\gamma,\eta}^{2\omega}} \underset{\eta}{\sim} \|b_\eta(D)J_\eta^{-\omega}\cdot\|_{\mathfrak{F}_{\gamma,\eta}}. \tag{2.3.8}$$
Since
$$D_j^{2\omega/\omega_j} J_\eta^{-\omega} = D_j^{\omega/\omega_j} J_\eta^{-\omega} D_j^{\omega/\omega_j}$$
and $(\xi^j)^{\omega/\omega_j}\Lambda^{-\omega} \in \mathcal{H}_0^\infty$ implies
$$D_j^{\omega/\omega_j} J_\eta^{-\omega} \in \mathcal{L}(\mathfrak{F}_{\gamma,\eta}), \quad \|D_j^{\omega/\omega_j} J_\eta^{-\omega}\|_{\mathcal{L}(\mathfrak{F}_{\gamma,\eta})} \le c, \qquad \eta \ne 0,$$

we deduce from

$$b_\eta(D) = |\eta|^{2\omega} + \sum_{j=1}^{d} D_j^{2\omega/\omega_j}$$

and (2.3.8)

$$\begin{aligned}\|u\|_{\mathfrak{F}^\omega_{\gamma,\eta}} \le c\,\|b_\eta(D)J_\eta^{-\omega}u\|_{\mathfrak{F}_{\gamma,\eta}} &\le c\Big(|\eta|^{2\omega}\,\|J_\eta^{-\omega}u\|_{\mathfrak{F}_{\gamma,\eta}} + \sum_{j=1}^{d}\|D_j^{2\omega/\omega_j}J_\eta^{-\omega}u\|_{\mathfrak{F}_{\gamma,\eta}}\Big)\\ &\le c\Big(|\eta|^{\omega}\,\|u\|_{\mathfrak{F}_{\gamma,\eta}} + \sum_{j=1}^{d}\|D_j^{\omega/\omega_j}u\|_{\mathfrak{F}_{\gamma,\eta}}\Big)\end{aligned}$$

for $\eta \in \dot{\mathsf{H}}$. This implies the claims. □

In Part 2, where we shall consider function spaces on manifolds, still another form of part (iii) of this theorem will be of importance. For this we denote for $k \in \mathbb{N}$

$$\nabla^k u := \{\,\partial^\alpha u\ ;\ |\alpha| = k\,\}, \quad \nabla := \nabla^1,$$

whenever $\partial^\alpha u$ is well-defined. We arrange this $m(k)$-tuple[5] by the lexicographical ordering. For smooth functions we set

$$|\nabla^k u| := \Big(\sum_{|\alpha|=k} |\partial^\alpha u|^2\Big)^{1/2}$$

so that $|\nabla^k u|$ is the Euclidean norm of the 'vector' $\nabla^k u$, and

$$\|\nabla^k u\| := \big\|\,|\nabla^k u|\,\big\|,$$

whenever $\|\cdot\|$ is a norm of a Banach space of which $|\nabla^k u|$ is a member.

Using this notation, the following corollary is obvious.

2.3.9 Corollary $\|\cdot\|_{\mathfrak{F}^{m\omega}_{\gamma,\eta}} \underset{\eta}{\sim} |\eta|^{m\omega}\,\|\cdot\|_{\mathfrak{F}_{\gamma,\eta}} + \sum_{i=1}^{\ell} \|\nabla_{x_i}^{m\nu/\nu_i}\cdot\|_{\mathfrak{F}_{\eta,\gamma}}$.

[5] $m(k) = \sum_{|\alpha|=k} 1 = \binom{d+k-1}{k}$.

CHAPTER 3

Fourier multipliers and function spaces

In this chapter the theory of anisotropic vector-valued Bessel potential and Besov spaces on $\mathbb{R}^d$ is developed. Similarly as in the isotropic scalar case, it is based on Fourier analysis. Whereas in the case of Besov spaces no restrictions on the Banach spaces have to be imposed, a powerful theory of vector-valued Bessel potential spaces requires a limitation of the class of admitted target spaces.

3.1 Marcinkiewicz type multiplier theorems

A Banach space E is a **UMD space** if the Hilbert transform is bounded on $L_2(\mathbb{R}, E)$. Then it is bounded on $L_q(\mathbb{R}, E)$ for each $q \in (1, \infty)$ and E is reflexive.

Every finite-dimensional Banach space is a UMD space. If E is a UMD space, then $L_q(X, \mu, E)$ is a UMD space as well whenever $1 < q < \infty$ and (X, μ) is a σ-finite measure space. Every Banach space isomorphic to a UMD space is such a space, and if E is a UMD space, then E' is one as well. Every Hilbert space is a UMD space, and so is every closed linear subspace of a UMD space. Finite products of UMD spaces are UMD spaces. If (E_0, E_1) is an interpolation couple of UMD spaces, then $E_{[\theta]}$ and $E_{\theta,q}$, $1 < q < \infty$, are UMD spaces for $0 < \theta < 1$. More details and proofs are found in H. Amann [4, Section III.4] (also see P.Ch. Kunstmann and L. Weis [44]).

Following G. Pisier [52], a Banach space E is said to have **property (α)** if there is a constant c such that for each $n \in \dot{\mathbb{N}}$ and $(e_{ij}, \alpha_{ij}) \in E \times \mathbb{C}$ with $|\alpha_{ij}| \le 1$,

$$\int_0^1 \int_0^1 \Big| \sum_{i,j=1}^n r_i(s) r_j(t) \alpha_{ij} e_{ij} \Big|_E \, ds\, dt \le c \int_0^1 \int_0^1 \Big| \sum_{i,j=1}^n r_i(s) r_j(t) e_{ij} \Big|_E \, ds\, dt$$

where (r_j) is the sequence of Rademacher functions,

$$r_j(t) = \operatorname{sign}(\sin 2^j \pi t).$$

Every finite-dimensional Banach space has property (α). If E has property (α), then each closed linear subspace of it, each Banach space isomorphic to E, and $L_q(X, \mu, E)$ have this property as well, where $1 \le q < \infty$ and (X, μ) is a σ-finite measure space. If E is a UMD space with property (α), then E' has property (α) as well. We refer to P.Ch. Kunstmann and L. Weis [44, Section 4] and the references therein for more details and proofs.

We denote by

$$\mathcal{M}a := \mathcal{M}a(\mathbb{R}^d)$$

the set of all $m \in C^d\big((\mathbb{R}^d)^\bullet\big)$ satisfying

$$\|m\|_{\mathcal{M}a} := \max_{\alpha \le (1,1,\dots,1)} \sup_{\xi \in (\mathbb{R}^d)^\bullet} |\xi^\alpha \partial^\alpha m(\xi)| < \infty$$

and

$$\mathcal{M}i := \mathcal{M}i(\mathbb{R}^d)$$

is the set of all $m \in C^d\big((\mathbb{R}^d)^\bullet\big)$ satisfying

$$\|m\|_{\mathcal{M}i} := \max_{\alpha \le (1,1,\dots,1)} \sup_{\xi \in (\mathbb{R}^d)^\bullet} |\xi|^{|\alpha|} \, |\partial^\alpha m(\xi)| < \infty.$$

Then

$$\mathcal{M}a = (\mathcal{M}a, \|\cdot\|_{\mathcal{M}a}) \text{ and } \mathcal{M}i = (\mathcal{M}i, \|\cdot\|_{\mathcal{M}i})$$

are Banach spaces. ($\mathcal{M}a$ and $\mathcal{M}i$ should remind the reader of Marcinkiewicz and Mikhlin, respectively.) Note $\mathcal{M}i \hookrightarrow \mathcal{M}a$.

3.1.1 Theorem *Suppose $1 < q < \infty$ and E is a UMD space. Then*

$$\big(m \mapsto m(D)\big) \in \mathcal{L}\big(\mathcal{M}i, \mathcal{L}(L_q(\mathbb{R}^d, E))\big).$$

If E has also property (α), then

$$\big(m \mapsto m(D)\big) \in \mathcal{L}\big(\mathcal{M}a, \mathcal{L}(L_q(\mathbb{R}^d, E))\big).$$

This result is due to F. Zimmermann [73]. Different proofs (in more general settings) have been given by L. Weis and coauthors and by R. Haller, H. Heck, and A. Noll [37]. F. Zimmermann has also shown that property (α) cannot be omitted if the vector-valued Marcinkiewicz multiplier theorem is to hold (also see [40]).

Suppose $E = \mathbb{C}$. Then the second assertion is a nonperiodic version of the Marcinkiewicz multiplier theorem [49] (see S.M. Nikol′skiĭ [51] or E.M. Stein [63, Section IV.6]). The first claim is a variant of Mikhlin's Fourier multiplier theorem (e.g., L. Hörmander [39]). Further historical details are given by H. Triebel in Remark 2.4.4.4 of [65].

The following theorem, an easy corollary to the preceding theorem, is of fundamental importance for what follows.

3.1.2 Theorem *Suppose $1 < q < \infty$. If either*

$$\boldsymbol{\omega} = \omega(1,1,\dots,1) = (\omega,\omega,\dots,\omega) \text{ and } E \text{ is a UMD space}$$

or

$$E \text{ is a UMD space with property } (\alpha),$$

then $L_q(\mathbb{R}^d, E)$ is $\mathcal{M}$-admissible.

Proof. Suppose $\alpha \le (1,1,\dots,1)$. Then

$$|(\xi^j)^{\alpha^j}| = (|\xi^j|^{2\omega/\omega_j})^{\alpha^j \omega_j / 2\omega} \le \Lambda_1(\xi)^{\alpha^j \omega_j}$$

for $1 \le j \le d$. Hence $|\xi^\alpha| \le \Lambda_1^{\alpha \cdot \boldsymbol{\omega}}(\xi)$ for $\xi \in \mathbb{R}^d$. Consequently $\|\cdot\|_{\mathcal{M}a} \le \|\cdot\|_{\mathcal{M}}$, that is $\mathcal{M} \hookrightarrow \mathcal{M}a$. If $\boldsymbol{\omega} = \omega(1,\dots,1)$, then

$$|\xi|^{|\alpha|} = \Big(\sum_{j=1}^d |\xi^j|^2\Big)^{|\alpha|/2} \le \Big(\sum_{j=1}^d |\xi^j|^2 + 1\Big)^{|\alpha|\,\omega/2\omega} = \Lambda_1^{|\alpha|\,\omega}(\xi) = \Lambda_1^{\alpha \cdot \boldsymbol{\omega}}(\xi), \qquad \xi \in \mathbb{R}^d,$$

so that $\mathcal{M} \hookrightarrow \mathcal{M}i$. Now the claim follows from Theorem 3.1.1. $\square$

Unfortunately, it is not true that Theorem 3.1.1 holds for operator-valued symbols. This follows from a result of Ph. Clément and J. Prüss [19], combining it with characterizations of maximal L_p-regularity for evolution equation due to N.J. Kalton and G. Lancien [42] (cf. [44] for more details).

Let E and F be Banach spaces. A subset $\mathcal{T}$ of $\mathcal{L}(E,F)$ is said to be $\mathfrak{R}$-bounded if

$$\Big\| \sum_{j=1}^{n} r_j T_j x_j \Big\|_{L_2((0,1),F)} \le c \Big\| \sum_{j=1}^{n} r_j x_j \Big\|_{L_2((0,1),E)}$$

for $n \in \dot{\mathbb{N}}$ and $(T_j, x_j) \in \mathcal{T} \times E$. The infimum of all such c is the **$\mathfrak{R}$-bound** of $\mathcal{T}$ and denoted by $\mathfrak{R}(\mathcal{T})$. It is obvious that an $\mathfrak{R}$-bounded set is bounded and

$$\sup_{T \in \mathcal{T}} \|T\|_{\mathcal{L}(E,F)} \le \mathfrak{R}(\mathcal{T}).$$

Using this concept we can formulate an extension of Theorem 3.1.1 for the case of operator-valued Fourier multipliers.

3.1.3 Theorem *Suppose $1 < q < \infty$, E and F are UMD spaces, and m belongs to $C^d\big((\mathbb{R}^d)^{\bullet}, \mathcal{L}(E,F)\big)$. If*

$$M := \big\{\, |\xi|^{|\alpha|}\, \partial^\alpha m(\xi) \; ; \; \alpha \le (1,\dots,1),\ \xi \in (\mathbb{R}^d)^{\bullet} \,\big\}$$

is $\mathfrak{R}$-bounded in $\mathcal{L}(E,F)$, then

$$m(D) \in \mathcal{L}\big(L_q(\mathbb{R}^d,E), L_q(\mathbb{R}^d,F)\big)$$

and

$$\|m(D)\| \le c\,\mathfrak{R}(M)$$

where c depends on E, F, d, and q only.

If, in addition, E and F have property (α), then M can be replaced by

$$\big\{\, \xi^\alpha \partial^\alpha m \; ; \; \alpha \le (1,\dots,1),\ \xi \in (\mathbb{R}^d)^{\bullet} \,\big\}.$$

In the 1-dimensional case the sufficiency part of the first assertion is due to L. Weis [70]. Extensions to n dimensions are given by Ž. Štrkalj and L. Weis [64] and R. Haller, H. Heck, and A. Noll [37]. Other proofs can be found in the memoir of R. Denk, M. Hieber, and J. Prüss [20] and in the survey of P.Ch. Kunstmann and L. Weis [44].

The extension of the Marcinkiewicz' multiplier theorem to operator-valued symbols, that is, the second part of Theorem 3.1.3, is due to R. Haller, H. Heck, and A. Noll [37]. A different proof appears in [44].

In the paper [19] by Ph. Clément and J. Prüss the $\mathfrak{R}$-boundedness condition is shown to be also necessary for the analogue to the Mikhlin theorem to hold (see also Section 3.13 in [44]).

There are many sufficient conditions for a family of bounded linear operators to be $\mathfrak{R}$-bounded (see R. Denk, M. Hieber, and J. Prüss [20] and P.Ch. Kunstmann and L. Weis [44]). We restrict ourselves to cite just one particularly simple and useful criterion.

3.1.4 Proposition *Let E and F be Banach spaces and K a compact subset of $\mathbb{C}$. Suppose m is a holomorphic map from a neighborhood of K into $\mathcal{L}(E,F)$. Then $m(K)$ is $\mathfrak{R}$-bounded in $\mathcal{L}(E,F)$.*

Proof. This is Proposition 3.10 in [20]. □

3.2 Dyadic decompositions and Fourier multipliers

Let Q be a quasi-norm and set

$$\Omega_0 = \Omega_0^{\mathsf{Q}} := [\mathsf{Q}_0 < 2], \quad \Omega_k = \Omega_k^{\mathsf{Q}} := [2^{k-1} < \mathsf{Q}_0 < 2^{k+1}], \qquad k \in \dot{\mathbb{N}}.$$

Note $\Omega_j \cap \Omega_k = \emptyset$ for $|j-k| \geq 2$.

Fix $\psi \in \mathcal{D}(\mathbb{R}^d)$ satisfying

$$\psi(\xi) = 1 \text{ for } \mathsf{Q}_0(\xi) \leq 3/2, \quad \operatorname{supp}(\psi) \subset \Omega_0. \tag{3.2.1}$$

Put

$$\widetilde{\psi}(\xi) := \psi(\xi) - \psi(2 \bullet \xi) = \psi(\xi) - \sigma_2\psi(\xi), \qquad \xi \in \mathbb{R}^d,$$

and

$$\psi_0 := \psi, \quad \psi_k := \sigma_{2^{-k}}\widetilde{\psi}, \qquad k \in \dot{\mathbb{N}}. \tag{3.2.2}$$

Then ψ_k is smooth with support in Ω_k and

$$\sum_{k=0}^{n} \psi_k = \sigma_{2^{-n}}\psi, \qquad n \in \mathbb{N}. \tag{3.2.3}$$

Hence (ψ_k) is a smooth partition of unity on $\mathbb{R}^d$, **induced by** ψ, subordinate to the **Q-dyadic** open covering (Ω_k).

Given $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $\alpha \in \mathbb{N}^d$,

$$\|\varphi\partial^\alpha(\sigma_{2^{-n}}\psi - \mathbf{1})\|_\infty \to 0, \qquad n \to \infty,$$

where $\mathbf{1}(\xi) = 1$ for $\xi \in \mathbb{R}^d$. Consequently, $\sigma_{2^{-n}}\psi \to \mathbf{1}$ in $\mathcal{O}_M(\mathbb{R}^d)$. Hence we infer from (3.2.3)

$$\sum_{k=0}^{\infty} \psi_k = \mathbf{1} \quad \text{in } \mathcal{O}_M(\mathbb{R}^d). \tag{3.2.4}$$

Thus, given $u \in \mathcal{S}' := \mathcal{S}'_E = \mathcal{S}'(\mathbb{R}^d, E)$,

$$\sum_{k=0}^{\infty} \psi_k \widehat{u} = \widehat{u} \quad \text{in } \mathcal{S}'$$

and, consequently,

$$\sum_{k=0}^{\infty} \psi_k(D)u = u \quad \text{in } \mathcal{S}', \tag{3.2.5}$$

due to $\mathcal{F} \in \mathcal{L}(\mathcal{S}')$.

In the following, we denote by

$$\mathcal{M}_0(\mathbb{R}^d, E)$$

the set of all $m \in C^{d+\ell}\big((\mathbb{R}^d)^{\bullet}, E\big)$ satisfying

$$\|m\|_{\mathcal{M}_0} := \max_{|\alpha| \leq d+\ell} \|\Lambda_0^{\alpha\cdot\boldsymbol{\omega}}\partial^\alpha m\|_\infty < \infty.$$

It is a Banach space with norm $\|\cdot\|_{\mathcal{M}_0}$ and

$$\mathcal{M}(\mathbb{R}^d, E) \hookrightarrow \mathcal{M}_0(\mathbb{R}^d, E).$$

3.2.1 Proposition *Let ψ satisfy (3.2.1) and let (ψ_k) be the anisotropic partition of unity on $\mathbb{R}^d$ induced by it, subordinate to the Q-dyadic open covering (Ω_k). Suppose*

$$\mathfrak{F} \in \{\, BUC, C_0, L_p \ ;\ 1 \le p < \infty \,\}$$

and set $\mathfrak{F}_E := \mathfrak{F}(\mathbb{R}^d, E)$. Also assume that $E_1 \times E_2 \to E_0$ is a multiplication of Banach spaces.

If $a \in \mathcal{M}(\mathbb{R}^d, E_1)$, then

$$(\psi_k a)(D) \in \mathcal{L}(\mathfrak{F}_{E_2}, \mathfrak{F}_{E_0})$$

and

$$(\psi_k a)(D) \in \mathcal{L}\big(L_\infty(\mathbb{R}^d, E_2), BUC(\mathbb{R}^d, E_0)\big)$$

for $k \in \mathbb{N}$ with

$$\sup_{k \ge 1} \|(\psi_k a)(D)\| \le c\, \|a\|_{\mathcal{M}_0}$$

and

$$\|(\psi_0 a)(D)\| \le c\, \|a\|_{\mathcal{M}},$$

where $c = c(\mathsf{Q}, \psi, d)$ is independent of a.

PROOF. (1) By the convolution theorem (see H. Amann [10], for example, for the vector-valued case)

$$(\psi_k a)(D)u = \mathfrak{F}^{-1}(\psi_k a) * u.$$

Hence the assertion follows from well-known elementary properties of convolutions (e.g., [5, Theorem 4.1]) provided we show $\psi_k a \in \mathcal{F}L_1(\mathbb{R}^d, E_1)$ and

$$\|\psi_k a\|_{\mathcal{F}L_1} \le c\, \|a\|_{\widetilde{\mathcal{M}}_k}, \qquad k \in \mathbb{N}, \quad a \in \mathcal{M}(\mathbb{R}^d, E_1), \tag{3.2.6}$$

where $\widetilde{\mathcal{M}}_k := \mathcal{M}$ for $k = 0$, and $\widetilde{\mathcal{M}}_k := \mathcal{M}_0$ if $k \in \dot{\mathbb{N}}$.

(2) Assume we have shown

$$\|D^\alpha(\psi_k a)\|_{L_1(\mathbb{R}^d, E_1)} \le c_0\, \|a\|_{\widetilde{\mathcal{M}}_k} \tag{3.2.7}$$

for $k \in \mathbb{N}$, $\alpha = (\alpha_1, \dots, \alpha_\ell)$ with $|\alpha_i| = d_i + 1$, and $a \in \mathcal{M}(\mathbb{R}^d, E_1)$. Then, by the Riemann–Lebesgue theorem,

$$x^\alpha \mathcal{F}^{-1}(\psi_k a) = \mathcal{F}^{-1}\big(D^\alpha(\psi_k a)\big) \in C_0(\mathbb{R}^d, E_1)$$

and

$$\|x^\alpha \mathcal{F}^{-1}(\psi_k a)\|_\infty \le c_1(c_0)\, \|a\|_{\widetilde{\mathcal{M}}_k} \tag{3.2.8}$$

for $k \in \mathbb{N}$, $\alpha = (\alpha_1, \dots, \alpha_\ell)$ with $|\alpha_i| = d_i + 1$, and $a \in \mathcal{M}(\mathbb{R}^d, E_1)$. The multinomial theorem implies

$$|x_i|^{d_i+1} = \big(((x_i^1)^2 + \cdots + (x_i^{d_i})^2)^{d_i+1}\big)^{1/2} \le c(d_i) \sum_{|\alpha_i| = d_i+1} |x_i^{\alpha_i}|$$

for $k \in \mathbb{N}$ and $1 \le i \le \ell$. Thus

$$|x_1|^{d_1+1} \cdot \cdots \cdot |x_\ell|^{d_\ell+1} \le c(\boldsymbol{d}) \prod_{i=1}^{\ell} \sum_{|\alpha_i| = d_i+1} |x_i^{\alpha_i}|, \qquad x \in \mathbb{R}^d.$$

Now we infer from (3.2.8), setting $c_2 := c_1(c_0)$,

$$|\mathcal{F}^{-1}(\psi_k a)(x)| \le c_2\, |x_1|^{-d_1-1} \cdot \cdots \cdot |x_\ell|^{-d_\ell-1}\, \|a\|_{\widetilde{\mathcal{M}}_k}, \qquad x \in (\mathbb{R}^d)^\bullet,$$

for $k \in \mathbb{N}$ and $a \in \mathcal{M}(\mathbb{R}^d, E_1)$. From this and the Riemann-Lebesgue theorem (3.2.6) follows, where $c_3 = c_2 c_1$ with

$$c_1 := \int_{[\,|x|\le 1]} dx + \int_{[\,|x|\ge 1]} |x_1|^{-d_1-1} \cdot \cdots \cdot |x_\ell|^{-d_\ell-1}\, dx < \infty.$$

Hence it remains to prove (3.2.7).

(3) From Leibniz' rule we deduce

$$|\partial^\alpha(\psi_0 a)| \le c(\psi) \sum_{\beta\le\alpha} |\partial^\beta a|\, \chi_{\Omega_0} \le c(\psi) \max_{|\beta|\le d+\ell} \|\partial^\beta a\|_\infty\, \chi_{\Omega_0} \le c(\psi)\, \|a\|_{\mathcal{M}}\, \chi_{\Omega_0},$$

hence

$$\|\partial^\alpha(\psi_0 a)\|_1 \le c(\psi, \mathsf{Q})\, \|a\|_{\mathcal{M}}, \qquad |\alpha| \le d+\ell, \quad a \in \mathcal{M}(\mathbb{R}^d, E_1).$$

(4) Suppose $k \ge 1$. Note

$$\alpha \bullet \boldsymbol{\omega} - |\boldsymbol{\omega}| = |\alpha_1|\, \nu_1 + \cdots + |\alpha_\ell|\, \nu_\ell - d_1\nu_1 - \cdots - d_\ell\nu_\ell = \nu_1 + \cdots + \nu_\ell \ge 1$$

for $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ with $|\alpha_i| = d_i + 1$. Thus (3.2.2) and Proposition 1.1.1 imply

$$\begin{aligned}\|\partial^\alpha(\psi_k a)\|_1 &= \big\|\partial^\alpha\big(\sigma_{2^{-k}}(\widetilde{\psi}\sigma_{2^k}a)\big)\big\|_1 = 2^{-k\alpha\cdot\boldsymbol{\omega}}\, \|\sigma_{2^{-k}}\partial^\alpha(\widetilde{\psi}\sigma_{2^k}a)\|_1\\ &= 2^{-k(\alpha\cdot\boldsymbol{\omega}-|\boldsymbol{\omega}|)}\, \|\partial^\alpha(\widetilde{\psi}\sigma_{2^k}a)\|_1 \le \|\partial^\alpha(\widetilde{\psi}\sigma_{2^k}a)\|_1\end{aligned}$$

for $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ with $|\alpha_i| = d_i + 1$. Using Leibniz' rule and Proposition 1.1.1 once more we find

$$|\partial^\alpha(\widetilde{\psi}\sigma_{2^k}a)| \le c \sum_{\beta\le\alpha} |\partial^\beta(\sigma_{2^k}a)|\, \chi_{\Omega_1} = c \sum_{\beta\le\alpha} 2^{k\beta\cdot\boldsymbol{\omega}}\, |\sigma_{2^k}\partial^\beta a|\, \chi_{\Omega_1}.$$

Lemma 1.2.1(iv) implies $\kappa^{-1}\Lambda \le \mathsf{Q} \le \kappa\Lambda$ for some $\kappa \ge 1$. Hence

$$\Omega_k \subset [2^{k-1}/\kappa < \Lambda_0 < 2^{k+1}\kappa], \qquad k \in \dot{\mathbb{N}}. \tag{3.2.9}$$

Since $2^k \bullet \xi \in \Omega_k$ for $\xi \in \Omega_1$ we thus find

$$2^{k\beta\cdot\boldsymbol{\omega}}\, |(\sigma_{2^k}\partial^\beta a)(\xi)| \le (2\kappa)^{\beta\cdot\boldsymbol{\omega}} \Lambda_0^{\beta\cdot\boldsymbol{\omega}}(2^k \bullet \xi)\, |\partial^\beta a(2^k \bullet \xi)| \le (2\kappa)^{\beta\cdot\boldsymbol{\omega}}\, \|a\|_{\mathcal{M}_0}$$

for $\xi \in \Omega_1$. This implies

$$|\partial^\alpha(\widetilde{\psi}\sigma_{2^k}a)| \le c(\psi, \mathsf{Q}, d)\, \|a\|_{\mathcal{M}_0}\, \chi_{\Omega_1}$$

for $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ with $|\alpha_i| = d_i + 1$ and $k \in \dot{\mathbb{N}}$. From this and step (3) we obtain (3.2.7). This proves the proposition. □

3.2.2 Remark Put $\varphi_k := \sigma_{2^{-k}}\widetilde{\psi}$ for $k \in \mathbb{Z}$. Then $(\varphi_k)_{k\in\mathbb{Z}}$ is a Q-dyadic resolution of the identity on $(\mathbb{R}^d)^\bullet$, subordinate to the open covering $(\Sigma_k)_{k\in\mathbb{Z}}$, where $\Sigma_k := [2^{k-1} < \mathsf{Q}_0 < 2^{k+1}]$ for $k \in \mathbb{Z}$. Then, given $a \in \mathcal{M}_0(\mathbb{R}^d, E_1)$,

$$(\varphi_k a)(D) \in \mathcal{L}(\mathfrak{F}_{E_2}, \mathfrak{F}_{E_0})$$

and

$$(\varphi_k a)(D) \in \mathcal{L}\big(L_\infty, (\mathbb{R}^d, E_2), BUC(\mathbb{R}^d, E_0)\big)$$

with

$$\sup_{k\in\mathbb{Z}} \|(\varphi_k a)(D)\| \le c\,\|a\|_{\mathcal{M}_0},$$

where $c = c(\psi, \mathsf{Q}, d)$ is independent of a.

Proof. This is obvious by the above proof. □

Proposition 3.2.1 is an extension of Proposition 4.5 of H. Amann [5] to the anisotropic situation. Besides of being used in the following section it is the basis for the proof of a general multiplier theorem for operator-valued symbols in Besov spaces which is given in Section 3.4 below. It is the proof of Proposition 3.2.1 in which we need $d+\ell$ derivatives of the symbol a, whereas the Marcinkiewicz and Mikhlin multiplier theorems require at most d of them. To have a unified treatment we use throughout $d+\ell$ derivatives in the definition of $\mathcal{M}(\mathbb{R}^d, E)$.

3.3 Besov spaces

Throughout this and the next three sections

- $p, p_0, p_1, q, q_0, q_1 \in [1,\infty]$ *and* $s, s_0, s_1, t \in \mathbb{R}$.

As usual,

$$\ell_q(E) = \ell_q(\mathbb{N}, E) := L_q(\mathbb{N}, \mu; E)$$

where μ is the counting measure, and

$$c_0(E) := C_0(\mathbb{N}, E)$$

is the closed subspace of ℓ_∞ of all null sequences in E.

It is convenient to put

- $\dfrac{1}{\boldsymbol{\nu}} := \Big(\dfrac{1}{\nu_1}, \dots, \dfrac{1}{\nu_\ell}\Big) \in \mathbb{R}^\ell.$

Set $\mathsf{Q} := \Lambda$ and suppose ψ satisfies (3.2.1). Denote by (ψ_k) the partition of unity induced by ψ, subordinate to the Λ-dyadic open covering (Ω_k). The (anisotropic) **Besov space**

$$B^{s/\boldsymbol{\nu}}_{p,q} = B^{s/\boldsymbol{\nu}}_{p,q}(\mathbb{R}^d, E)$$

associated with the (anisotropic) dilation (1.1.2) is the vector subspace of

$$\mathcal{S}' = \mathcal{S}'_E = \mathcal{S}'(\mathbb{R}^d, E)$$

of all u for which

$$\big(2^{ks}\psi_k(D)u\big) \in \ell_q\big(L_p(\mathbb{R}^d, E)\big),$$

endowed with its natural norm

$$\|u\|_{B^{s/\boldsymbol{\nu}}_{p,q}} := \big\|\big(2^{ks}\psi_k(D)u\big)\big\|_{\ell_q(L_p)} = \big\|\big(2^{ks}\,\|\psi_k(D)u\|_{L_p(\mathbb{R}^d,E)}\big)\big\|_{\ell_q}. \tag{3.3.1}$$

We denote by

$$\overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,\infty} := \overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,\infty}(\mathbb{R}^d, E), \qquad p \ne \infty,$$

the closed linear subspace of $B^{s/\boldsymbol{\nu}}_{p,\infty}$ consisting of all u satisfying

$$\big(2^{ks}\psi_k(D)u\big) \in c_0\big(L_p(\mathbb{R}^d, E)\big),$$

and

$$\overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{\infty,q} = \overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{\infty,q}(\mathbb{R}^d, E)$$

is the closed linear subspace of $B^{s/\boldsymbol{\nu}}_{\infty,q}$ made up by those u for which

$$\left(2^{ks}\psi_k(D)u\right) \in \begin{cases} \ell_q\bigl(C_0(\mathbb{R}^d, E)\bigr), & q < \infty, \\ c_0\bigl(C_0(\mathbb{R}^d, E)\bigr), & q = \infty. \end{cases}$$

In the isotropic case $\boldsymbol{\omega} = (1, \dots, 1)$, where $\boldsymbol{\nu} = 1$, we write, of course, $B^s_{p,q}$ for $B^{s/\boldsymbol{\nu}}_{p,q}$.

Clearly, (3.3.1) depends on the choice of ψ and the particular selection of the special quasi-norm Λ. The following lemma shows that a different choice of (Q, ψ) leads to an equivalent norm.

3.3.1 Lemma *Let Q be a quasi-norm, let ψ^{Q} satisfy* (3.2.1)*, and denote by (ψ^{Q}_k) the partition of unity induced by ψ^{Q}, subordinate to the Q-dyadic open covering (Ω^{Q}_k) of $\mathbb{R}^d$. Denote by $B^{s/\boldsymbol{\nu},\mathsf{Q}}_{p,q}$ the Besov space defined as above, but with (Λ, ψ) replaced by $(\mathsf{Q}, \psi^{\mathsf{Q}})$. Then $B^{s/\boldsymbol{\nu},\mathsf{Q}}_{p,q} \doteq B^{s/\boldsymbol{\nu}}_{p,q}$.*

Proof. Fix $\kappa \geq 1$ such that (3.2.9) is true and choose $m \in \mathbb{N}$ with $\kappa \leq 2^m$. Then

$$\Omega^{\mathsf{Q}}_k \subset [2^{k-m-1} < \Lambda_0 < 2^{k+m+1}] =: \Omega_{k,m}, \qquad k \in \dot{\mathbb{N}}, \tag{3.3.2}$$

and, similarly,

$$\Omega^{\mathsf{Q}}_0 \subset [\Lambda_0 < 2^{m+1}] =: \Omega_{0,m}. \tag{3.3.3}$$

Set

$$\chi_{k,m} := \sum_{i=-m-1}^{m+1} \psi_{k+i}, \quad \psi_j := 0 \text{ for } j < 0.$$

Then we see by (3.2.3) that

$$\chi_{k,m} \,|\, \Omega_{k,m} = 1, \quad \operatorname{supp}(\chi_{k,m}) \subset \Omega_{k,m+1}.$$

Consequently, by (3.3.2) and (3.3.3),

$$\psi^{\mathsf{Q}}_k = \psi^{\mathsf{Q}}_k \chi_{k,m}, \qquad k \in \mathbb{N}. \tag{3.3.4}$$

Choosing $a = \mathbf{1}$ in Proposition 3.2.1 it thus follows

$$\|\psi^{\mathsf{Q}}_k(D)u\|_p \leq c\, \|\chi_{k,m}(D)u\|_p \leq c \sum_{i=-m-1}^{m+1} \|\psi_{k+i}(D)u\|_p$$

for $k \in \mathbb{N}$. This implies

$$\bigl\|\bigl(2^{ks}\psi^{\mathsf{Q}}_k(D)u\bigr)\bigr\|_{\ell_q(L_p)} \leq c\, \bigl\|\bigl(2^{ks}\psi_k(D)u\bigr)\bigr\|_{\ell_q(L_p)},$$

that is, $B^{s/\boldsymbol{\nu}}_{p,q} \hookrightarrow B^{s/\boldsymbol{\nu},\mathsf{Q}}_{p,q}$. By interchanging the rôles of (Λ, ψ) and $(\mathsf{Q}, \psi^{\mathsf{Q}})$ we obtain $B^{s/\boldsymbol{\nu},\mathsf{Q}}_{p,q} \hookrightarrow B^{s/\boldsymbol{\nu}}_{p,q}$. □

Anisotropic Besov spaces have been intensively studied — in the scalar case — by S.M. Nikol′skiĭ and O.V. Besov by classical methods (see the monographs of S.M. Nikol′skiĭ [51] and O.V. Besov, V.P. Il′in, and S.M. Nikol′skiĭ [17] for additional references). Fourier-analytic approaches, as the one used here, are due to H. Triebel [66, Section 10.1], H.-J. Schmeisser and H. Triebel [57], M. Yamazaki [71], [72] (also see W. Farkas, J. Johnsen, and W. Sickel [23] for further references), who used various quasi-norms. In particular, M. Yamazaki based his comprehensive treatment on the Euclidean quasi-norm E, and most later writers followed this usage.

All the above references deal with the scalar case. Isotropic vector-valued Besov spaces have been investigated by P. Grisvard [27] and H.-J. Schmeißer [53] under various restrictions on s and p. The general situation is dealt with in H. Amann [5] (also see Section 15 in H. Triebel's book [67] and, for further historic references, H.-J. Schmeißer and W. Sickel [55]).

In the following, we collect the basic properties of anisotropic vector-valued Besov spaces and give only brief hints concerning proofs. Many theorems carry over without alteration from the scalar case to the vector-valued situation. To a large extent we can follow the isotropic approach in H. Amann [5]. Complete details will be given in the forthcoming second volume of H. Amann [4].

We set

$$\ell_q^s(E) := L_q(\mathbb{N}, \mu^s; E),$$

where μ^s is the weighted counting measure assigning the value 2^{ks} to $\{k\} \subset \mathbb{N}$. Then the proof of Lemma 5.1 in H. Amann [5] carries over to show that

$$\ell_q^s(L_p) := \ell_q^s\big(L_p(\mathbb{R}^d, E)\big) \to B_{p,q}^{s/\boldsymbol{\nu}}, \quad (v_k) \mapsto \sum_k \chi_k(D) v_k, \tag{3.3.5}$$

with

$$\chi_k := \psi_{k-1} + \psi_k + \psi_{k+1}, \tag{3.3.6}$$

is a retraction and $u \mapsto \big(\psi_k(D)u\big)$ is a coretraction. Hence $B_{p,q}^{s/\boldsymbol{\nu}}$ is a Banach space, since it is isomorphic to a closed linear subspace of the Banach space $\ell_q^s(L_p)$ (cf. Proposition I.2.3.2 in H. Amann [4]).

The next theorem collects some of the most important embedding results.

3.3.2 Theorem *The following embeddings are valid:*

$$\mathcal{S} \hookrightarrow B_{p,q_1}^{s/\boldsymbol{\nu}} \hookrightarrow B_{p,q_0}^{s/\boldsymbol{\nu}} \hookrightarrow \mathcal{S}', \qquad q_1 \leq q_0, \tag{3.3.7}$$

and

$$B_{p,q_1}^{s_1/\boldsymbol{\nu}} \hookrightarrow B_{p,q_0}^{s_0/\boldsymbol{\nu}}, \qquad s_1 > s_0. \tag{3.3.8}$$

Furthermore,

$$B_{p_1,q}^{s_1/\boldsymbol{\nu}} \hookrightarrow B_{p_0,q}^{s_0/\boldsymbol{\nu}}, \qquad s_1 > s_0, \quad s_1 - |\boldsymbol{\omega}|/p_1 = s_0 - |\boldsymbol{\omega}|/p_0. \tag{3.3.9}$$

Proof. (1) Assertions (3.3.7) and (3.3.8) are obtained by obvious modifications of the proof of Proposition 2.3.2.2 in H. Triebel [66].

(2) Statement (3.3.9) follows from the anisotropic version of the Nikol′skiĭ inequality: if $p_1 < p_0$ and $r > 0$, then

$$\|\varphi(D)u\|_{p_0} \leq c r^{|\boldsymbol{\omega}|\,(1/p_1 - 1/p_0)} \, \|\varphi(D)u\|_{p_1}, \qquad u \in \mathcal{S}', \tag{3.3.10}$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$ with $\operatorname{supp}(\varphi) \subset [\Lambda_0 \leq r]$.

To prove (3.3.10) we first suppose $r = 1$. Then

$$\varphi(D)u = (\psi\varphi)(D)u = \psi(D)\varphi(D)u,$$

the convolution theorem, and Young's inequality imply $\varphi(D)u \in C_0 = C_0(\mathbb{R}^d, E)$ and

$$\|\varphi(D)u\|_\infty \le \|\mathcal{F}^{-1}\psi\|_{p_1'} \|\varphi(D)u\|_{p_1} = c\,\|\varphi(D)u\|_{p_1}$$

for $u \in \mathcal{S}'$, since $\mathcal{F}^{-1}\psi \in \mathcal{S}(\mathbb{R}^d) \hookrightarrow L_{p_1'}(\mathbb{R}^d)$. Hence

$$\|\varphi(D)u\|_{p_0} \le \|\varphi(D)u\|_\infty^{1-p_1/p_0} \|\varphi(D)u\|_{p_1}^{p_1/p_0} \le c\,\|\varphi(D)u\|_{p_1}.$$

This proves (3.3.10) in this case.

(3) Now suppose $r \ne 1$. Then $\operatorname{supp}(\sigma_r\varphi) \subset [\Lambda_0 \le 1]$. By Proposition 1.1.1,

$$(\sigma_r\varphi)(D)u = \mathcal{F}^{-1}\big((\sigma_r\varphi)\widehat{u}\big) = \sigma_{1/r}\mathcal{F}^{-1}\varphi\mathcal{F}\sigma_r u$$

and, consequently,

$$\|(\sigma_r\varphi)(D)u\|_p = r^{|\boldsymbol{\omega}|/p} \|\varphi(D)(\sigma_r u)\|_p.$$

Thus, by step (2),

$$r^{|\boldsymbol{\omega}|/p_0} \|\varphi(D)(\sigma_r u)\|_{p_0} \le c\,r^{|\boldsymbol{\omega}|/p_1} \|\varphi(D)(\sigma_r u)\|_{p_1}$$

for $u \in \mathcal{S}'$. This implies (3.3.10) since $\sigma_r \in \mathcal{L}\mathrm{aut}(\mathcal{S}')$. $\square$

This embedding theorem is in the scalar case due to M. Yamazaki [71, Theorem 3.4], where the proof of Nikol′skiĭ's inequality is based, as usual, on the maximal function. Also see Proposition 7 in W. Farkas, J. Johnsen, and W. Sickel [23] where a more general result involving anisotropic Triebel–Lizorkin spaces is given.

Using retraction (3.3.5) it is not difficult to see that $\mathcal{S}$ is dense in $B_{p,q}^{s/\boldsymbol{\nu}}$ for $p \vee q < \infty$, and in $\overset{\vartriangle}{B}{}_{p,q}^{s/\boldsymbol{\nu}}$ if $p \vee q = \infty$. Thus, in order to allow for a unified treatment, we put

$$\overset{\vartriangle}{B}{}_{p,q}^{s/\boldsymbol{\nu}} := B_{p,q}^{s/\boldsymbol{\nu}}, \qquad q \vee p < \infty,$$

so that, in general,

$$\overset{\vartriangle}{B}{}_{p,q}^{s/\boldsymbol{\nu}} \text{ is the closure of } \mathcal{S} \text{ in } B_{p,q}^{s/\boldsymbol{\nu}}. \tag{3.3.11}$$

Retraction (3.3.5) and well-known facts from interpolation theory (cf. J. Bergh and J. Löfström [16, Theorem 5.6.1] and H. Triebel [65, Theorem 1.18.2]) imply

$$(B_{p,q_0}^{s_0/\boldsymbol{\nu}}, B_{p,q_1}^{s_1/\boldsymbol{\nu}})_{\theta,q} \doteq B_{p,q}^{s_\theta/\boldsymbol{\nu}} \tag{3.3.12}$$

and

$$(B_{p,q_0}^{s_0/\boldsymbol{\nu}}, B_{p,q_1}^{s_1/\boldsymbol{\nu}})_{\theta,q}^0 \doteq \overset{\vartriangle}{B}{}_{p,q}^{s_\theta/\boldsymbol{\nu}} \tag{3.3.13}$$

for $s_0 \ne s_1$ and $0 < \theta < 1$. Thus the reiteration theorem for the continuous interpolation functor gives

$$(\overset{\vartriangle}{B}{}_{p,q}^{s_0/\boldsymbol{\nu}}, \overset{\vartriangle}{B}{}_{p,q}^{s_1/\boldsymbol{\nu}})_{\theta,\infty}^0 \doteq \overset{\vartriangle}{B}{}_{p,\infty}^{s_\theta/\boldsymbol{\nu}} \tag{3.3.14}$$

$s_0 \ne s_1$ and $0 < \theta < 1$.

Next we study duality properties of anisotropic vector-valued Besov spaces. Note that the density of $\mathcal{S}(\mathbb{R}^d)$ in $\overset{\vartriangle}{B}{}_{p,q}^{s/\boldsymbol{\nu}}(\mathbb{R}^d)$ implies $\big(\overset{\vartriangle}{B}{}_{p,q}^{s/\boldsymbol{\nu}}(\mathbb{R}^d)\big)' \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$ with respect to the $\mathcal{S}'(\mathbb{R}^d)$-$\mathcal{S}(\mathbb{R}^d)$-duality pairing. An analogous result is true in the vector-valued case if E is either reflexive or has a separable dual.

First we note that L. Schwartz' theory of vector-valued distributions guarantees the existence of a unique separately (in fact: hypo-)continuous separating bilinear form

$$\langle\cdot,\cdot\rangle : \mathcal{S}'_{E'} \times \mathcal{S}_E \to \mathbb{C}, \quad (u',u) \mapsto \langle u',u\rangle,$$

the **$\mathcal{S}'$-$\mathcal{S}$-duality pairing**, such that

$$\langle u',u\rangle = \int_{\mathbb{R}^d} \bigl\langle u'(x), u(x)\bigr\rangle_E \, dx, \quad (u',u) \in \mathcal{S}_{E'} \times \mathcal{S}_E$$

(cf. H. Amann [10, Theorem 1.7.5]). We also call it **L_q-duality pairing**.

3.3.3 Theorem *Let either E be reflexive or E' separable. Then*

$$\overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q}(\mathbb{R}^d,E)' \doteq B^{-s/\boldsymbol{\nu}}_{p',q'}(\mathbb{R}^d,E')$$

with respect to the $\mathcal{S}'$-$\mathcal{S}$-duality pairing.

Proof. The proof of this result is based on retraction (3.3.5) and known duality properties of vector-valued L_p-spaces. To guarantee $L_p(\mathbb{R}^d,E)' \doteq L_{p'}(\mathbb{R}^d,E')$ we have imposed the above assumptions on E, although it would suffice to suppose E has the Radon–Nikodym property. Details will be given in Volume II of [4]. □

3.3.4 Corollary *Let E be reflexive.*

(i) *If $1 < p,q < \infty$, then $B^{s/\boldsymbol{\nu}}_{p,q}$ is also reflexive and*

$$\bigl(B^{s/\boldsymbol{\nu}}_{p,q}(\mathbb{R}^d,E)\bigr)' \doteq B^{-s/\boldsymbol{\nu}}_{p',q'}(\mathbb{R}^d,E').$$

(ii) *If $p \wedge q > 1$, then $B^{s/\boldsymbol{\nu}}_{p,q}$ is the bidual of $\overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q}$ with respect to the $\mathcal{S}'$-$\mathcal{S}$-duality pairing.*

It should also be remarked that

$$\bigl(\overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{\infty,q}(\mathbb{R}^d,E)\bigr)' = B^{-s/\boldsymbol{\nu}}_{1,q'}(\mathbb{R}^d,E')$$

is true without any restriction on E.

3.4 Fourier multipliers in Besov spaces

In this section we establish a general Fourier multiplier theorem with operator-valued symbols, an extension of Theorem 6.2 in [5] to anisotropic Banach spaces. Using it we can derive several additional interpolation results and prove important renorming theorems.

First we present an isomorphism theorem which, together with the multiplier theorem, will allow us to make use of the general results on admissible Banach scales established in earlier chapters.

3.4.1 Theorem *Suppose $\mathcal{B} \in \{B, \overset{\vartriangle}{B}\}$. Then*

$$J^t \in \mathcal{L}\mathrm{is}(\mathcal{B}^{(s+t)/\boldsymbol{\nu}}_{p,q}, \mathcal{B}^{s/\boldsymbol{\nu}}_{p,q}),$$

uniformly with respect to s, p, and q; and $(J^t)^{-1} = J^{-t}$.

Proof. This follows by modifying the proof of Theorem 6.1 in H. Amann [5] in a by now obvious way (defining χ_k by (3.3.6)). □

It should be observed that the following multiplier theorem does not require any restriction on the underlying Banach space E. This is in stark contrast to the Marcinkiewicz multiplier theorems of Section 3.1.

3.4.2 Theorem *Suppose* $\mathcal{B} \in \{B, \overset{\scriptscriptstyle\Delta}{B}\}$ *and let* $E_1 \times E_2 \to E_0$ *be a multiplication of Banach spaces, Then*

$$\bigl(m \mapsto m(D)\bigr) \in \mathcal{L}\bigl(\mathcal{M}(\mathbb{R}^d, E_1), \mathcal{L}\bigl(\mathcal{B}_{p,q}^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E_2), \mathcal{B}_{p,q}^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E_0)\bigr)\bigr).$$

Proof. Due to Proposition 3.2.1 the proof of Theorem 6.2 in H. Amann [5] carries literally over to the present anisotropic setting. □

It should be noted that Remark 3.2.2 can be used to prove a multiplier theorem for homogeneous Besov spaces using operator-valued multipliers of Marcinkiewicz type. We refrain here from giving details.

On the basis of Theorem 3.4.1 we impose the following convention:

We fix any $\psi \in \mathcal{D}(\mathbb{R}^d)$ *satisfying* (3.2.1) *with* $\mathsf{Q}_0 = \Lambda_0$
and endow $B_{p,q}^{0/\boldsymbol{\nu}}$ *with the norm* $u \mapsto \bigl\|\bigl(\psi_k(D)u\bigr)\bigr\|_{\ell_q(L_p)}$.
If $s \neq 0$, *then* $B_{p,q}^{s/\boldsymbol{\nu}}$ *is given the norm* $u \mapsto \|J^s u\|_{B_{p,q}^{0/\boldsymbol{\nu}}}$.

The following theorem will be most important for the rest of this treatise.

3.4.3 Theorem *The anisotropic* ***Besov scale*** $[\,\overset{\scriptscriptstyle\Delta}{B}{}_{p,q}^{s/\boldsymbol{\nu}}\ ;\ s \in \mathbb{R}\,]$ *is the fractional power scale generated by* $(\overset{\scriptscriptstyle\Delta}{B}{}_{p,q}^{0/\boldsymbol{\nu}}, J)$. *It consists of* $\mathcal{M}$*-admissible Banach spaces.*

Proof. This follows from (3.3.7), (3.3.11), Theorems 3.4.1 and 3.4.2, and Theorem 2.3.2(iv). □

As an immediate consequence we obtain from Theorem 2.3.2(v)

$$[\overset{\scriptscriptstyle\Delta}{B}{}_{p,q}^{s_0/\boldsymbol{\nu}}, \overset{\scriptscriptstyle\Delta}{B}{}_{p,q}^{s_1/\boldsymbol{\nu}}]_\theta \doteq \overset{\scriptscriptstyle\Delta}{B}{}_{p,q}^{s_\theta/\boldsymbol{\nu}}, \qquad s_0 \neq s_1, \quad 0 < \theta < 1. \tag{3.4.1}$$

3.5 Anisotropic Sobolev and Hölder spaces

In this section we study anisotropic Besov spaces of positive order and investigate their relation to classical function spaces. For this we suppose[1]

$$\mathfrak{F} \in \{\, BUC, C_0, L_p\ ;\ p \neq \infty \,\}$$

and put for $k \in \mathbb{N}$

$$\mathfrak{F}^{k\nu/\boldsymbol{\nu}} = \mathfrak{F}^{k\nu/\boldsymbol{\nu}}(\mathbb{R}^d, E) := \{\, u \in \mathfrak{F}\ ;\ \partial^\alpha u \in \mathfrak{F},\ \alpha \boldsymbol{\cdot} \boldsymbol{\omega} \leq k\nu \,\},$$

equipped with the norm

$$u \mapsto \|u\|_{\mathfrak{F}^{k\nu/\boldsymbol{\nu}}} := \begin{cases} \Bigl(\sum_{\alpha\cdot\boldsymbol{\omega}\leq k\nu} \|\partial^\alpha u\|_p^p\Bigr)^{1/p}, & \mathfrak{F} = L_p, \\ \max_{\alpha\cdot\boldsymbol{\omega}\leq k\nu} \|\partial^\alpha u\|_\infty & \text{otherwise.} \end{cases}$$

[1]Here we mean by $\mathfrak{F} = BUC$, for example, the symbol BUC and not the Banach space $BUC(\mathbb{R}^d, E)$, etc. This dual use of $\mathfrak{F}$ should cause no confusion to the attentive reader.

Thus $\mathfrak{F}^{0\nu/\boldsymbol{\nu}} = \mathfrak{F}$. Also recall that $\nu = \omega$ is the least common multiple of $\omega_1, \ldots, \omega_d$, hence of $\nu_1, \ldots, \nu_\ell$. For $\mathfrak{F} = L_p$ we write $W_p^{k\nu/\boldsymbol{\nu}} := \mathfrak{F}^{k\nu/\boldsymbol{\nu}}$. If $E = \mathbb{C}$, then $W_p^{k\nu/\boldsymbol{\nu}}$ is the classical anisotropic Sobolev space of S.M. Nikol′skiĭ [51].

3.5.1 Example Let $\boldsymbol{\omega}$ be the $2m$-parabolic weight vector. Then, if $p \neq \infty$,

$$u \in W_p^{(2m,1)}(\mathbb{R}^d, E) \text{ iff } u, \partial_t u, \partial_x^\alpha u \in L_p(\mathbb{R}^d, E), \qquad |\alpha| \leq 2m.$$

Moreover,

$$u \mapsto \|u\|_p + \|\partial_t u\|_p + \sum_{k=1}^{2m} \|\nabla_x^k u\|_p$$

is an equivalent norm for $W_p^{(2m,1)}(\mathbb{R}^d, E)$.

Similarly, $u \in BUC^{(2m,1)}(\mathbb{R}^d, E)$, respectively $u \in C_0^{(2m,1)}(\mathbb{R}^d, E)$, iff u, $\partial_t u$, and $\partial_x^\alpha u$ with $|\alpha| \leq 2m$ belong to $BUC(\mathbb{R}^d, E)$, respectively $C_0(\mathbb{R}^d, E)$, and

$$u \mapsto \|u\|_\infty + \|\partial_t u\|_\infty + \sum_{k=1}^{2m} \|\nabla_x^k u\|_\infty$$

is an equivalent norm. □

The following 'sandwich theorem' gives important inclusions between Besov and classical function spaces.

3.5.2 Theorem *If $k \in \mathbb{N}$, then*

$$B_{p,1}^{k\nu/\boldsymbol{\nu}} \overset{d}{\hookrightarrow} W_p^{k\nu/\boldsymbol{\nu}} \overset{d}{\hookrightarrow} \overset{\scriptscriptstyle\triangle}{B}{}_{p,\infty}^{k\nu/\boldsymbol{\nu}} \hookrightarrow B_{p,\infty}^{k\nu/\boldsymbol{\nu}}, \qquad p \neq \infty, \tag{3.5.1}$$

and

$$B_{\infty,1}^{k\nu/\boldsymbol{\nu}} \hookrightarrow BUC^{k\nu/\boldsymbol{\nu}} \hookrightarrow B_{\infty,\infty}^{k\nu/\boldsymbol{\nu}} \tag{3.5.2}$$

and

$$\overset{\scriptscriptstyle\triangle}{B}{}_{\infty,1}^{k\nu/\boldsymbol{\nu}} \overset{d}{\hookrightarrow} C_0^{k\nu/\boldsymbol{\nu}} \overset{d}{\hookrightarrow} \overset{\scriptscriptstyle\triangle}{B}{}_{\infty,\infty}^{k\nu/\boldsymbol{\nu}}. \tag{3.5.3}$$

Proof. It follows from $u = \sum_k \psi_k(D)u$ for $u \in \mathcal{S}'$ that

$$\|u\|_p \leq \sum_k \|\psi_k(D)u\|_p = \|u\|_{B_{p,1}^{0/\boldsymbol{\nu}}}.$$

Consequently,

$$B_{p,1}^{0/\boldsymbol{\nu}} \hookrightarrow L_p. \tag{3.5.4}$$

If $u \in B_{\infty,1}^{0/\boldsymbol{\nu}}$, then $\psi_k(D)u \in L_\infty$. Hence, recalling (3.3.6), it follows from (3.3.4) that

$$\psi_k(D)u = \psi_k(D)\chi_k(D)u \in BUC,$$

setting $a = 1$ in Proposition 3.2.1. This improves (3.5.4) in the case $p = \infty$ to

$$B_{\infty,1}^{0/\boldsymbol{\nu}} \hookrightarrow BUC.$$

On the other side, using Proposition 3.2.1 once more,

$$\|u\|_{B_{p,\infty}^{0/\boldsymbol{\nu}}} = \sup_k \|\psi_k(D)u\|_p \leq c\,\|u\|_p.$$

This proves $L_p \hookrightarrow B_{p,\infty}^{0/\boldsymbol{\nu}}$ and, as above, $BUC \hookrightarrow B_{\infty,\infty}^{0/\boldsymbol{\nu}}$. Thus (3.5.1) and (3.5.2) are true for $k = 0$. Moreover, (3.5.3) follows for $k = 0$ from this and (3.3.11).

The assertion for $k > 0$ is now obtained by Theorems 2.3.8 and 3.4.2, observing that the density of $\mathcal{S}$ in $\mathfrak{F}$ is not essential in the proof of Theorem 2.3.8. $\Box$

The following proposition shows that anisotropic Besov spaces of positive order can be obtained by interpolation from classical function spaces.

3.5.3 Proposition *Suppose* $0 < s < k\nu$ *and* $k \in \mathbb{N}$. *Then*

$$B_{p,q}^{s/\boldsymbol{\nu}} \doteq (L_p, W_p^{k\nu/\boldsymbol{\nu}})_{s/k\nu,q}, \qquad p \neq \infty, \tag{3.5.5}$$

and

$$B_{\infty,q}^{s/\boldsymbol{\nu}} \doteq (BUC, BUC^{k\nu/\boldsymbol{\nu}})_{s/k\nu,q}, \tag{3.5.6}$$

and

$$\overset{\scriptscriptstyle\triangle}{B}{}_{\infty,\infty}^{s/\boldsymbol{\nu}} \doteq (C_0, C_0^{k\nu/\boldsymbol{\nu}})_{s/k\nu,\infty}^0. \tag{3.5.7}$$

Proof. This is implied by (3.3.12), (3.3.14), and the preceding theorem. $\Box$

3.6 Renorming theorems

First we introduce equivalent norms for Besov spaces of positive order. To do this we need some preparation.

We denote by $\{\,\tau_h\ ;\ h \in \mathbb{R}^d\,\}$ the translation group, defined on $\mathcal{S}$ by

$$(\tau_h u)(x) := u(x+h), \qquad x \in \mathbb{R}^d, \quad u \in \mathcal{S},$$

and extended to $\mathcal{S}'$ by

$$(\tau_h u)(\varphi) := u(\tau_{-h}\varphi), \qquad u \in \mathcal{S}', \quad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Then the difference operators

$$\triangle_h := \tau_h - 1, \quad \triangle_h^{k+1} := \triangle_h \triangle_h^k, \qquad k \in \dot{\mathbb{N}},$$

are well-defined on $\mathcal{S}'$.

For $h = (h_1, \dots, h_\ell) \in \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_\ell}$ we set $\check{h}_i := (0, \dots, 0, h_i, 0, \dots, 0)$. If $s > 0$, then we put[2]

$$[u]_{s/\boldsymbol{\nu},p,q} := \Bigl(\sum_{i=1}^{\ell} \bigl\|\, |h_i|^{-s/\nu_i}\, \|\triangle_{\check{h}_i}^{[s/\nu_i]+1} u\|_p \,\bigr\|_{L_q((\mathbb{R}^{d_i})^\bullet, dh_i/|h_i|^{d_i})}^q\Bigr)^{1/q} \tag{3.6.1}$$

if $q \neq \infty$, and

$$[u]_{s/\boldsymbol{\nu},p,\infty} := \max_{1 \le i \le \ell} \bigl\|\, |h_i|^{-s/\nu_i}\, \|\triangle_{\check{h}_i}^{[s/\nu_i]+1} u\|_p \,\bigr\|_\infty. \tag{3.6.2}$$

Then the following important renorming theorem is valid.

3.6.1 Theorem *Suppose* $s > 0$. *Then*

$$\|\cdot\|_{B_{p,q}^{s/\boldsymbol{\nu}}} \sim \|\cdot\|_p + [\cdot]_{s/\boldsymbol{\nu},p,q}.$$

---

[2] $[\xi]$ is the greatest integer less than or equal to ξ for $\xi \in \mathbb{R}^+$.

PROOF. Let $e_1, \ldots, e_d$ be the standard basis of $\mathbb{R}^d$. It is not difficult to see that $\{\, \tau_{te_j} \ ;\ t \geq 0 \,\}$, $1 \leq j \leq d$, is a pair-wise commuting family of strongly continuous contraction semigroups on $\mathfrak{F} \in \{\, BUC, L_p \ ;\ 1 \leq p < \infty \,\}$. Moreover, A_j, the infinitesimal generator of $\{\, \tau_{te_j} \ ;\ t \geq 0 \,\}$, is the $\mathfrak{F}$-realization of $\partial_j \in \mathcal{L}(\mathcal{S}')$. Using this, Theorem 1 in H.-J. Schmeißer and H. Triebel [56], Proposition 3.5.3, and Theorem 2.5.1 of H. Triebel [65] we obtain the assertion. $\square$

3.6.2 Remark The integer $[s/\nu_i] + 1$ in (3.6.1) and (3.6.2) has been chosen to avoid a further parameter dependence. It could be replaced by any $k_i \in \mathbb{N}$ satisfying $k_i > s/\nu_i$. $\square$

For $1 \leq i \leq \ell$ we set

$$\mathbb{R}^{(d,d_i)} := \mathbb{R}^{d_1} \times \cdots \times \widehat{\mathbb{R}^{d_i}} \times \cdots \times \mathbb{R}^{d_\ell}$$

where, as usual and in related situations, the hat over a factor (or component) means that the corresponding entry is absent. The general point of $\mathbb{R}^{(d,d_i)}$ is denoted by

$$x_{\widehat{\imath}} := (x_1, \ldots, \widehat{x_i}, \ldots, x_\ell), \tag{3.6.3}$$

and we put

$$u(x_{\widehat{\imath}}, \cdot) := u(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_\ell). \tag{3.6.4}$$

If $\ell = 1$, then $d_\ell = d$ and we set $\mathbb{R}^{(d,d)} := \{0\} = \mathbb{R}^0$. Note

$$\mathfrak{F}(\mathbb{R}^0, E) := E \text{ for } \mathfrak{F} \in \{L_p, BUC, C_0\}.$$

Suppose $u \in L_p(\mathbb{R}^d, E)$ with $p \neq \infty$. Then, by Fubini's theorem,

$$u(x_{\widehat{\imath}}, \cdot) \in L_p(\mathbb{R}^{d_i}, E) \qquad \text{a.a. } x_{\widehat{\imath}} \in \mathbb{R}^{(d,d_i)},$$

and

$$U_i u := \big(x_{\widehat{\imath}} \mapsto u(x_{\widehat{\imath}}, \cdot)\big) \in L_p\big(\mathbb{R}^{(d,d_i)}, L_p(\mathbb{R}^{d_i}, E)\big).$$

In fact,

$$U_i :\ L_p(\mathbb{R}^d, E) \to L_p\big(\mathbb{R}^{(d,d_i)}, L_p(\mathbb{R}^{d_i}, E)\big)$$

is an isometric isomorphism. This is not true if $p = \infty$ (e.g., H. Amann and J. Escher [11, Theorem X.6.22 and Remark X.6.23]). However, it is not difficult to see that

$$U_i :\ C_0(\mathbb{R}^d, E) \to C_0\big(\mathbb{R}^{(d,d_i)}, C_0(\mathbb{R}^{d_i}, E)\big) \tag{3.6.5}$$

is an isometric isomorphism.

To simplify the writing we set

$$B_p^{s/\boldsymbol{\nu}} = B_p^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E) := B_{p,p}^{s/\boldsymbol{\nu}}$$

and, if $s > 0$,

$$[\cdot]_{s/\boldsymbol{\nu},p} := [\cdot]_{s/\boldsymbol{\nu},p,p}.$$

The next theorem shows that $B_p^{s/\boldsymbol{\nu}}$ can be characterized as an intersection space for $s > 0$ (cf. S.M. Nikol′skiĭ [51] for related results in the scalar case).

3.6.3 Theorem *If $s > 0$, then*

$$B_p^{s/\boldsymbol{\nu}} \doteq \bigcap_{i=1}^{\ell} L_p\big(\mathbb{R}^{(d,d_i)}, B_p^{s/\nu_i}(\mathbb{R}^{d_i}, E)\big), \qquad p \neq \infty,$$

and

$$\overset{\scriptscriptstyle\triangle}{B}{}_\infty^{s/\boldsymbol{\nu}} \doteq \bigcap_{i=1}^{\ell} C_0\big(\mathbb{R}^{(d,d_i)}, \overset{\scriptscriptstyle\triangle}{B}{}_\infty^{s/\nu_i}(\mathbb{R}^{d_i}, E)\big).$$

Proof. Set $k_i := [s/\nu_i] + 1$. If $p \neq \infty$, then, by Fubini's theorem,

$$\begin{aligned}\Big\| \, |h_i|^{-s/\nu_i} \, \|\triangle_{\check{h}_i}^{k_i} u\|_p \, \Big\|_{L_p((\mathbb{R}^{d_i})^\bullet, dh_i/|h_i|^{d_i})} \\ = \Big(\int_{\mathbb{R}^{(d,d_i)}} \big[u(x_{\hat{\imath}}, \cdot)\big]_{s/\nu_i,p}^p \, dx_{\hat{\imath}}\Big)^{1/p}.\end{aligned} \tag{3.6.6}$$

This implies

$$\|\cdot\|_p + [u]_{s/\boldsymbol{\nu},p} \sim \max_{1 \leq i \leq \ell} \|\cdot\|_{L_p(\mathbb{R}^{(d,d_i)}, B_p^{s/\nu_i})}$$

so that the first assertion follows.

Similarly,

$$\Big\| \, |h_i|^{-s/\nu_i} \, \|\triangle_{\check{h}_i}^{k_i} u\|_\infty \, \Big\|_{L_\infty((\mathbb{R}^{d_i})^\bullet)} = \sup_{x_{\hat{\imath}} \in \mathbb{R}^{(d,d_i)}} \big[u(x_{\hat{\imath}}, \cdot)\big]_{s/\nu_i,\infty}. \tag{3.6.7}$$

From this and (3.6.5) we obtain the second assertion. □

Let $E_1, \ldots, E_n$ be Banach spaces satisfying $E_i \hookrightarrow \mathcal{Y}$ for some LCS $\mathcal{Y}$. The **sum space**, ΣE_i, of $E_1, \ldots, E_n$ is defined by

$$\Sigma E_i := \sum_{i=1}^n E_i := \{\, y \in \mathcal{Y} \; ; \; \exists \, x_i \in E_i \text{ with } y = x_1 + \cdots + x_n \,\}$$

and is equipped with the norm

$$y \mapsto \|y\|_{\Sigma E_i} := \inf\Big\{ \sum_{i=1}^n \|x_i\|_{E_i} \; ; \; y = x_1 + \cdots + x_n \Big\}.$$

For the reader's convenience we include a precise formulation and a proof of the following duality theorem which seems to be folklore and is often rather vaguely stated (see, e.g., Theorem 2.7.1 in J. Bergh and J. Löfström [16]).

3.6.4 Proposition *Let $\mathcal{X}$ be an LCS such that $\mathcal{X} \overset{d}{\hookrightarrow} E_i$ for $1 \leq i \leq n$.*

(i) *$E_i' \hookrightarrow \mathcal{X}'$, so that $\Sigma E_i'$ is well-defined and*

$$\Sigma E_i' = (\cap E_i)'$$

with respect to the duality pairing

$$\langle x', x \rangle_{\cap E_i} := \langle x_1', x \rangle_{E_1} + \cdots + \langle x_n', x \rangle_{E_n}, \qquad (x', x) \in \Sigma E_i' \times \cap E_i$$

and any $x_i' \in E_i'$ with $x_1' + \cdots + x_n' = x'$.

(ii) *If $E_1, \ldots, E_n$ are reflexive, then $\cap E_i$ and ΣE_i are so too.*

Proof. (1) Set $F_\infty := \prod_{i=1}^n E_i$, endowed with the ℓ_∞-norm. Denote by M the closed subspace of all $(x_1, \ldots, x_n)$ satisfying $x_1 = \cdots = x_n$. Then

$$f : M \to \mathcal{Y}, \quad (y, y, \ldots, y) \mapsto y$$

is a continuous linear map whose image equals $\bigcap_{i=1}^n E_i$. Hence f is an isometric isomorphism from M onto $\cap E_i$.

(2) Put $F_1 := \prod_{i=1}^n E_i$ and equip it with the ℓ_1-norm. Then

$$\mathrm{add} : F_1 \to \mathcal{Y}, \quad (x_1, \ldots, x_n) \mapsto \sum_{i=1}^n x_i$$

is a continuous linear map and $\Sigma E_i = \mathrm{add}\, F_1$. Hence ΣE_i is a Banach space by Remark 2.2.1(a).

(3) Let the hypotheses of (i) be satisfied. Set $F_1^\sharp := \prod_{i=1}^n E_i'$, endowed with the ℓ_1-norm. Then $F_1^\sharp = (F_\infty)'$ by means of the duality pairing

$$\langle y^\sharp, y \rangle_{F_\infty} = \langle y_1^\sharp, y_1 \rangle_{E_1} + \cdots + \langle y_n^\sharp, y_n \rangle_{E_n}, \qquad (y^\sharp, y) \in F_1^\sharp \times F_1.$$

By step (1),

$$\cap E_i \xleftarrow[\cong]{f} M \overset{i}{\hookrightarrow} F_\infty$$

where $i(M) = M$ is closed in F_∞. Thus, by duality,

$$F_1^\sharp \xrightarrow{i'} M' \xleftarrow[\cong]{f'} (\cap E_i)'.$$

From $i(m) = m$ for $m \in M$ and $i'(y')(m) = y'(im) = y'(m)$ it follows that $i'(y')$ is the restriction $y' | M$ to M for $y' \in F_1^\sharp$.

Since i is injective and has closed range,

$$\mathrm{im}(i') = \ker(i)^\perp = M', \quad \ker(i') = \mathrm{im}(i)^\perp = M^\perp.$$

Hence there exists a unique $\psi^\sharp$ for which the diagram

$$\begin{tikzcd}
F_1^\sharp \arrow[rr, "i'"] \arrow[dr] & & M' \\
& F_1^\sharp / M^\perp \arrow[ur, "\cong", "\psi^\sharp"'] &
\end{tikzcd}$$

is commuting.

(4) Suppose $x' \in \big(\cap E_i\big)'$ and set $m' = f'(x') \in M'$. Then

$$\begin{aligned}
\langle x', x \rangle_{\cap E_i} &= \big\langle {f'}^{-1}(m'), x \big\rangle_{\cap E_i} = \big\langle m', f^{-1}(x) \big\rangle_M \\
&= \langle y_1^\sharp, x \rangle_{E_1} + \cdots + \langle y_n^\sharp, x \rangle_{E_n}
\end{aligned} \tag{3.6.8}$$

for $x \in \cap E_i$ and $y^\sharp = (y_1^\sharp, \ldots, y_n^\sharp) \in F_1^\sharp$ with $y^\sharp | M = m'$. Note that $E_i' \hookrightarrow \mathcal{X}'$ implies

$$\langle e_i', x \rangle_{E_i} = \langle e_i', x \rangle_{\mathcal{X}}, \qquad x \in \mathcal{X}.$$

Hence it follows from $\mathcal{X} \subset \cap E_i$ and (3.6.8) that

$$\langle x', x\rangle_{\cap E_i} = \langle y_1^\sharp + \cdots + y_n^\sharp, x\rangle_{\mathcal{X}}, \qquad x \in \mathcal{X}. \tag{3.6.9}$$

Thus, if $\widetilde{y}^\sharp \in F_1^\sharp$ also satisfies $\widetilde{y}^\sharp \,|\, M = m'$, we find

$$y_1^\sharp + \cdots + y_n^\sharp = \widetilde{y}_1^\sharp + \cdots + \widetilde{y}_n^\sharp.$$

(5) Since f^{-1} is an isometry from $\cap E_i$ onto M we obtain from

$$\langle x', x\rangle_{\cap E_i} = \big\langle m', f^{-1}(x)\big\rangle_M$$

that

$$\|m'\|_{M'} = \|x'\|_{(\cap E_i)'}.$$

The Hahn-Banach theorem guarantees the existence of $m^\sharp \in F_1^\sharp$ with $m^\sharp \,|\, M = m'$ and $\|m^\sharp\|_{F_1^\sharp} = \|m'\|_{M'}$. Set $y' := m_1^\sharp + \cdots + m_n^\sharp$. Then $y' \in \Sigma E_i'$ and

$$\|y'\|_{\Sigma E_i'} \le \|m^\sharp\|_{F_1^\sharp} = \|x'\|_{(\cap E_i)'}.$$

Hence we see from (4) that

$$g : \ (\cap E_i)' \to \Sigma E_i', \quad x' \mapsto y'$$

is a well-defined bounded linear map of norm at most 1.

(6) Suppose $y' \in \Sigma E_i'$ and $y^\sharp \in F_1^\sharp$ satisfy $y' = y_1^\sharp + \cdots + y_n^\sharp$. Then

$$h(y')(x) := \langle y_1^\sharp, x\rangle_{E_1} + \cdots + \langle y_n^\sharp, x\rangle_{E_n}, \qquad x \in \cap E_i,$$

defines

$$h(y') \in (\cap E_i)', \quad \|h(y')\|_{(\cap E_i)'} \le \|y^\sharp\|_{F_1^\sharp}.$$

This being true for every such $y^\sharp$, it follows that h maps $\Sigma E_i'$ onto $(\cap E_i)'$ and has norm at most 1. Combining this with the result of (5) proves claim (i).

(7) Let $E_1, \ldots, E_n$ be reflexive. Then M, being a closed linear subspace of the reflexive Banach space F_∞, is reflexive. Hence its isomorphic image $\cap E_i$ is also reflexive. Now (i) implies the reflexivity of ΣE_i. □

In order to apply this proposition we first prove a density theorem which will repeatedly be useful.

3.6.5 Lemma *Suppose $m, n \in \dot{\mathbb{N}}$ and $p \ne \infty$. Let $\mathfrak{F}$ be a Banach space with*

$$\mathcal{S}(\mathbb{R}^n, E) \overset{d}{\hookrightarrow} \mathfrak{F} \overset{d}{\hookrightarrow} \mathcal{S}'(\mathbb{R}^n, E).$$

Then $\mathcal{D}(\mathbb{R}^{m+n}, E)$ is dense in $L_p(\mathbb{R}^m, \mathfrak{F})$.

Proof. Let $\varepsilon > 0$ and $u \in L_p(\mathbb{R}^m, \mathfrak{F})$ be given. Since $\mathcal{D}(\mathbb{R}^m, \mathfrak{F})$ is dense in $L_p(\mathbb{R}^m, \mathfrak{F})$ there exists $v \in \mathcal{D}(\mathbb{R}^m, \mathfrak{F})$ with

$$\|u - v\|_{L_p(\mathbb{R}^m, \mathfrak{F})} < \varepsilon/2.$$

Set $K := \mathrm{supp}(v)$, denote by C_K the measure of the 1-neighborhood of K in $\mathbb{R}^m$, and fix $C_1 > 2C_K^{1/p}$. By continuity and compactness we find $\delta \in (0,1)$ and $y_1, \ldots, y_r$ in K such that

$$\|v(y) - v(y_k)\|_{\mathfrak{F}} < \varepsilon/C_1, \qquad y \in U_k := \{\, z \in \mathbb{R}^m \ ; \ |z - y_k| < \delta \,\},$$

and such that $\{ U_k \ ;\ 1 \le k \le r \}$ is an open covering of K. By the density of $\mathcal{D}(\mathbb{R}^n, E)$ in $\mathcal{S}(\mathbb{R}^n, E)$, hence in $\mathfrak{F}$, we can assume $v(y_k) \in \mathcal{D}(\mathbb{R}^n, E)$. Denote by $\{ \psi_k \ ;\ 1 \le k \le r \}$ a smooth partition of unity on K subordinate to this covering. Then

$$w := \sum \psi_k v(y_k) \in \mathcal{D}(\mathbb{R}^m, \mathfrak{F}).$$

Since $w - v = \sum \psi_k \bigl(v(y_k) - v\bigr)$ and $0 \le \psi_k \le 1$, it follows

$$\|w(x) - v(x)\|_{\mathfrak{F}} \le (\varepsilon/C_1) \sum \psi_k(x) \le \varepsilon/C_1, \qquad x \in \bigcup U_k,$$

hence

$$\|w - v\|_{L_p(\mathbb{R}^m, \mathfrak{F})} \le (\varepsilon/C_1) C_K^{1/p} < \varepsilon/2.$$

Due to $w \in \mathcal{D}(\mathbb{R}^{m+n}, E)$, this proves the claim. □

By means of this result and duality we can now give another representation for some Besov spaces of negative order.

3.6.6 Theorem *Let E be reflexive and suppose $1 < p < \infty$ and $s > 0$. Then*

$$B_p^{-s/\boldsymbol{\nu}} \doteq \sum_{i=1}^{\ell} L_p\bigl(\mathbb{R}^{(d,d_i)}, B_p^{-s/\nu_i}(\mathbb{R}^{d_i}, E)\bigr).$$

Proof. This is a consequence of Duality Theorem 3.3.3, Theorem 3.6.3, and Proposition 3.6.4. □

If p is finite, then there is still another useful representation of $B_{p,q}^{s/\boldsymbol{\nu}}$. For this we set

$$\boldsymbol{\omega}' := (\omega_2, \ldots, \omega_d),$$

provided $d \ge 2$, of course.

3.6.7 Theorem *Suppose $s > 0$. Then*[3]

$$B_p^{s/\boldsymbol{\nu}} \doteq B_p^{s/\nu_1}\bigl(\mathbb{R}, L_p(\mathbb{R}^{d-1}, E)\bigr) \cap L_p\bigl(\mathbb{R}, B_p^{s/\boldsymbol{\omega}'}(\mathbb{R}^{d-1}, E)\bigr), \qquad p \ne \infty,$$

and

$$\overset{\vartriangle}{B}{}_\infty^{s/\boldsymbol{\nu}} \doteq \overset{\vartriangle}{B}{}_\infty^{s/\nu_1}\bigl(\mathbb{R}, C_0(\mathbb{R}^{d-1}, E)\bigr) \cap C_0\bigl(\mathbb{R}, \overset{\vartriangle}{B}{}_\infty^{s/\boldsymbol{\omega}'}(\mathbb{R}^{d-1}, E)\bigr).$$

If E is reflexive and $1 < p < \infty$, then

$$B_p^{-s/\boldsymbol{\nu}} \doteq B_p^{-s/\nu_1}\bigl(\mathbb{R}, L_p(\mathbb{R}^{d-1}, E)\bigr) + L_p\bigl(\mathbb{R}, B_p^{-s/\boldsymbol{\omega}'}(\mathbb{R}^{d-1}, E)\bigr).$$

Proof. The first assertion is clear by Theorems 3.6.1 and 3.6.3 (by (3.6.6) and (3.6.7) in particular), and by the density of $\mathcal{S}\bigl(\mathbb{R}, \mathcal{S}(\mathbb{R}^{d-1}, E)\bigr)$ in $\mathcal{S}(\mathbb{R}^d, E)$ which is easily verified (cf. Lemma 1.3.7 in H. Amann [10]). The second claim follows from Theorem 3.3.3, Proposition 3.6.4, and Lemma 3.6.5. □

Clearly, there is nothing which singles the first coordinate out. Thus everything above remains valid mutatis mutandis if we replace x^1 by another coordinate x^j. Such a **relabeling of coordinates** will be frequently used in what follows, often without explicit mention, as in the following example.

[3] It should be observed that for this theorem vector-valued Besov spaces are needed, even in the scalar case $E = \mathbb{C}$.

3.6.8 Example Let $\boldsymbol{\omega}$ be the $2m$-parabolic weight vector. Suppose $s > 0$. Then

$$B_p^{(s,s/2m)} \doteq L_p\big(\mathbb{R}, B_p^s(\mathbb{R}^{d-1}, E)\big) \cap B_p^{s/2m}\big(\mathbb{R}, L_p(\mathbb{R}^{d-1}, E)\big), \qquad p \neq \infty,$$

and

$$\overset{\scriptscriptstyle\vartriangle}{B}{}_\infty^{(s,s/2m)} \doteq C_0\big(\mathbb{R}, \overset{\scriptscriptstyle\vartriangle}{B}{}_\infty^s(\mathbb{R}^{d-1}, E)\big) \cap \overset{\scriptscriptstyle\vartriangle}{B}{}_\infty^{s/2m}\big(\mathbb{R}, C_0(\mathbb{R}^{d-1}, E)\big).$$

Furthermore,

$$B_p^{-(s,s/2m)} \doteq L_p\big(\mathbb{R}, B_p^{-s}(\mathbb{R}^{d-1}, E)\big) + B_p^{-s/2m}\big(\mathbb{R}, L_p(\mathbb{R}^{d-1}, E)\big),$$

provided $1 < p < \infty$ and E is reflexive. □

3.7 Bessel potential spaces

Throughout this section we suppose

- *$p, p_0, p_1 \in (1, \infty)$, $\quad s, s_0, s_1, t \in \mathbb{R}$;*
- *E is a UMD space which has property (α) if $\boldsymbol{\omega} \neq \omega(1, \dots, 1)$.*

Anisotropic Bessel potential spaces are defined by[4]

$$H_p^{s/\boldsymbol{\nu}} = H_p^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E) := J^{-s} L_p. \tag{3.7.1}$$

In other words, $[\, H_p^{s/\boldsymbol{\nu}} \ ;\ s \in \mathbb{R}\,]$ is the fractional power scale generated by (L_p, J).

3.7.1 Theorem

(i) *$H_p^{s/\boldsymbol{\nu}}$ is an $\mathcal{M}$-admissible reflexive Banach space and*

$$H_p^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E)' = H_{p'}^{-s/\boldsymbol{\nu}}(\mathbb{R}^d, E')$$

with respect to the L_p-duality pairing.

(ii) *$H_p^{k\nu/\boldsymbol{\nu}} \doteq W_p^{k\nu/\boldsymbol{\nu}}$ for $k \in \mathbb{N}$.*

(iii) *$B_{p,1}^{s/\boldsymbol{\nu}} \overset{d}{\hookrightarrow} H_p^{s/\boldsymbol{\nu}} \overset{d}{\hookrightarrow} \overset{\scriptscriptstyle\vartriangle}{B}{}_{p,\infty}^{s/\boldsymbol{\nu}}$.*

(iv) *For $s_0 \neq s_1$ and $0 < \theta < 1$*

$$[H_p^{s_0/\boldsymbol{\nu}}, H_p^{s_1/\boldsymbol{\nu}}]_\theta \doteq H_p^{s_\theta/\boldsymbol{\nu}}$$

and

$$(H_p^{s_0/\boldsymbol{\nu}}, H_p^{s_1/\boldsymbol{\nu}})_{\theta,q} \doteq B_{p,q}^{s_\theta/\boldsymbol{\nu}}, \qquad 1 \leq q \leq \infty.$$

Proof. (i) The first assertion follows from Theorems 2.3.2 and 3.1.2. The proof of the duality assertion is similar to the one for Besov spaces and will also be given in Volume II of H. Amann [4]. The reflexivity is then a consequence of the one of E.

(ii) is a consequence of Theorem 2.3.8.

[4]This definition makes sense for arbitrary Banach spaces, and much of the theory developed below remains true in such a general situation (cf. H.-J. Schmeißer and W. Sickel [54]). A noteworthy exception is (ii) of Theorem 3.7.1. Since this property renders the Bessel potential scale useful in practice, we restrict ourselves throughout to the UMD space case.

(iii) Theorem 3.5.2 guarantees

$$B^0_{p,1} \overset{d}{\hookrightarrow} L_p \overset{d}{\hookrightarrow} \overset{\vartriangle}{B}{}^0_{p,\infty}.$$

From Theorem 3.4.1 we infer $B^s_{p,q} \doteq J^{-s} B^0_{p,q}$ for $1 \le q \le \infty$. Now the claim is clear.

(iv) This statement follows from (iii), Theorem 2.3.2(v), and (3.3.12). □

Our next theorem is an analogue for anisotropic Bessel potential spaces to the characterizations of anisotropic Besov spaces given in Theorems 3.6.3 and 3.6.6.

3.7.2 Theorem

$$H_p^{s/\boldsymbol{\nu}} \doteq \begin{cases} \displaystyle\bigcap_{i=1}^{\ell} L_p\big(\mathbb{R}^{(d,d_i)}, H_p^{s/\nu_i}(\mathbb{R}^{d_i}, E)\big), & s > 0, \\ \displaystyle\sum_{i=1}^{\ell} L_p\big(\mathbb{R}^{(d,d_i)}, H_p^{s/\nu_i}(\mathbb{R}^{d_i}, E)\big), & s < 0. \end{cases}$$

Proof. (1) Fix $s > 0$. For $1 \le i \le \ell$ set

$$a_i(\xi_i) := (1 + |\xi_i|^{2\nu/\nu_i})^{s/2\nu}, \qquad \xi_i \in \mathbb{R}^{d_i}.$$

Note $a_i(\xi_i) \sim \langle \xi_i \rangle^{s/\nu_i}$. Hence

$$a_i(D_{x_i}) \in \mathcal{L}\mathrm{is}\big(H_p^{s/\nu_i}(\mathbb{R}^{d_i}, E), L_p(\mathbb{R}^{d_i}, E)\big).$$

Denote by A_i the point-wise extension of $a_i(D_{x_i})$ over $L_p\big(\mathbb{R}^{(d,d_i)}, H_p^{s/\nu_i}(\mathbb{R}^{d_i}, E)\big)$. Then

$$A_i \in \mathcal{L}\mathrm{is}\Big(L_p\big(\mathbb{R}^{(d,d_i)}, H_p^{s/\nu_i}(\mathbb{R}^{d_i}, E)\big), L_p(\mathbb{R}^d, E)\Big).$$

(2) Define

$$a(\xi) := a_1(\xi_1) + \cdots + a_\ell(\xi_\ell), \qquad \xi = (\xi_1, \ldots, \xi_\ell) \in \mathbb{R}^d = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_\ell}.$$

On the basis of Lemmas 1.4.1(i) and 1.4.3 one verifies

$$a/\Lambda_1^s, \Lambda_1^s/a \in \mathcal{M}(\mathbb{R}^d).$$

Consequently,

$$\begin{aligned} \|u\|_{H_p^{s/\nu}} = \|J^s u\|_p &\le c\,\|\Lambda_1^s/a\|_{\mathcal{M}}\,\|a(D)u\|_p \le c\sum_{i=1}^{\ell} \|a_i(D_{x_i})u\|_p \\ &= c\sum_{i=1}^{\ell} \|A_i u\|_p \le c\sum_{i=1}^{\ell} \|u\|_{L_p(\mathbb{R}^{(d,d_i)}, H_p^{s/\nu_i}(\mathbb{R}^{d_i}, E))}. \end{aligned}$$

This shows

$$I := \bigcap_{i=1}^{\ell} L_p\big(\mathbb{R}^{(d,d_i)}, H_p^{s/\nu_i}(\mathbb{R}^{d_i}, E)\big) \hookrightarrow H_p^{s/\boldsymbol{\nu}}.$$

(3) We also find $a_i/\Lambda_1^s \in \mathcal{M}(\mathbb{R}^d)$ for $1 \le i \le \ell$. Thus, similarly as in the preceding step,

$$\|A_i u\|_p \le c\,\|J^s u\|_p = c\,\|u\|_{H_p^{s/\nu}}.$$

Hence

$$\sum_{i=1}^{\ell} \|A_i u\|_p \le c\,\|u\|_{H_p^{s/\boldsymbol{\nu}}}$$

so that $H_p^{s/\boldsymbol{\nu}} \hookrightarrow I$. This proves the theorem for $s > 0$.

(4) If $s < 0$, then the claim follows from what has just been shown, Proposition 3.6.4, Theorem 3.7.1(i), and Lemma 3.6.5. □

There is an analogue to Theorem 3.6.7 for Bessel potential spaces.

3.7.3 Theorem *If* $s > 0$, *then*

$$H_p^{s/\boldsymbol{\nu}} \doteq L_p\big(\mathbb{R}, H_p^{s/\boldsymbol{\omega}'}(\mathbb{R}^{d-1}, E)\big) \cap H_p^{s/\nu_1}\big(\mathbb{R}, L_p(\mathbb{R}^{d-1}, E)\big)$$

and

$$H_p^{-s/\boldsymbol{\nu}} \doteq L_p\big(\mathbb{R}, H_p^{-s/\boldsymbol{\omega}'}(\mathbb{R}^{d-1}, E)\big) + H_p^{-s/\nu_1}\big(\mathbb{R}, L_p(\mathbb{R}^{d-1}, E)\big).$$

Proof. Since

$$\|u(\cdot, x')\|_{H_p^{s/\nu_1}(\mathbb{R},E)} = \big\|\langle D_1\rangle^{s/\nu_1} u(\cdot, x')\big\|_{L_p(\mathbb{R},E)}, \qquad x' \in \mathbb{R}^{d-1},$$

Fubini's theorem implies

$$\begin{aligned}
\|u\|^p_{L_p(\mathbb{R}^{d-1}, H_p^{s/\nu_1}(\mathbb{R},E))} &= \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \big|\langle D_1\rangle^{s/\nu_1} u(x^1, x')\big|^p \,dx^1\,dx' \\
&= \int_{\mathbb{R}} \big\|\langle D_1\rangle^{s/\nu_1} u(x^1, \cdot)\big\|^p_{L_p(\mathbb{R}^{d-1})}\,dx^1 \\
&= \|u\|_{H_p^{s/\nu_1}(\mathbb{R}, L_p(\mathbb{R}^{d-1}, E))}.
\end{aligned}$$

Hence Theorem 3.7.2 implies the assertion. □

3.7.4 Example Let $\boldsymbol{\omega}$ be the $2m$-parabolic weight vector. Then

$$H_p^{(s,s/2m)}(\mathbb{R}^d, E) \doteq L_p\big(\mathbb{R}, H_p^{s}(\mathbb{R}^{d-1}, E)\big) \cap H_p^{s/2m}\big(\mathbb{R}, L_p(\mathbb{R}^{d-1}, E)\big)$$

and

$$H_p^{-(s,s/2m)}(\mathbb{R}^d, E) \doteq L_p\big(\mathbb{R}, H_p^{-s}(\mathbb{R}^{d-1}, E)\big) + H_p^{-s/2m}\big(\mathbb{R}, L_p(\mathbb{R}^{d-1}, E)\big)$$

for $s > 0$. If $s = 2mk$ for some $k \in \dot{\mathbb{N}}$, then

$$u \mapsto \|u\|_p + \|\partial_t^k u\|_p + \|\nabla_{x'}^{2mk} u\|_p$$

is an equivalent norm for $H_p^{(2mk,k)}(\mathbb{R}^d, E)$.

Proof. This follows from Corollary 2.3.9 and Theorems 3.7.1(ii) and 3.7.3. □

As a first application of this Theorem 3.7.2 we prove a Sobolev-type embedding theorem for anisotropic Bessel potential scales, an analogue to (3.3.9).

3.7.5 Theorem *Suppose*

$$s_1 - |\boldsymbol{\omega}|/p_1 = s_0 - |\boldsymbol{\omega}|/p_0, \qquad s_1 > s_0.$$

Then $H_{p_1}^{s_1/\boldsymbol{\nu}} \hookrightarrow H_{p_0}^{s_0/\boldsymbol{\nu}}$.

PROOF. (1) Suppose $\boldsymbol{\omega}$ is isotropic. Then the assertion has been shown (even without assuming that E is a UMD space) by H.-J. Schmeißer and W. Sickel [54].

(2) Suppose $s_0 > 0$. Then we obtain the claim from Theorem 3.7.2 and the isotropic case.

(3) If $s_0 \le 0$, fix $t > -s_0$. Then $J^t \in \mathcal{L}\mathrm{is}(H_{p_j}^{s_j+t}, H_{p_j}^{s_j})$ for $j = 0, 1$. Thus the statement is a consequence of step (2). □

In the scalar case the results of this section are well-known, except perhaps for the second part of Theorem 3.7.3; see S.M. Nikol′skiĭ [51] and P.I. Lizorkin [48], and the references therein.

3.8 Sobolev–Slobodeckii and Nikol′skiĭ scales

In this section

- *E is a UMD space which possesses property (α) if $\boldsymbol{\omega} \neq \omega(1, \dots, 1)$.*
- $1 < p < \infty$.

As in the scalar case, E-valued isotropic Sobolev–Slobodeckii spaces are defined by

$$W_p^s = W_p^s(\mathbb{R}^d, E) := \begin{cases} H_p^s, & s \in \mathbb{Z}, \\ B_p^s, & s \in \mathbb{R} \backslash \mathbb{Z}. \end{cases} \tag{3.8.1}$$

On this basis we introduce the **anisotropic Sobolev–Slobodeckii scale**

$$[\, W_p^{s/\boldsymbol{\nu}} \ ; \ s \in \mathbb{R} \,]$$

by setting

$$W_p^{s/\boldsymbol{\nu}} = W_p^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E) := \begin{cases} \displaystyle\bigcap_{i=1}^{\ell} L_p\big(\mathbb{R}^{(d,d_i)}, W_p^{s/\nu_i}(\mathbb{R}^{d_i}, E)\big), & s > 0, \\ L_p, & s = 0, \\ \displaystyle\sum_{i=1}^{\ell} L_p\big(\mathbb{R}^{(d,d_i)}, W_p^{s/\nu_i}(\mathbb{R}^{d_i}, E)\big), & s < 0. \end{cases}$$

The following theorem is an analogue of (3.8.1)

3.8.1 Theorem *For $s \in \mathbb{R}$*

$$W_p^{s/\boldsymbol{\nu}} \doteq \begin{cases} H_p^{m\nu/\boldsymbol{\nu}}, & \text{if } s = m\nu \text{ with } m \in \mathbb{Z}, \\ B_p^{s/\boldsymbol{\nu}}, & \text{if } s/\nu_i \notin \mathbb{Z},\ 1 \le i \le \ell. \end{cases}$$

PROOF. This follows from (3.8.1) and Theorems 3.6.3 and 3.7.2. □

Note that $W_p^{s/\boldsymbol{\nu}}$ is neither an anisotropic Bessel potential nor a Besov space if $s \neq m\nu$ and $s/\nu_i \in \mathbb{Z}$ for at least one $i \in \{1, \dots, \ell\}$.

3.8.2 Theorem *For $s \in \mathbb{R}$*

$$W_p^{-s/\boldsymbol{\nu}} \doteq W_{p'}^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E')'$$

with respect to the L_p-duality pairing $\langle \cdot, \cdot \rangle$, and $W_p^{s/\boldsymbol{\nu}}$ is reflexive.

PROOF. This is an easy consequence of Proposition 3.6.4 and Lemma 3.6.5. □

For $s > 0$ we set

$$[u]_{s,p;i} := \big\| \, |h_i|^{-s} \, \|\triangle_{\check{h}_i}^{[s]+1} u\|_p \, \big\|_{L_p((\mathbb{R}^{d_i})^\bullet, dh_i/|h_i|^{d_i})}$$

and

$$[\![u]\!]_{s,p;i} := \begin{cases} [u]_{s,p;i}, & s \notin \dot{\mathbb{N}}, \\ \|\nabla_{x_i}^s u\|_p, & s \in \dot{\mathbb{N}}, \end{cases} \tag{3.8.2}$$

where, of course, the index i is omitted if $\ell = i = 1$. Furthermore,

$$[\![\cdot]\!]_{s/\boldsymbol{\nu},p} := \max_{1 \le i \le \ell} [\![\cdot]\!]_{s/\nu_i,p;i} \tag{3.8.3}$$

and

$$\|\cdot\|_{s/\boldsymbol{\nu},p} := \|\cdot\|_p + [\![\cdot]\!]_{s/\boldsymbol{\nu},p}.$$

It is now easy to prove a useful renorming theorem for the Sobolev–Slobodeckii spaces of positive order.

3.8.3 Proposition *If $s > 0$, then $\|\cdot\|_{W_p^{s/\nu}} \sim \|\cdot\|_{s/\boldsymbol{\nu},p}$.*

Proof. It is an immediate consequence of (3.8.1) and Theorems 3.6.1 and 3.7.1(ii) that $\|\cdot\|_p + [\![\cdot]\!]_{s/\nu_i,p;i}$ is an equivalent norm for $L_p(\mathbb{R}^{(d,d_i)}, W_p^{s/\nu_i})$. Now the claim is clear. □

3.8.4 Theorem *If $s_0 \ne s_1$ and $0 < \theta < 1$, then*

$$(W_p^{s_0/\boldsymbol{\nu}}, W_p^{s_1/\boldsymbol{\nu}})_{\theta,p} \doteq B_p^{s_\theta/\boldsymbol{\nu}}.$$

Proof. This follows from Theorem 3.8.2, formula (3.3.12), Theorem 3.7.1(iv), and the reiteration and duality theorems of interpolation theory (cf. I.2.6.1 and I.2.8.2 in H. Amann [4]). □

The next result is an analogue of Theorem 3.6.7 for the Sobolev–Slobodeckii scale.

3.8.5 Theorem *Suppose $s > 0$. Then*

$$W_p^{s/\boldsymbol{\nu}} \doteq L_p\big(\mathbb{R}, W_p^{s/\boldsymbol{\omega}'}(\mathbb{R}^{d-1}, E)\big) \cap W_p^{s/\omega_1}\big(\mathbb{R}, L_p(\mathbb{R}^{d-1}, E)\big)$$

and

$$W_p^{-s/\boldsymbol{\nu}} \doteq L_p\big(\mathbb{R}, W_p^{-s/\boldsymbol{\omega}'}(\mathbb{R}^{d-1}, E)\big) + W_p^{-s/\omega_1}\big(\mathbb{R}, L_p(\mathbb{R}^{d-1}, E)\big).$$

Proof. Since Λ is equivalent to the natural ω-quasi-norm it follows

$$W_p^{s/\boldsymbol{\nu}} \doteq L_p\big(\mathbb{R}^{d-1}, W_p^{s/\omega_1}(\mathbb{R}, E)\big) \cap \bigcap_{j=2}^{\ell} L_p\big(\mathbb{R}^{d-1}, W_p^{s/\omega_j}(\mathbb{R}, E)\big).$$

If $s/\omega_1 \in \dot{\mathbb{N}}$, then we deduce from Theorem 3.7.1(ii) and Fubini's theorem

$$L_p\big(\mathbb{R}^{d-1}, W_p^{s/\omega_1}(\mathbb{R}, E)\big) = W_p^{s/\omega_1}\big(\mathbb{R}, L_p(\mathbb{R}^{d-1}, E)\big). \tag{3.8.4}$$

If $s/\omega_1 \in \mathbb{R}\backslash\mathbb{N}$, then (3.8.4) follows from (3.6.6).

By definition,

$$W_p^{s/\boldsymbol{\omega}'}(\mathbb{R}^{d-1}, E) = \bigcap_{j=2}^{\ell} L_p\big(\mathbb{R}^{d-1}, W_p^{s/\omega_j}(\mathbb{R}, E)\big).$$

This proves the statement for $W_p^{s/\boldsymbol{\nu}}$. The assertion about $W_p^{-s/\boldsymbol{\nu}}$ follows now by duality. □

3.8.6 Example Let $\boldsymbol{\omega}$ be the $2m$-parabolic weight vector. Then

$$W_p^{(s,s/2m)} \doteq L_p\big(\mathbb{R}, W_p^s(\mathbb{R}^{d-1},E)\big) \cap W_p^{s/2m}\big(\mathbb{R}, L_p(\mathbb{R}^{d-1},E)\big)$$

and

$$W_p^{-(s,s/2m)} \doteq L_p\big(\mathbb{R}, W_p^{-s}(\mathbb{R}^{d-1},E)\big) + W_p^{-s/2m}\big(\mathbb{R}, L_p(\mathbb{R}^{d-1},E)\big)$$

for $s > 0$. Furthermore, if $s = 2mk + j + \theta$ for some $j,k \in \mathbb{N}$ with $j \le 2m-1$ and $0 \le \theta < 1$, then

$$u \in W_p^{(s,s/2m)}(\mathbb{R}^d,E) \text{ iff } u, \partial_t^k u, \nabla_{x'}^{2mk+j} u \in L_p(\mathbb{R}^d,E)$$

and, provided $(j,\theta) \ne (0,0)$,

$$[\partial_t^k u]_{(j+\theta)/2m,p;t} := \Big(\int_{\mathbb{R}^{d-1}} \big[\partial_t^k u(x',\cdot)\big]_{(j+\theta)/2m,p}^p \, dx'\Big)^{1/p} < \infty$$

and, if $\theta \ne 0$, also

$$[\nabla_{x'}^{2mk+j} u]_{\theta,p;x'} := \Big(\int_{\mathbb{R}} \big[\nabla_{x'}^{2mk+j} u(\cdot,t)\big]_{\theta,p}^p \, dt\Big)^{1/p} < \infty.$$

In particular, if $m = 1$ and $0 < \theta < 1$, then

$$u \in W_p^{(\theta,\theta/2)} \text{ iff } u \in L_p \text{ and } [u]_{\theta,p;x'} + [u]_{\theta/2,p;t} < \infty$$

and

$$u \in W_p^{(1,1/2)} \text{ iff } u \in L_p \text{ and } \|\nabla_{x'} u\|_p + [u]_{1/2,p;t} < \infty,$$

and

$$u \in W_p^{(1+\theta,(1+\theta)/2)} \text{ iff } u \in L_p \text{ and } \|\nabla_{x'} u\|_p + [\nabla_{x'} u]_{\theta,p;x'} + [u]_{(1+\theta)/2,p;t} < \infty,$$

where $\|\cdot\|_p$ is the norm in $L_p = L_p(\mathbb{R}^d,E)$.

Proof. This follows from the preceding theorem and well-known renorming results for isotropic Slobodeckii spaces (e.g., Theorem 2.5.1 and Remark 2.5.1.4 in H. Triebel [65], whose proofs carry over without change to the E-valued case). □

Besides the Slobodeckii spaces the Nikol′skiĭ spaces form an important subclass of Besov spaces. As in the scalar case they are defined in the isotropic situation by

$$N_p^s = N_p^s(\mathbb{R}^d,E) := \begin{cases} H_p^s, & s \in \mathbb{N}, \\ B_{p,\infty}^s, & s \in \mathbb{R}^+ \backslash \mathbb{N}. \end{cases}$$

In analogy to the definition of the Sobolev–Slobodeckii scale the **anisotropic Nikol′skiĭ scale** $[\, N_p^{s/\boldsymbol{\nu}} \ ; \ s \ge 0\,]$ is defined by[5]

$$N_p^{s/\boldsymbol{\nu}} = N_p^{s/\boldsymbol{\nu}}(\mathbb{R}^d,E) := \bigcap_{i=1}^{\ell} L_p\big(\mathbb{R}^{(d,d_i)}, N_p^{s/\nu_i}(\mathbb{R}^{d_i},E)\big). \tag{3.8.5}$$

Defining $[\![\cdot]\!]_{s/\boldsymbol{\nu},p,\infty}$ by replacing p in (3.8.2), (3.8.3) by ∞, it follows from Theorem 3.6.1 and Proposition 3.8.3 that

$$\|\cdot\|_{s/\boldsymbol{\nu},p,\infty} := \|\cdot\|_p + [\![\cdot]\!]_{s/\boldsymbol{\nu},p,\infty} \tag{3.8.6}$$

[5]Note that this scale is defined for $s \ge 0$ only and that we do not treat the case $p = 1$.

is an equivalent norm for $N_p^{s/\boldsymbol{\nu}}$. Furthermore, the following analogue to the first part of Theorem 3.8.5 is valid:

$$N_p^{s/\boldsymbol{\nu}} \doteq L_p\big(\mathbb{R}, N_p^{s/\boldsymbol{\omega}'}(\mathbb{R}^{d-1}, E)\big) \cap N_p^{s/\omega_1}\big(\mathbb{R}, L_p(\mathbb{R}^{d-1}, E)\big). \tag{3.8.7}$$

We leave it to the reader to write down explicitly the meaning of these facts in the case of the $2m$-parabolic weight vector.

As usual, $\overset{\scriptscriptstyle\triangle}{N}{}_p^{s/\boldsymbol{\nu}}$ is the closure of $\mathcal{S}$ in $N_p^{s/\boldsymbol{\nu}}$. This defines the **anisotropic small Nikol′skiĭ scale** whose importance lies in the fact that it is a densely injected Banach space scale. Scalar isotropic and anisotropic Nikol′skiĭ spaces have been introduced by S.M. Nikol′skiĭ in [50] and have been intensively studied by him and his school; cf., in particular, his book [51].

3.9 Hölder scales

Now we suppose

- *E is an arbitrary Banach space.*

For $k \in \mathbb{N}$ we set

$$C_0^k = C_0^k(\mathbb{R}^d, E) := \big(\{\, u \in C_0 \ ;\ \partial^\alpha u \in C_0,\ |\alpha| \le k \,\},\ \|\cdot\|_{k,\infty}\big).$$

It follows that C_0^k is a closed linear subspace of BUC^k, hence a Banach space. It is not difficult to see that

$$C_0^k \text{ is the closure of } \mathcal{S} \text{ in } BUC^k.$$

In analogy to (3.8.1), but restricting ourselves to $s \ge 0$, we define an isotropic Banach scale, the **small Hölder scale**, by

$$C_0^s = C_0^s(\mathbb{R}^d, E) := \begin{cases} C_0^s, & s \in \mathbb{N}, \\ \overset{\scriptscriptstyle\triangle}{B}{}_\infty^s, & s \in \mathbb{R}^+ \backslash \mathbb{N}. \end{cases}$$

On this basis the **anisotropic small Hölder scale** is introduced by

$$C_0^{s/\boldsymbol{\nu}} = C_0^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E) := \bigcap_{i=1}^{\ell} C_0\big(\mathbb{R}^{(d,d_i)}, C_0^{s/\nu_i}(\mathbb{R}^{d_i}, E)\big)$$

for $s > 0$, and $C_0^{0/\boldsymbol{\nu}} := C_0$.

Similarly as in the proof of Proposition 3.8.3, we see

$$\|\cdot\|_{C_0^{s/\boldsymbol{\nu}}} \sim \|\cdot\|_{s/\boldsymbol{\nu},\infty} := \|\cdot\|_\infty + [\![\cdot]\!]_{s/\boldsymbol{\nu},\infty}. \tag{3.9.1}$$

It also follows from Theorem 3.6.3

$$C_0^{s/\boldsymbol{\nu}} \doteq \overset{\scriptscriptstyle\triangle}{B}{}_\infty^{s/\boldsymbol{\nu}}, \qquad s/\nu_i \notin \mathbb{N}, \quad 1 \le i \le \ell. \tag{3.9.2}$$

From (3.3.13), (3.5.7), and the reiteration theorem for the continuous interpolation functor we deduce

$$(C_0^{s_0/\boldsymbol{\nu}}, C_0^{s_1/\boldsymbol{\nu}})_{\infty,\theta}^0 \doteq \overset{\scriptscriptstyle\triangle}{B}{}_\infty^{s_\theta/\boldsymbol{\nu}}, \qquad 0 \le s_0 < s_1, \quad 0 < \theta < 1. \tag{3.9.3}$$

There is also an analogue to the first part of Theorem 3.8.5, namely,

$$C_0^{s/\boldsymbol{\nu}} \doteq C_0\big(\mathbb{R}, C_0^{s/\boldsymbol{\omega}'}(\mathbb{R}^{d-1}, E)\big) \cap C_0^{s/\omega_1}\big(\mathbb{R}, C_0(\mathbb{R}^{d-1}, E)\big) \tag{3.9.4}$$

for $s > 0$.

Using (3.9.2) we can now complement our embedding results by the following anisotropic version of the SOBOLEV EMBEDDING THEOREM.

3.9.1 Theorem *Suppose $k \in \mathbb{N}$, $1 \le p, q \le \infty$, and $s > t + |\boldsymbol{\omega}|/p$. Then*

$$\overset{\scriptscriptstyle\triangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q} \overset{d}{\hookrightarrow} C_0^{t/\boldsymbol{\nu}}.$$

If $1 < p < \infty$, then

$$H_p^{s/\boldsymbol{\nu}} \overset{d}{\hookrightarrow} C_0^{t/\boldsymbol{\nu}},$$

provided E is a UMD space which has property (α) if $\boldsymbol{\omega} \neq \omega(1, \dots, 1)$.

PROOF. This is a consequence of Theorems 3.3.2 and 3.7.5, and of (3.9.2). □

In the following example we consider the particularly important case where $\boldsymbol{\omega}$ is the $2m$-parabolic weight vector. In this situation we set for $0 < \theta < 1$

$$[v]_{\theta,\infty;x'} := \sup_{t \in \mathbb{R}} \sup_{x',y' \in \mathbb{R}^{d-1}} \frac{|v(x',t) - v(y',t)|}{|x'-y'|^\theta}$$

and

$$[v]_{\theta,\infty;t} := \sup_{x' \in \mathbb{R}^{d-1}} \sup_{s,t \in \mathbb{R}} \frac{|v(x',s) - v(x',t)|}{|s-t|^\theta}$$

where here and in similar situations it is understood that $x' \neq y'$ and $s \neq t$, respectively. We also put

$$[v]_{\theta,\infty;\mathrm{par}} := \sup_{(x',s),(y',t)} \frac{|v(x',s) - v(y',t)|}{(|x'-y'|^2 + |s-t|)^{\theta/2}}.$$

Thus $[\cdot]_{\theta,\infty;\mathrm{par}}$ is the θ-Hölder seminorm with respect to the parabolic metric (1.3.3) on $\mathbb{R} = \mathbb{R}^{d-1} \times \mathbb{R}$. It is not difficult to verify

$$[\cdot]_{\theta,\infty;\mathrm{par}} \sim [\cdot]_{\theta,\infty;x'} + [\cdot]_{\theta/2,\infty;t}. \tag{3.9.5}$$

3.9.2 Example Let $\boldsymbol{\omega}$ be the $2m$-parabolic weight vector. Then

$$C_0^{(s,s/2m)}(\mathbb{R}^d, E) \doteq C_0\big(\mathbb{R}, C_0^s(\mathbb{R}^{d-1}, E)\big) \cap C_0^{s/2}\big(\mathbb{R}, C_0(\mathbb{R}^{d-1}, E)\big)$$

for $s > 0$. If $m = 1$ and $0 < \theta < 1$, then

$$u \mapsto \|u\|_\infty + [u]_{\theta,\infty;\mathrm{par}}$$

is an equivalent norm for $C_0^{(\theta,\theta/2)}(\mathbb{R}^d, E)$,

$$u \mapsto \|u\|_\infty + \|\nabla_{x'} u\|_\infty + [u]_{1/2,\infty;t}$$

is an equivalent norm for $C_0^{(1,1/2)}(\mathbb{R}^d, E)$, and

$$u \mapsto \|u\|_\infty + \|\nabla_{x'} u\|_\infty + [\nabla_{x'} u]_{\theta,\infty;x'} + [u]_{(1+\theta)/2,\infty;t}$$

is an equivalent norm for $C_0^{(1+\theta,(1+\theta)/2)}(\mathbb{R}^d, E)$, where $\|\cdot\|_\infty$ is the maximum norm on $\mathbb{R}^d$.

PROOF. This follows by easy arguments from (3.9.4) and (3.9.5). □

CHAPTER 4

Distributions on half-spaces and corners

This last chapter of Part 1 contains an in-depth study of trace and extension theorems for anisotropic Bessel potential and Besov spaces. Of particular relevance for the weak theory of parabolic problems are such theorems for spaces on corners. Most of these results are new and far from being straightforward extensions of known theorems.

4.1 Restrictions and extensions of smooth functions

We denote by

$$\mathbb{H}^d := \mathbb{R}^+ \times \mathbb{R}^{d-1}$$

the *closed* right half-space in $\mathbb{R}^d$ and by $\partial\mathbb{H}^d = \{0\} \times \mathbb{R}^{d-1}$ its boundary.[1] Given any open subset X of $\mathbb{H}^d$, a point $x \in X \cap \partial\mathbb{H}^d$, and a map f from X into some Banach space, by the partial derivative $\partial_1 f(x)$ at x we mean the right derivative

$$\partial_1 f(x) = \lim_{t \to 0+} \big(f(x + te_1) - f(x) \big)/t,$$

of course.

Let F be a Banach space. We write

$$\mathcal{S}(\mathbb{H}^d, F)$$

for the Fréchet space of all **smooth rapidly decreasing** F-valued **functions on** $\mathbb{H}^d$. Its topology is induced by the family of seminorms

$$u \mapsto q_{k,m}(u) := \max_{|\alpha| \le m} \sup_{x \in \mathbb{H}^d} \langle x \rangle^k \, |\partial^\alpha u(x)|, \qquad k, m \in \mathbb{N}, \tag{4.1.1}$$

thus by restricting the usual seminorms of $\mathcal{S}(\mathbb{R}^d, F)$ to $\mathbb{H}^d$. It contains $\mathcal{D}(\mathbb{H}^d, F)$, the space of **test functions on** $\mathbb{H}^d$, as a closed linear subspace, where $\mathcal{D}(\mathbb{H}^d, F)$ is given the usual LF-topology. This space has to be carefully distinguished from

$$\mathcal{D}(\mathring{\mathbb{H}}^d, F)$$

which is identified with a subspace of $\mathcal{D}(\mathbb{R}^d, F)$ by extending $u \in \mathcal{D}(\mathring{\mathbb{H}}^d, F)$ by zero over $\mathbb{H}^d$.

We shall now construct an extension operator from $\mathcal{S}(\mathbb{H}^d, F)$ into $\mathcal{S}(\mathbb{R}^d, F)$. For this we employ the following lemma which is taken from R. Hamilton [38].

[1]We often identify $\partial\mathbb{H}^d$ with $\mathbb{R}^{d-1}$. The reader will easily recognize which representation is used in a given formula.

4.1.1 Lemma *There exists $h \in C^\infty(\dot{\mathbb{R}}^+, \mathbb{R})$ satisfying*

$$\int_0^\infty t^s\, |h(t)|\, dt < \infty, \qquad s \in \mathbb{R}, \tag{4.1.2}$$

and

$$(-1)^k \int_0^\infty t^k h(t)\, dt = 1, \qquad k \in \mathbb{Z}, \tag{4.1.3}$$

as well as

$$h(1/t) = -t h(t), \qquad t > 0. \tag{4.1.4}$$

Proof. Denote by

$$\mathbb{C}\backslash\mathbb{R}^+ \to \mathbb{C}, \quad z \mapsto z^{1/4}$$

the branch of $z^{1/4}$ which, for $x \in \mathbb{R}^+$, satisfies $(x + i0)^{1/4} = x^{1/4}$. Then

$$(x - i0)^{1/4} = i\, x^{1/4}, \qquad x \geq 0.$$

Put

$$f(z) := (1+z)^{-1} \exp\bigl(-(1-i)z^{1/4} - (1+i)z^{-1/4}\bigr), \qquad z \in \mathbb{C}\backslash\mathbb{R}^+.$$

Then

$$f(x+i0) = (1+x)^{-1} e^{-(x^{1/4}+x^{-1/4})}\bigl(\cos(x^{1/4} - x^{-1/4}) + i \sin(x^{1/4} - x^{-1/4})\bigr)$$

and

$$f(x - i0) = \overline{f(x+i0)}, \qquad x \in \mathbb{R}^+.$$

Let Γ be a piece-wise smooth path in $\mathbb{C}\backslash\mathbb{R}^+$ running from $\infty - i0$ to $\infty + i0$. Then, by Cauchy's theorem,

$$\int_\Gamma z^k f(z)\, dz = 2i \int_0^\infty (1+x)^{-1} x^k e^{-(x^{1/4}+x^{-1/4})} \sin(x^{1/4} - x^{-1/4})\, dx \tag{4.1.5}$$

for $k \in \mathbb{Z}$. Since $z^\ell f(z) \to 0$ for each $\ell \in \mathbb{N}$ as $|z| \to \infty$, we can apply the residue theorem to deduce that

$$\int_\Gamma z^k f(z)\, dz = 2\pi i \operatorname{Res}(f, -1) = 2\pi i (-1)^k e^{-2\sqrt{2}},$$

thanks to $(-1)^{1/4} = (1+i)\big/\sqrt{2}$. Thus, putting

$$h(t) := \pi^{-1} e^{2\sqrt{2}} (1+t)^{-1} e^{-(t^{1/4}+t^{-1/4})} \sin(t^{1/4} - t^{-1/4})$$

for $t > 0$, we see that (4.1.3) is true. The remaining assertions are obvious. □

We write $x = (y, x')$ for the general point of $\mathbb{H}^d$ with $y \in \mathbb{R}^+$. Then, given u in $(L_1 + L_\infty)(\mathbb{H}^d, F)$, we set

$$\varepsilon u(x) := \int_0^\infty h(t) u(-ty, x')\, dt, \qquad \text{a.a. } x \in -\mathbb{H}^d, \tag{4.1.6}$$

and

$$e^+ u(x) := \begin{cases} u(x), & \text{a.a. } x \in \mathbb{H}^d, \\ \varepsilon u(x), & \text{a.a. } x \in -\mathring{\mathbb{H}}^d. \end{cases} \tag{4.1.7}$$

We do not indicate dimension d in this notation since it will be clear from the context.

4.1.2 Lemma *Suppose* $1 \le p \le \infty$. *Then*

$$e^+ \in \mathcal{L}\big(\mathcal{S}(\mathbb{H}^d, F), \mathcal{S}(\mathbb{R}^d, F)\big) \cap \mathcal{L}\big(L_p(\mathbb{H}^d, F), L_p(\mathbb{R}^d, F)\big).$$

PROOF. (1) The assertion concerning the L_p-spaces is immediate by (4.1.2) and

$$\|v\|^p_{L_p(\mathbb{R}^d,F)} = \|v\|^p_{L_p(\mathbb{H}^d,F)} + \|v\|^p_{L_p(-\mathbb{H}^d,F)} \tag{4.1.8}$$

for $1 \le p < \infty$, and

$$\|v\|_{L_\infty(\mathbb{R}^d,F)} \le \|v\|_{L_\infty(\mathbb{H}^d,F)} + \|v\|_{L_\infty(-\mathbb{H}^d,F)}.$$

(2) Suppose $u \in \mathcal{S}(\mathbb{H}^d, F)$. Then, given $\alpha \in \mathbb{N}^d$, it follows from (4.1.2) that

$$\partial^\alpha(\varepsilon u)(x) = (-1)^{\alpha^1} \int_0^\infty t^{\alpha^1} h(t) \partial^\alpha u(-ty, x')\, dt, \qquad x \in -\mathbb{H}^d. \tag{4.1.9}$$

Thus $\varepsilon u \in C^\infty(-\mathbb{H}^d, F)$ and (4.1.3) implies $\partial^\alpha \varepsilon u(0, x') = \partial^\alpha u(0, x')$ for $x' \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}^d$. This proves $e^+u \in C^\infty(\mathbb{R}^d, F)$.

From (4.1.2) and (4.1.9) we easily deduce $e^+ \in \mathcal{L}\big(\mathcal{S}(\mathbb{H}^d, F), \mathcal{S}(\mathbb{R}^d, F)\big)$. □

We denote by

$$r^+ : \mathcal{S}(\mathbb{R}^d, F) \to \mathcal{S}(\mathbb{H}^d, F), \quad u \mapsto u \,|\, \mathbb{H}^d$$

the **point-wise restriction operator for** $\mathbb{H}^d$.

4.1.3 Theorem r^+ *is a retraction from* $\mathcal{S}(\mathbb{R}^d, F)$ *onto* $\mathcal{S}(\mathbb{H}^d, F)$, *and* e^+ *is a coretraction.*

PROOF. This follows immediately from Lemma 4.1.2. □

This is the well-known extension theorem of R. Seeley [59], who based its proof on a discrete version of Lemma 4.1.1, extending the classical reflection method (e.g., Lemma 2.9.1.1 in H. Triebel [65]).

The **trivial extension operator for** $\mathbb{H}^d$,

$$e_0^+ : (L_1 + L_\infty)(\mathbb{H}^d, F) \to (L_1 + L_\infty)(\mathbb{R}^d, F),$$

is defined by

$$e_0^+ u := \begin{cases} u & \text{on} \quad \mathbb{H}^d, \\ 0 & \text{on} \ -\mathring{\mathbb{H}}^d. \end{cases}$$

We set

$$\mathcal{S}(\mathring{\mathbb{H}}^d, F) := \big\{\, u \in \mathcal{S}(\mathbb{H}^d, F) \ ; \ \partial^\alpha u \,|\, \partial\mathbb{H}^d = 0,\ \alpha \in \mathbb{N}^d \,\big\}.$$

It is a closed linear subspace of $\mathcal{S}(\mathbb{H}^d, F)$, hence a Fréchet space.

4.1.4 Lemma $\mathcal{D}(\mathring{\mathbb{H}}^d, F)$ *is dense in* $\mathcal{S}(\mathring{\mathbb{H}}^d, F)$.

PROOF. (1) Set $\tau_h u(x) := u(x-h)$ for $x, h \in \mathbb{R}^d$. For $t \geq 0$ put

$$\rho_t := r^+ \circ \tau_{te_1} \circ e_0^+$$

where $e_1 = (1, 0, \dots, 0) \in \mathbb{H}^d$. Since differentiation and translation commute it is easily verified that $\{\, \rho_t \ ; \ t \geq 0 \,\}$ is a strongly continuous semigroup on $\mathcal{S}(\mathring{\mathbb{H}}^d, F)$. Note

$$\operatorname{supp}(\rho_t u) \subset te_1 + \mathbb{H}^d, \qquad u \in \mathcal{S}(\mathring{\mathbb{H}}^d, F), \quad t > 0.$$

(2) Suppose

$$u \in \mathcal{S}(\mathring{\mathbb{H}}^d, F), \quad \operatorname{supp}(u) \subset te_1 + \mathbb{H}^d \text{ for some } t > 0.$$

Fix $\varphi \in \mathcal{D}(\mathbb{R}^d)$ with $\varphi(x) = 1$ for $|x| \leq 1$ and put $\varphi_r := \varphi(x/r)$ for $r > 0$. Then

$$u_r := \varphi_r u \in \mathcal{D}(\mathring{\mathbb{H}}^d, F), \qquad r > 0.$$

Since $\langle x \rangle^k \leq \langle x \rangle^{k+1}/r$ for $|x| \geq r$ and since $u_r - u = (\varphi_r - 1)u$ and $\partial^\beta \varphi_r$ vanish for $|x| < r$ and $\beta \in \mathbb{N}^d \setminus \{0\}$ we obtain from Leibniz' rule

$$\begin{aligned} &\sup_{x \in \mathbb{H}^d} \langle x \rangle^k \left| \partial^\alpha (u_r - u)(x) \right| \\ &\qquad \leq c(\alpha, k) \sum_{\beta \leq \alpha} r^{-|\beta|} \sup_{\substack{x \in \mathbb{H}^d \\ |x| \geq r}} \langle x \rangle^k \left| \partial^{\alpha - \beta} u(x) \right| \leq c(k, m) r^{-1} q_{k+1,m}(u) \end{aligned}$$

for $|\alpha| \leq m$, $k, m \in \mathbb{N}$, and $r \geq 1$. Thus $u_r \to u$ in $\mathcal{S}(\mathbb{H}^d, F)$ for $r \to \infty$. This and (1) imply the statement. □

It is clear that in the above definitions and theorems we could have replaced $\mathbb{H}^d$ by the left half-space $-\mathbb{H}^d$ using obvious modifications. We denote the corresponding extension and restriction operators by e^-, e_0^-, and r^-, respectively.

Following essentially R. Hamilton [38], we set

$$r_0^+ := r^+(1 - e^- r^-), \quad r_0^- := r^-(1 - e^+ r^+). \tag{4.1.10}$$

Then

$$r_0^\pm \in \mathcal{L}\big(\mathcal{S}(\mathbb{R}^d, F), \mathcal{S}(\pm\mathring{\mathbb{H}}^d, F)\big). \tag{4.1.11}$$

However, a more precise statement is true. For this we first note some simple but fundamental observations (cf. Proposition I.2.3.2 in H. Amann [4]).

4.1.5 Lemma *Let $\mathcal{X}$ and $\mathcal{Y}$ be LCSs and suppose $r : \mathcal{X} \to \mathcal{Y}$ is a retraction and e a coretraction. Then*

$$p := er \in \mathcal{L}(\mathcal{X}) \tag{4.1.12}$$

is a projection,

$$\mathcal{X} = \operatorname{im}(p) \oplus \ker(p) = e\mathcal{Y} \oplus \ker(r),$$

and

$$r \in \mathcal{L}\mathrm{is}(e\mathcal{Y}, \mathcal{Y}). \tag{4.1.13}$$

PROOF. Clearly, $p^2 = (er)(er) = e(re)r = er = p$. Hence

$$\mathcal{X} = \operatorname{im}(p) \oplus \ker(p) = p\mathcal{X} \oplus (1-p)\mathcal{X}.$$

Moreover, $rp = r$ implies $\ker(p) \subset \ker(r)$. Thus, since $p = er$ gives the converse inclusion, $\ker(p) = \ker(r)$. From this we deduce that r is a continuous bijection from $p\mathcal{X}$ onto $\mathcal{Y}$, and e is a continuous inverse for $r \,|\, p\mathcal{X}$. Hence $p\mathcal{X} = e\mathcal{Y}$ and (4.1.13) is true. $\square$

4.1.6 Lemma *Let $\mathcal{Y}$ and $\mathcal{Z}$ be LCSs and suppose $r : \mathcal{Y} \to \mathcal{Z}$ is a retraction. If $\mathcal{X}$ is a dense subset of $\mathcal{Y}$, then $r(\mathcal{X})$ is dense in $\mathcal{Z}$.*

Proof. Choose a coretraction r^c for r. Suppose $z \in \mathcal{Z}$ and U is a neighborhood of z in $\mathcal{Z}$. Then $r^{-1}(U)$ is a neighborhood of $r^c(z)$ in $\mathcal{Y}$. Hence there exists $x \in \mathcal{X} \cap r^{-1}(U)$ due to the density of $\mathcal{X}$ in $\mathcal{Y}$. Then $r(x) \in r(\mathcal{X}) \cap U$. This proves the claim. $\square$

4.1.7 Theorem *$r_0^{\pm}$ is a retraction from $\mathcal{S}(\mathbb{R}^d, F)$ onto $\mathcal{S}(\pm\mathring{\mathbb{H}}^d, F)$, and $e_0^{\pm}$ is a coretraction.*

Proof. Theorem 4.1.3 and Lemma 4.1.5 imply, with $p^- = e^- r^-$,

$$\ker(r^-) = \ker(p^-) = \operatorname{im}(1 - p^-).$$

Clearly, if $u \in \ker(r^-)$, then $u(x) = 0$ for $x \in -\mathbb{H}^d$. Hence

$$r_0^+ = r^+(1 - p^-) \in \mathcal{L}\big(\mathcal{S}(\mathbb{R}^d, F), \mathcal{S}(\mathring{\mathbb{H}}^d, F)\big)$$

and $r_0^+ e_0^+ u = u$ for $u \in \mathcal{S}(\mathring{\mathbb{H}}^d, F)$. This proves the $\mathring{\mathbb{H}}^d$-claim. The proof for $-\mathring{\mathbb{H}}^d$ is similar. $\square$

4.1.8 Theorem *The following direct sum decomposition is valid:*

$$\mathcal{S}(\mathbb{R}^d, F) = e^+ \mathcal{S}(\mathbb{H}^d, F) \oplus e_0^- \mathcal{S}(-\mathring{\mathbb{H}}^d, F).$$

Proof. Due to Theorem 4.1.3 and Lemma 4.1.5 it is enough to show that, setting $\mathcal{S} := \mathcal{S}(\mathbb{R}^d, F)$,

$$(1 - p^+)\mathcal{S} = e_0^- \mathcal{S}(-\mathring{\mathbb{H}}^d, F). \tag{4.1.14}$$

Since $(1 - p^+)u$ vanishes on $\mathbb{H}^d$ for $u \in \mathcal{S}$, its restriction $r^-(1 - p^+)u$ to $-\mathbb{H}^d$ belongs to $\mathcal{S}(-\mathring{\mathbb{H}}^d, F)$. Hence $e_0^- r^-(1 - p^+)u = (1 - p^+)u$ which proves (4.1.14). $\square$

Clearly, there is a similar decomposition if $\mathbb{H}^d$ is replaced by $-\mathbb{H}^d$.

$e^+ r^+ + e_0^- r_0^- = 1_{\mathcal{S}}$

4.1.9 Corollary *, $r^{\pm} e_0^{\mp} = 0$, and $r_0^{\pm} e^{\mp} = 0$.*

4.1.10 Remark Of course, the extension operators e^+, e_0^+ and the restriction operators r^+, r_0^+ depend on the Banach space F as well which we do not notationally indicate. This is justified by the following observation: suppose $F_1 \hookrightarrow F_0$ and denote by $r_{(j)}^+$ the retraction from $\mathcal{S}(\mathbb{R}^d, F_j)$ onto $\mathcal{S}(\mathbb{H}^d, F_j)$ and by $e_{(j)}^+$ the corresponding extension. Then the following diagram is commuting:

$$\begin{tikzcd}
\mathcal{S}(\mathbb{R}^d, F_1) \arrow[r, hook] \arrow[d, shift right, "r_{(1)}^+"'] & \mathcal{S}(\mathbb{R}^d, F_0) \arrow[d, shift right, "r_{(0)}^+"'] \\
\mathcal{S}(\mathbb{H}^d, F_1) \arrow[r, hook] \arrow[u, shift right, "e_{(1)}^+"'] & \mathcal{S}(\mathbb{H}^d, F_0) \arrow[u, shift right, "e_{(0)}^+"']
\end{tikzcd}$$

In this sense r^+ and e^+ and, consequently, r_0^+ and e_0^+ are said to be **independent** of F, or **universal**.

Proof. This is obvious by the construction of these operators. $\square$

4.2 Distributions on half-spaces

Let F be a Banach space, $\mathcal{X}$ and $\mathcal{Y}$ LCSs, and f a continuous linear map from $\mathcal{X}$ into $\mathcal{Y}$. Then, given $u \in \mathcal{L}(\mathcal{Y}, F)$, the pull-back $f^*u = u \circ f$ of u with f belongs to $\mathcal{L}(\mathcal{X}, F)$, and

$$f^* \in \mathcal{L}\big(\mathcal{L}(\mathcal{Y}, F), \mathcal{L}(\mathcal{X}, F)\big).$$

In particular,

$$i : \mathcal{X} \hookrightarrow \mathcal{Y} \;\Rightarrow\; i^* : \mathcal{L}(\mathcal{Y}, F) \to \mathcal{L}(\mathcal{X}, F), \quad u \mapsto u|\mathcal{X},$$

which means that $i^*u = u|\mathcal{X}$ is continuous on $\mathcal{X}$ for $u \in \mathcal{L}(\mathcal{Y}, F)$. If $\mathcal{X}$ is dense in $\mathcal{Y}$, then i^* is injective. Thus

$$i : \mathcal{X} \overset{d}{\hookrightarrow} \mathcal{Y} \;\Rightarrow\; i^* : \mathcal{L}(\mathcal{Y}, F) \hookrightarrow \mathcal{L}(\mathcal{X}, F). \tag{4.2.1}$$

Also note

$$f \in \mathcal{L}\mathrm{is}(\mathcal{X}, \mathcal{Y}) \;\Rightarrow\; f^* \in \mathcal{L}\mathrm{is}\big(\mathcal{L}(\mathcal{Y}, F), \mathcal{L}(\mathcal{X}, F)\big), \quad (f^*)^{-1} = (f^{-1})^*. \tag{4.2.2}$$

In the special case $F = \mathbb{C}$ we have $\mathcal{L}(\mathcal{X}, \mathbb{C}) = \mathcal{X}'$. Thus (4.2.1) generalizes

$$\mathcal{X} \overset{d}{\hookrightarrow} \mathcal{Y} \;\Rightarrow\; \mathcal{Y}' \hookrightarrow \mathcal{X}'.$$

Suppose $r : \mathcal{X} \to \mathcal{Y}$ is a retraction and $e : \mathcal{Y} \to \mathcal{X}$ a coretraction, that is, the diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[rr, "r"] & & \mathcal{Y} \\
& \mathcal{Y} \arrow[ul, "e"] \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

is commuting. Then

$$\begin{tikzcd}
\mathcal{L}(\mathcal{X}, F) \arrow[dr, "e^*"'] & & \mathcal{L}(\mathcal{Y}, F) \arrow[ll, "r^*"'] \arrow[dl, "\mathrm{id}"] \\
& \mathcal{L}(\mathcal{Y}, F) &
\end{tikzcd} \tag{4.2.3}$$

is also commuting. Hence e^* is a retraction from $\mathcal{L}(\mathcal{X}, F)$ onto $\mathcal{L}(\mathcal{Y}, F)$, and r^* is a coretraction. Moreover, setting $p := er$,

$$r^*e^* = (er)^* = p^*$$

shows that p^*, the pull-back of p, equals the projection r^*e^* associated with (4.2.3).

Recall that $\mathcal{D}'(\mathring{\mathbb{H}}^d, F)$, the space of F-valued distributions on $\mathring{\mathbb{H}}^d$, is defined by

$$\mathcal{D}'(\mathring{\mathbb{H}}^d, F) := \mathcal{L}\big(\mathcal{D}(\mathring{\mathbb{H}}^d), F\big).$$

In analogy, we denote by

$$\mathcal{D}'(\mathbb{H}^d, F) := \mathcal{L}\big(\mathcal{D}(\mathbb{H}^d), F\big) \tag{4.2.4}$$

the space of F-valued **distributions on** $\mathbb{H}^d$, and by

$$\mathcal{S}'(\mathbb{H}^d, F) := \mathcal{L}\big(\mathcal{S}(\mathbb{H}^d), F\big) \quad \text{and} \quad \mathcal{S}'(\mathring{\mathbb{H}}^d, F) := \mathcal{L}\big(\mathcal{S}(\mathring{\mathbb{H}}^d), F\big) \tag{4.2.5}$$

the space of F-valued **tempered distributions on** $\mathbb{H}^d$, and **on** $\mathring{\mathbb{H}}^d$, respectively.

An obvious modification of step (2) of the proof of Lemma 4.1.4 shows

$$\mathcal{D}(\mathbb{H}^d, F) \overset{d}{\hookrightarrow} \mathcal{S}(\mathbb{H}^d, F).$$

Using this (with $F = \mathbb{C}$) it follows from (4.2.1)

$$\mathcal{S}'(\mathbb{H}^d, F) \hookrightarrow \mathcal{D}'(\mathbb{H}^d, F).$$

Similarly, we deduce from Lemma 4.1.4

$$\mathcal{S}'(\mathring{\mathbb{H}}^d, F) \hookrightarrow \mathcal{D}'(\mathring{\mathbb{H}}^d, F).$$

Thus the elements of $\mathcal{S}'(\mathbb{H}^d, F)$ and $\mathcal{S}'(\mathring{\mathbb{H}}^d, F)$ are indeed distributions on $\mathbb{H}^d$ and $\mathring{\mathbb{H}}^d$, respectively.

Note that

$$\mathcal{S}(\mathbb{H}^d, F) \times \mathcal{S}(\mathring{\mathbb{H}}^d) \to F, \quad (u, \varphi) \mapsto \int_{\mathbb{H}^d} u\varphi \, dx \tag{4.2.6}$$

and

$$\mathcal{S}(\mathring{\mathbb{H}}^d, F) \times \mathcal{S}(\mathbb{H}^d) \to F, \quad (u, \varphi) \mapsto \int_{\mathbb{H}^d} u\varphi \, dx \tag{4.2.7}$$

are bilinear continuous maps satisfying

$$\int_{\mathbb{H}^d} \varphi \partial^\alpha u \, dx = (-1)^{|\alpha|} \int_{\mathbb{H}^d} u \partial^\alpha \varphi \, dx, \qquad \alpha \in \mathbb{N}^d, \tag{4.2.8}$$

in either case. Given $u \in \mathcal{S}(\mathbb{H}^d, F)$, respectively $u \in \mathcal{S}(\mathring{\mathbb{H}}^d, F)$, denote the map

$$\varphi \mapsto \int_{\mathbb{H}^d} u\varphi \, dx$$

in the first case by T_u and in the second one by $\mathring{T}_u$. Then

$$T := (u \mapsto T_u) : \ \mathcal{S}(\mathbb{H}^d, F) \to \mathcal{S}'(\mathring{\mathbb{H}}^d, F)$$

and

$$\mathring{T} := (u \mapsto \mathring{T}_u) : \ \mathcal{S}(\mathring{\mathbb{H}}^d, F) \to \mathcal{S}'(\mathbb{H}^d, F)$$

are continuous linear injections. By means of T we identify $\mathcal{S}(\mathbb{H}^d, F)$ with a linear subspace of $\mathcal{S}'(\mathring{\mathbb{H}}^d, F)$. In other words,

$$\mathcal{S}(\mathbb{H}^d, F) \hookrightarrow \mathcal{S}'(\mathring{\mathbb{H}}^d, F)$$

by *identifying* $u \in \mathcal{S}(\mathbb{H}^d, F)$ *with the* F*-valued distribution*

$$\varphi \mapsto u(\varphi) := T_u \varphi = \int_{\mathbb{H}^d} u\varphi \, dx, \qquad \varphi \in \mathcal{S}(\mathring{\mathbb{H}}^d).$$

It follows from (4.2.8) that $\partial^\alpha u$ is identified with the distributional derivative $\partial^\alpha T_u$ of T_u. Similarly,

$$\mathcal{S}(\mathring{\mathbb{H}}^d, F) \hookrightarrow \mathcal{S}'(\mathbb{H}^d, F)$$

by identifying u *with* $\mathring{T}_u$. These injections correspond to the canonical embedding

$$\mathcal{S}(\mathbb{R}^d, F) \overset{d}{\hookrightarrow} \mathcal{S}'(\mathbb{R}^d, F)$$

which identifies $u \in \mathcal{S}(\mathbb{R}^d, F)$ with

$$\varphi \mapsto \int_{\mathbb{R}^d} u\varphi \, dx, \qquad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

These **canonical embeddings** will be used throughout.

4.2.1 Lemma *If* $(u, \varphi) \in \mathcal{S}(\mathbb{H}^d, F) \times \mathcal{S}(\mathbb{R}^d)$, *then* $e^+ u(\varphi) = u(r_0^+ \varphi)$.

Proof. The definition of $e^+ u \in \mathcal{S}(\mathbb{R}^d, F)$ gives

$$(e^+ u)(\varphi) = \int_{\mathbb{R}^d} \varphi e^+ u \, dx = \int_{\mathbb{H}^d} \varphi u \, dx + \int_{-\mathbb{H}^d} \varphi \varepsilon u \, dx. \tag{4.2.9}$$

By Fubini's theorem

$$\begin{aligned}
\int_{-\infty}^0 \varphi(y, \cdot)\varepsilon u(y, \cdot) \, dy \quad &= \quad \int_{-\infty}^0 \varphi(y, \cdot) \int_0^\infty h(t) u(-ty, \cdot) \, dt \, dy \\
&\underset{y \mapsto -y}{=} \quad \int_0^\infty \int_0^\infty \varphi(-y, \cdot) u(ty, \cdot) \, dy \, h(t) \, dt \\
&\underset{y \mapsto z := ty}{=} \quad \int_0^\infty \int_0^\infty \varphi\Bigl(-\frac{z}{t}, \cdot\Bigr) u(z, \cdot) \, dz \, h(t) \, \frac{dt}{t} \\
&\underset{t \mapsto s := 1/t}{=} \quad \int_0^\infty \int_0^\infty \varphi(-sz, \cdot) h\Bigl(\frac{1}{s}\Bigr) \, \frac{ds}{s} \, u(z, \cdot) \, dz \\
&= \quad -\int_0^\infty \int_0^\infty h(s) \varphi(-sz, \cdot) \, ds \, u(z, \cdot) \, dz \\
&= \quad -\int_0^\infty (\varepsilon\varphi)(z, \cdot) u(z, \cdot) \, dz,
\end{aligned}$$

where we used (4.1.4) in the next to the last step. Thus

$$\int_{-\mathbb{H}^d} \varphi \varepsilon u \, dx = -\int_{\mathbb{H}^d} (\varepsilon\varphi) u \, dx. \tag{4.2.10}$$

Define εu on $\mathbb{H}^d$ by replacing $x \in -\mathbb{H}^d$ in (4.1.6) by $x \in \mathbb{H}^d$. Then

$$e^- r^- \varphi(y, x') = \begin{cases} \varphi(y, x'), & y \le 0, \\ \varepsilon\varphi(y, x'), & y \ge 0. \end{cases}$$

Hence

$$r_0^+ \varphi(y, x') = r^+ (1 - e^- r^-) \varphi(y, x') = \varphi(y, x') - \varepsilon\varphi(y, x') \tag{4.2.11}$$

for $y \ge 0$ and $x' \in \mathbb{R}^{d-1}$. Now the assertion follows from (4.2.9) and (4.2.10). $\square$

After these preparations we can prove the following fundamental retraction theorem for tempered distributions on $\mathbb{H}^d$.

4.2.2 Theorem *The following diagram is commuting:*

$$\begin{tikzcd}
\mathcal{S}(\mathbb{H}^d,F) \arrow[r, shift left, "e^+"] \arrow[d, hook, "d"] & \mathcal{S}(\mathbb{R}^d,F) \arrow[l, shift left, "r^+"] \arrow[r, shift right, "r_0^+"'] \arrow[d, hook, "d"] & \mathcal{S}(\mathring{\mathbb{H}}^d,F) \arrow[l, shift right, "e_0^+"'] \arrow[d, hook, "d"] \\
\mathcal{S}'(\mathring{\mathbb{H}}^d,F) \arrow[r, shift left, "(r_0^+)^*"] & \mathcal{S}'(\mathbb{R}^d,F) \arrow[l, shift left, "(e_0^+)^*"] \arrow[r, shift right, "(e^+)^*"'] & \mathcal{S}'(\mathbb{H}^d,F) \arrow[l, shift right, "(r^+)^*"']
\end{tikzcd}$$

and $(e_0^+)^*$ *and* $(e^+)^*$ *are retractions with respective coretractions* $(r_0^+)^*$ *and* $(r^+)^*$.

Proof. (1) Lemma 4.2.1 guarantees the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{S}(\mathbb{H}^d,F) \arrow[r, "e^+"] \arrow[d, hook] & \mathcal{S}(\mathbb{R}^d,F) \arrow[d, hook] \\
\mathcal{S}'(\mathring{\mathbb{H}}^d,F) \arrow[r, "(r_0^+)^*"] & \mathcal{S}'(\mathbb{R}^d,F)
\end{tikzcd}$$

Given $v \in \mathcal{S}(\mathbb{R}^d,F)$ and $\psi \in \mathcal{S}(\mathring{\mathbb{H}}^d)$,

$$(r^+v)(\psi) = \int_{\mathbb{H}^d} v\psi\,dx = \int_{\mathbb{R}^d} ve_0^+\psi\,dx = (e_0^+)^*v(\psi).$$

This proves the commutativity of

$$\begin{tikzcd}
\mathcal{S}(\mathbb{H}^d,F) \arrow[d, hook] & \mathcal{S}(\mathbb{R}^d,F) \arrow[l, "r^+"'] \arrow[d, hook] \\
\mathcal{S}'(\mathring{\mathbb{H}}^d,F) & \mathcal{S}'(\mathbb{R}^d,F) \arrow[l, "(e_0^+)^*"']
\end{tikzcd}$$

Now the assertions for the left half of the diagram of the statement follow from Theorems 4.1.3 and 4.1.7, the fact that $\mathcal{S}(\mathbb{R}^d,F)$ is dense in $\mathcal{S}'(\mathbb{R}^d,F)$, and Lemma 4.1.6.

(2) Suppose $u \in \mathcal{S}(\mathring{\mathbb{H}}^d,F)$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then

$$(e_0^+u)(\varphi) = \int_{\mathbb{H}^d} u\varphi\,dx = \int_{\mathbb{H}^d} ur^+\varphi\,dx.$$

Thus $e_0^+u = (r^+)^*u$ for $u \in \mathcal{S}(\mathring{\mathbb{H}}^d,F)$ so that

$$\begin{tikzcd}
\mathcal{S}(\mathring{\mathbb{H}}^d,F) \arrow[r, "e_0^+"] \arrow[d, hook] & \mathcal{S}(\mathbb{R}^d,F) \arrow[d, hook] \\
\mathcal{S}'(\mathbb{H}^d,F) \arrow[r, "(r^+)^*"] & \mathcal{S}'(\mathbb{R}^d,F)
\end{tikzcd}$$

is commuting. As in the proof of Lemma 4.2.1 we obtain from (4.2.11)

$$(r_0^+u)(\varphi) = u(e^+\varphi), \qquad u \in \mathcal{S}(\mathbb{R}^d,F), \quad \varphi \in \mathcal{S}(\mathbb{H}^d).$$

This shows the commutativity of

$$\begin{tikzcd}
\mathcal{S}(\mathring{\mathbb{H}}^d, F) \arrow[d, hook] & & \mathcal{S}(\mathbb{R}^d, F) \arrow[ll, "r_0^+"'] \arrow[d, hook] \\
\mathcal{S}'(\mathbb{H}^d, F) & & \mathcal{S}'(\mathbb{R}^d, F) \arrow[ll, "(e^+)^*"']
\end{tikzcd}$$

Now the assertions for the right half of the diagram of the claim follow again from Theorems 4.1.3 and 4.1.7, the density of $\mathcal{S}(\mathbb{R}^d, F)$ in $\mathcal{S}'(\mathbb{R}^d, F)$, and Lemma 4.1.6.

(3) The last part of the statement follows from Theorems 4.1.3 and 4.1.7 and from (4.2.3). $\square$

This theorem shows that $(r_0^+)^*$ and $(e_0^+)^*$ are the unique continuous extensions of e^+ and r^+, respectively. Thus we can use the same symbols for them without fearing confusion, that is, we set

$$r^+ := (e_0^+)^*, \quad e^+ := (r_0^+)^*, \quad r_0^+ := (e^+)^*, \quad e_0^+ := (r^+)^*.$$

Evidently, corresponding results are valid for r^-, e^-, r_0^-, and e_0^-.

4.2.3 Corollary *The diagram*

$$\begin{tikzcd}
\mathcal{S}(\mathbb{R}^d, F) \arrow[d, hook, "d"] = & e^+\mathcal{S}(\mathbb{H}^d, F) \arrow[d, hook, "d"] \oplus & e_0^-\mathcal{S}(-\mathring{\mathbb{H}}^d, F) \arrow[d, hook, "d"] \\
\mathcal{S}'(\mathbb{R}^d, F) = & e^+\mathcal{S}'(\mathring{\mathbb{H}}^d, F) \oplus & e_0^-\mathcal{S}'(-\mathbb{H}^d, F)
\end{tikzcd}$$

is commuting. An analogous statement holds if $\mathbb{H}^d$ is replaced by $-\mathbb{H}^d$.

Proof. This is a consequence of Theorem 4.1.8 and the preceding consideration. $\square$

The crucial observation formulated in Lemma 4.2.1 and Theorem 4.2.2 are due (in the scalar case, of course) to R. Hamilton [38].

4.2.4 Theorem

(i) *If $u \in \mathcal{S}'(\mathbb{R}^d, F)$, then r^+u is the restriction of u to $\mathring{\mathbb{H}}^d$ in the sense of distributions, that is,*

$$r^+u(\varphi) = u(\varphi), \qquad \varphi \in \mathcal{D}(\mathring{\mathbb{H}}^d).$$

(ii) *Set*

$$\mathcal{S}'_{\mathbb{H}^d}(\mathbb{R}^d, F) := \left\{ v \in \mathcal{S}'(\mathbb{R}^d, F) \ ; \ \operatorname{supp}(v) \subset \mathbb{H}^d \right\}.$$

Then

$$\mathcal{S}'_{\mathbb{H}^d}(\mathbb{R}^d, F) = e_0^+\mathcal{S}'(\mathbb{H}^d, F).$$

Proof. (i) Since $r^+u \in \mathcal{S}'(\mathring{\mathbb{H}}^d, F)$ and $r^+u(\varphi) = u(e_0^+\varphi)$ for $\varphi \in \mathcal{S}(\mathring{\mathbb{H}}^d)$, the assertion follows from the density of $\mathcal{D}(\mathring{\mathbb{H}}^d)$ in $\mathcal{S}(\mathring{\mathbb{H}}^d)$ and the identification of $\varphi \in \mathcal{D}(\mathring{\mathbb{H}}^d)$ with $e_0^+\varphi$.

(ii) For $u \in \mathcal{S}'(\mathbb{H}^d, F)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$ with $\operatorname{supp}(\varphi) \subset -\mathring{\mathbb{H}}^d$ we obtain

$$e_0^+ u(\varphi) = u(r^+ \varphi) = 0.$$

Hence $e_0^+ \mathcal{S}'(\mathbb{H}^d, F) \subset \mathcal{S}'_{\mathbb{H}^d}(\mathbb{R}^d, F)$.

Conversely, suppose $v \in \mathcal{S}'_{\mathbb{H}^d}(\mathbb{R}^d, F)$. Then, given $\psi \in \mathcal{S}(-\mathring{\mathbb{H}}^d)$, the density of $\mathcal{D}(-\mathring{\mathbb{H}}^d)$ in $\mathcal{S}(-\mathring{\mathbb{H}}^d)$ and $\psi = e_0^- \psi$ for $\psi \in \mathcal{D}(-\mathring{\mathbb{H}}^d)$ imply $0 = v(e_0^- \psi) = r^- v(\psi)$. Thus $r^- v = 0$ and, consequently, $e^- r^- v = 0$. Hence

$$v = (1 - e^- r^-) v \in e_0^+ \mathcal{S}'(\mathbb{H}^d, F),$$

since (the analogue of) Corollary 4.2.3 shows

$$\mathcal{S}'(\mathbb{R}^d, F) = e_0^+ \mathcal{S}'(\mathbb{H}^d, F) \oplus e^- \mathcal{S}'(-\mathring{\mathbb{H}}^d, F).$$

From this we infer $\mathcal{S}'_{\mathbb{H}^d}(\mathbb{R}^d, F) \subset e_0^+ \mathcal{S}'(\mathbb{H}^d, F)$. $\square$

4.2.5 Corollary *r_0^+ is an isomorphism from $\mathcal{S}'_{\mathbb{H}^d}(\mathbb{R}^d, F)$ onto $\mathcal{S}'(\mathbb{H}^d, F)$.*

Proof. This follows from (4.1.13). $\square$

By means of this isomorphism $\mathcal{S}'(\mathbb{H}^d, F)$ is often identified with the space of F-valued tempered distributions on $\mathbb{R}^d$ which are supported in $\mathbb{H}^d$.

We introduce bilinear forms

$$\langle \cdot, \cdot \rangle_{\mathring{\mathbb{H}}^d} \quad \text{on } \mathcal{S}'(\mathring{\mathbb{H}}^d, F') \times \mathcal{S}(\mathring{\mathbb{H}}^d, F)$$

and

$$\langle \cdot, \cdot \rangle_{\mathbb{H}^d} \quad \text{on } \mathcal{S}'(\mathbb{H}^d, F') \times \mathcal{S}(\mathbb{H}^d, F)$$

by

$$\langle u', u \rangle_{\mathring{\mathbb{H}}^d} := \langle e^+ u', e_0^+ u \rangle$$

and

$$\langle v', v \rangle_{\mathbb{H}^d} := \langle e_0^+ v', e^+ v \rangle,$$

respectively.

4.2.6 Theorem *These bilinear forms are separately continuous and satisfy*

$$\langle u', u \rangle_{\mathring{\mathbb{H}}^d} = \int_{\mathbb{H}^d} \bigl\langle u'(x), u(x) \bigr\rangle_F \, dx, \qquad (u', u) \in \mathcal{S}(\mathbb{H}^d, F') \times \mathcal{S}(\mathring{\mathbb{H}}^d, F),$$

and

$$\langle v', v \rangle_{\mathbb{H}^d} = \int_{\mathbb{H}^d} \bigl\langle v'(x), v(x) \bigr\rangle_F \, dx, \qquad (v', v) \in \mathcal{S}(\mathring{\mathbb{H}}^d, F') \times \mathcal{S}(\mathbb{H}^d, F).$$

Proof. The continuity assertion is immediate from the corresponding property of $\langle \cdot, \cdot \rangle$ on $\mathcal{S}'(\mathbb{R}^d, F') \times \mathcal{S}(\mathbb{R}^d, F)$ and the continuity of e^+ and e_0^+.

Suppose $u' \in \mathcal{S}(\mathbb{H}^d, F')$. Then $e^+ u' \in \mathcal{S}(\mathbb{R}^d, F')$ and, consequently,

$$\begin{aligned} \langle u', u \rangle_{\mathring{\mathbb{H}}^d} = \langle e^+ u', e_0^+ u \rangle &= \int_{\mathbb{R}^d} \bigl\langle e^+ u'(x), e_0^+ u(x) \bigr\rangle_F \, dx \\ &= \int_{\mathbb{H}^d} \bigl\langle u'(x), u(x) \bigr\rangle \, dx, \end{aligned}$$

since $e_0^+ u$ vanishes on $-\mathbb{H}^d$. This proves the assertion for $\langle \cdot, \cdot \rangle_{\mathring{\mathbb{H}}^d}$. The one for $\langle \cdot, \cdot \rangle_{\mathbb{H}^d}$ follows by an analogous argument. $\square$

4.2.7 Corollary *For* $(w', w) \in \mathcal{S}'(\mathbb{R}^d, F') \times \mathcal{S}(\mathbb{R}^d, F)$,

$$\langle w', w\rangle = \langle r^+ w', r_0^+ w\rangle_{\mathring{\mathbb{H}}^d} + \langle r_0^- w', r^- w\rangle_{-\mathbb{H}^d}$$

and

$$\langle w', w\rangle = \langle r_0^+ w', r^+ w\rangle_{\mathbb{H}^d} + \langle r^- w', r_0^- w\rangle_{-\mathring{\mathbb{H}}^d}.$$

Proof. Suppose $w = \varphi \otimes f$ with $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $f \in F$. Then the assertion is an easy consequence of Corollaries 4.2.3 and 4.1.9. Hence it holds for w belonging to $\mathcal{S}(\mathbb{R}^d) \otimes F$ by linear extension. Now we obtain the statement by continuity and the density of $\mathcal{S}(\mathbb{R}^d) \otimes F$ in $\mathcal{S}(\mathbb{R}^d, F)$ (cf. H.Amann [10, Theorem 1.3.6(v)]). □

4.3 Corners

For $k \in \{1, \dots, d\}$ we set

$$\mathbb{K}_k^d := (\mathbb{R}^+)^k \times \mathbb{R}^{d-k}$$

and call it **standard closed k-corner in $\mathbb{R}^d$**. Note $\mathbb{K}_1^d = \mathbb{H}^d$. The interior of $\mathbb{K}_k^d$,

$$\mathring{\mathbb{K}}_k^d = (\dot{\mathbb{R}}^+)^k \times \mathbb{R}^{d-k},$$

is the standard **open k-corner**. Any subset of $\mathbb{K}_k^d$ of the form $I_1 \times \cdots \times I_k \times \mathbb{R}^{d-k}$ with $I_j \in \{\mathbb{R}^+, \dot{\mathbb{R}}^+\}$ and being different from $\mathbb{K}_k^d$ and $\mathring{\mathbb{K}}_k^d$ is called standard **partially open k-corner**. A 2-corner is also called **wedge**.

For $1 \le j \le k$ the **closed j-face** of $\mathbb{K}_k^d$ is defined to be

$$\partial_j \mathbb{K}_k^d := \{\, x \in \mathbb{K}_k^d \ ;\ x^j = 0 \,\}.$$

It is linearly[2] isometrically diffeomorphic to $\mathbb{K}_{k-1}^{d-1}$ by means of the **natural diffeomorphism**

$$\partial_j \mathbb{K}_k^d \cong \mathbb{K}_{k-1}^{d-1}, \quad x \mapsto (x^1, \dots, \widehat{x^j}, \dots, x^d) \tag{4.3.1}$$

by which we often identify $\partial_j \mathbb{K}_k^d$ with $\mathbb{K}_{k-1}^{d-1}$ without fearing confusion. Note

$$\partial \mathbb{K}_k^d = \bigcup_{j=1}^k \partial_j \mathbb{K}_k^d$$

and

$$\partial_{ij} \mathbb{K}_k^d := \partial_i \mathbb{K}_k^d \cap \partial_j \mathbb{K}_k^d = \{\, x \in \mathbb{K}_k^d \ ;\ x^i = x^j = 0 \,\} \cong \mathbb{K}_{k-2}^{d-2}$$

for $i \ne j$. Also observe that $\mathbb{K}$ is a partially open standard k-corner in $\mathbb{R}^d$ iff

$$\mathbb{K} = \mathbb{K}_k^d \Big\backslash \bigcup_{j \in J^*} \partial_j \mathbb{K}_k^d, \tag{4.3.2}$$

where J^* is a nonempty proper subset of $\{1, \dots, k\}$. Set[3] $J := \{1, \dots, k\} \backslash J^*$. Then

$$\mathbb{K}^* = \mathbb{K}_k^d \Big\backslash \bigcup_{j \in J} \partial_j \mathbb{K}_k^d$$

---

[2]**Linearly (isometrically) diffeomorphic** means, of course, that the diffeomorphism is the restriction of an (isometric) automorphism of $\mathbb{R}^d$.

[3]Our notation implies that face $\partial_j \mathbb{K}_k^d$ belongs to $\mathbb{K}$ iff $j \in J$. These faces are the **essential faces** of $\mathbb{K}$.

is the **complementary corner** of $\mathbb{K}$. We also put

$$(\mathbb{K}_k^d)^* := \mathring{\mathbb{K}}_k^d, \quad (\mathring{\mathbb{K}}_k^d)^* := \mathbb{K}_k^d.$$

Clearly, $\mathcal{S}(\mathbb{K}_k^d, F)$ is the space of all smooth F-valued rapidly decreasing functions on $\mathbb{K}_k^d$, where $\partial_j u(x)$ for $x \in \partial_j \mathbb{K}_k^d$ and $1 \le j \le k$ is the right derivative of $u : \mathbb{K}_k^d \to F$. It is a Fréchet space with the topology induced by the seminorms obtained by restricting (4.1.1) to $\mathbb{K}_k^d$.

Let now $\mathbb{K}$ be any (closed, open, or partially open) **standard k-corner in $\mathbb{R}^d$**, that is, $\mathbb{K}$ is given by (4.3.2), where now J^* is any (possibly empty or not proper) subset of $\{1, \dots, k\}$. Given a Banach space F, we denote by $\mathcal{S}(\mathbb{K}, F)$ the closed linear subspace of $\mathcal{S}(\mathbb{K}_k^d, F)$ consisting of all u satisfying

$$\partial_j^m u|_{x^j=0} = 0, \qquad m \in \mathbb{N}, \quad j \in J^*.$$

It is a Fréchet space. The space of tempered F-valued **distributions on** $\mathbb{K}$ is defined by

$$\mathcal{S}'(\mathbb{K}, F) := \mathcal{L}\big(\mathcal{S}(\mathbb{K}), F\big).$$

It follows from (4.2.6)–(4.2.8) that

$$\mathcal{S}(\mathbb{K}, F) \times \mathcal{S}(\mathbb{K}^*) \to F, \quad (u, \varphi) \mapsto \int_{\mathbb{K}} u\varphi \, dx$$

is a continuous bilinear map satisfying

$$\int_{\mathbb{K}} \varphi \partial^\alpha u \, dx = (-1)^{|\alpha|} \int_{\mathbb{K}} u \partial^\alpha \varphi \, dx, \qquad \alpha \in \mathbb{N}^d. \tag{4.3.3}$$

Similarly as in Section 4.2 we identify $u \in \mathcal{S}(\mathbb{K}, F)$ with the F-valued distribution $T_u \in \mathcal{S}'(\mathbb{K}^*, F)$ given by

$$\varphi \mapsto u(\varphi) := T_u \varphi := \int_{\mathbb{K}} u\varphi \, dx, \qquad \varphi \in \mathcal{S}(\mathbb{K}^*).$$

This is possible since $u \mapsto T_u$ is injective. Thus

$$\mathcal{S}(\mathbb{K}, F) \hookrightarrow \mathcal{S}'(\mathbb{K}^*, F).$$

For $u \in \mathcal{S}'(\mathbb{K}^*, F)$ we define **distributional derivatives** $\partial^\alpha u$ for $\alpha \in \mathbb{N}^d$ by

$$\partial^\alpha u(\varphi) := (-1)^{|\alpha|} u(\partial^\alpha \varphi), \qquad \varphi \in \mathcal{S}(\mathbb{K}^*).$$

By (4.3.3) this definition is meaningful in the sense that it extends the classical derivative, that is, $T_{\partial^\alpha u} = \partial^\alpha T_u$ for $u \in \mathcal{S}(\mathbb{K}, F)$.

The purpose of the following considerations is to generalize the results of the preceding sections from half-spaces to corners. For clarity, we first consider the case of wedges (so that $d \ge 2$).

It is convenient to set $X = Y := \mathbb{R}$ and $Z := \mathbb{R}^{d-2}$. Moreover, $X^\pm$ and $Y^\pm$ equal $\mathbb{R}^\pm$ with $\mathbb{R}^- := -\mathbb{R}^+$. Then

$$\mathbb{K}_2^d = X^+ \times Y^+ \times Z, \quad \mathring{\mathbb{K}}_2^d = \dot{X}^+ \times \dot{Y}^+ \times Z.$$

The partially open standard wedges are $\dot{X}^+ \times Y^+ \times Z$ and $X^+ \times \dot{Y}^+ \times Z$. We also need to consider the three closed wedges $X^- \times Y^+ \times Z$, $X^- \times Y^- \times Z$, and $X^+ \times Y^- \times Z$ as well as their open and partially open subcorners. Related to these four wedges are the four half-spaces $X^{\pm} \times Y \times Z$ and $X \times Y^{\pm} \times Z$. All these wedges and half-spaces are obviously linearly diffeomorphic to standard wedges or $\mathbb{H}^d$.

For abbreviation we write

$$\mathcal{S}_{\widetilde{X} \times \widetilde{Y}} := \mathcal{S}(\widetilde{X} \times \widetilde{Y} \times Z, F), \qquad \widetilde{X}, \widetilde{Y} \in \{\mathbb{R}, \mathbb{R}^{\pm}, \overset{\bullet}{\mathbb{R}}{}^{\pm}\}.$$

We also set

$$(r_X, e_X) := (r_X^+, e_X^+), \quad (r_X^0, e_X^0) := (r_{0,X}^+, e_{0,X}^+),$$

where $r_X^+ : \mathcal{S}_{X \times Y} \to \mathcal{S}_{X^+ \times Y}$ is the point-wise restriction, and $e_{0,X}^+$ is the trivial extension $\mathcal{S}_{\overset{\bullet}{X}{}^+ \times Y} \to \mathcal{S}_{X \times Y} = \mathbb{R}^d$, etc. Of course, there are analogous retractions

$$r_Y : \mathcal{S}_{X \times Y} \to \mathcal{S}_{X \times Y^+}, \quad r_Y^0 : \mathcal{S}_{X \times Y} \to \mathcal{S}_{X \times \overset{\bullet}{Y}{}^+},$$

and corresponding coretractions e_Y and e_Y^0.

Using Theorems 4.1.3 and 4.1.7 and definitions (4.1.6) and (4.1.7) it is not difficult to verify that the following diagram of retractions and corresponding coretractions is commuting:

$$\begin{tikzcd}
\mathcal{S}_{\overset{\bullet}{X}{}^+ \times \overset{\bullet}{Y}{}^+} \arrow[r, shift left, leftarrow, "r_Y^0"] \arrow[r, shift right, "e_Y^0"'] \arrow[d, shift right, leftarrow, "r_X^0"'] \arrow[d, shift left, "e_X^0"] & \mathcal{S}_{\overset{\bullet}{X}{}^+ \times Y} \arrow[r, shift left, "r_Y"] \arrow[r, shift right, leftarrow, "e_Y"'] \arrow[d, shift right, leftarrow, "r_X^0"'] \arrow[d, shift left, "e_X^0"] & \mathcal{S}_{\overset{\bullet}{X}{}^+ \times Y^+} \arrow[d, shift right, leftarrow, "r_X^0"'] \arrow[d, shift left, "e_X^0"] \\
\mathcal{S}_{X \times \overset{\bullet}{Y}{}^+} \arrow[r, shift left, leftarrow, "r_Y^0"] \arrow[r, shift right, "e_Y^0"'] \arrow[d, shift right, "r_X"'] \arrow[d, shift left, leftarrow, "e_X"] & \mathcal{S}_{X \times Y} \arrow[r, shift left, "r_Y"] \arrow[r, shift right, leftarrow, "e_Y"'] \arrow[d, shift right, "r_X"'] \arrow[d, shift left, leftarrow, "e_X"] & \mathcal{S}_{X \times Y^+} \arrow[d, shift right, "r_X"'] \arrow[d, shift left, leftarrow, "e_X"] \\
\mathcal{S}_{X^+ \times \overset{\bullet}{Y}{}^+} \arrow[r, shift left, leftarrow, "r_Y^0"] \arrow[r, shift right, "e_Y^0"'] & \mathcal{S}_{X^+ \times Y} \arrow[r, shift left, "r_Y"] \arrow[r, shift right, leftarrow, "e_Y"'] & \mathcal{S}_{X^+ \times Y^+}
\end{tikzcd} \tag{4.3.4}$$

Then we can apply Theorem 4.2.2 to obtain the following commuting diagram of retractions:

$$\begin{tikzcd}
\mathcal{S}'_{X^+ \times Y^+} \arrow[r, shift left, leftarrow, "r_Y^0"] \arrow[r, shift right, "e_Y^0"'] \arrow[d, shift right, leftarrow, "r_X^0"'] \arrow[d, shift left, "e_X^0"] & \mathcal{S}'_{X^+ \times Y} \arrow[r, shift left, "r_Y"] \arrow[r, shift right, leftarrow, "e_Y"'] \arrow[d, shift right, leftarrow, "r_X^0"'] \arrow[d, shift left, "e_X^0"] & \mathcal{S}'_{X^+ \times \overset{\bullet}{Y}{}^+} \arrow[d, shift right, leftarrow, "r_X^0"'] \arrow[d, shift left, "e_X^0"] \\
\mathcal{S}'_{X \times Y^+} \arrow[r, shift left, leftarrow, "r_Y^0"] \arrow[r, shift right, "e_Y^0"'] \arrow[d, shift right, "r_X"'] \arrow[d, shift left, leftarrow, "e_X"] & \mathcal{S}'_{X \times Y} \arrow[r, shift left, "r_Y"] \arrow[r, shift right, leftarrow, "e_Y"'] \arrow[d, shift right, "r_X"'] \arrow[d, shift left, leftarrow, "e_X"] & \mathcal{S}'_{X \times \overset{\bullet}{Y}{}^+} \arrow[d, shift right, "r_X"'] \arrow[d, shift left, leftarrow, "e_X"] \\
\mathcal{S}'_{\overset{\bullet}{X}{}^+ \times Y^+} \arrow[r, shift left, leftarrow, "r_Y^0"] \arrow[r, shift right, "e_Y^0"'] & \mathcal{S}'_{\overset{\bullet}{X}{}^+ \times Y} \arrow[r, shift left, "r_Y"] \arrow[r, shift right, leftarrow, "e_Y"'] & \mathcal{S}'_{\overset{\bullet}{X}{}^+ \times \overset{\bullet}{Y}{}^+}
\end{tikzcd} \tag{4.3.5}$$

Moreover, given any space of the first diagram, it is densely injected in that space of the second diagram which sits at the same position. For example, looking at the left lower corners, $\mathcal{S}_{X^+ \times \overset{\bullet}{Y}{}^+} \overset{d}{\hookrightarrow} \mathcal{S}'_{\overset{\bullet}{X}{}^+ \times Y^+}$.

For $1 \le i \le d$ we denote by

$$r_i : \mathcal{S}'(\mathbb{R}^d, F) \to \mathcal{S}'(\mathbb{R}^{i-1} \times \mathbb{R}^+ \times \mathbb{R}^{d-1}, F)$$

the point-wise restriction, and by e_i its coretraction, constructed in Sections 4.1 and 4.2 (modulo a relabeling of coordinates). Similarly,

$$e_i^0 : \mathcal{S}'(\mathbb{R}^{i-1} \times \dot{\mathbb{R}}^+ \times \mathbb{R}^{d-1}, F) \to \mathcal{S}'(\mathbb{R}^d, F)$$

is the trivial extension, and r_i^0 is the corresponding coretraction.

Let $\mathbb{K}$ be any standard k-corner in $\mathbb{R}^d$. Using notation (4.3.2) set[4]

$$r_{\mathbb{K}} := \prod_{j \in J} r_j \prod_{j^* \in J^*} r_{j^*}^0, \quad e_{\mathbb{K}} := \prod_{j \in J} e_j \prod_{j^* \in J^*} e_{j^*}^0. \tag{4.3.6}$$

Then the preceding considerations extend easily to imply the validity of the following generalization of Theorem 4.2.2.

4.3.1 Theorem *The diagram*

$$\begin{tikzcd}
\mathcal{S}(\mathbb{R}^d, F) \arrow[r, shift left, "r_{\mathbb{K}}"] \arrow[d, hook, "d"] & \mathcal{S}(\mathbb{K}, F) \arrow[l, shift left, "e_{\mathbb{K}}"] \arrow[d, hook, "d"] \\
\mathcal{S}'(\mathbb{R}^d, F) \arrow[r, shift left, "r_{\mathbb{K}}"] & \mathcal{S}'(\mathbb{K}^*, F) \arrow[l, shift left, "e_{\mathbb{K}}"]
\end{tikzcd}$$

is commuting, $r_{\mathbb{K}}$ is a retraction and $e_{\mathbb{K}}$ is a coretraction for it.

Similarly as in the case of half-spaces, we introduce a bilinear form

$$\langle \cdot, \cdot \rangle_{\mathbb{K}} : \mathcal{S}'(\mathbb{K}, F') \times \mathcal{S}(\mathbb{K}, F) \to \mathbb{C}$$

by

$$\langle u', u \rangle_{\mathbb{K}} := \langle e_{\mathbb{K}^*} u', e_{\mathbb{K}} u \rangle. \tag{4.3.7}$$

Then we obtain a generalization of Theorem 4.2.6:

4.3.2 Theorem *The bilinear form $\langle \cdot, \cdot \rangle_{\mathbb{K}}$ is separately continuous and determined by its values on $\mathcal{S}(\mathbb{K}^*, F') \times \mathcal{S}(\mathbb{K}, F)$ which are given by*

$$\langle u', u \rangle_{\mathbb{K}} = \int_{\mathbb{K}} \big\langle u'(x), u(x) \big\rangle_F \, dx, \qquad (u', u) \in \mathcal{S}(\mathbb{K}^*, F') \times \mathcal{S}(\mathbb{K}, F). \tag{4.3.8}$$

Proof. The first two statements are clear. Write

$$e_{\mathbb{K}} = \varepsilon_1 \circ \cdots \circ \varepsilon_k, \quad e_{\mathbb{K}^*} = \varepsilon_1^* \circ \cdots \circ \varepsilon_k^*$$

with $\varepsilon_i, \varepsilon_i^* \in \{e_i, e_i^0\}$ and note that for each i either ε_i or ε_i^* is the trivial extension. Thus formula (4.3.8) is a consequence of Fubini's theorem. $\square$

[4] In any product of maps or spaces we always use — unless explicitly indicated otherwise — the natural ordering such that the object with the lowest index stands on the left.

4.4 Function spaces on corners

Throughout this section we suppose[5]

- *E is a UMD space which possesses property (α) if $\boldsymbol{\omega} \neq \omega(1,\dots,1)$;*
- *$1 < p < \infty,\ \ 1 \leq q \leq \infty$;*
- *$1 \leq k \leq d$ and $\mathbb{K}$ is a standard k-corner in $\mathbb{R}^d$.*

Suppose $\mathfrak{F}(\mathbb{R}^d,E)$ is a Banach space satisfying

$$\mathcal{S}(\mathbb{R}^d,E) \overset{d}{\hookrightarrow} \mathfrak{F}(\mathbb{R}^d,E) \overset{d}{\hookrightarrow} \mathcal{S}'(\mathbb{R}^d,E). \tag{4.4.1}$$

It follows from Theorem 4.3.1 and Remarks 2.2.1 that

$$\mathfrak{F}(\mathbb{K},E) := r_{\mathbb{K}}\mathfrak{F}(\mathbb{R}^d,E) \tag{4.4.2}$$

is a well-defined Banach space satisfying

$$\mathcal{S}(\mathbb{K},E) \overset{d}{\hookrightarrow} \mathfrak{F}(\mathbb{K},E) \overset{d}{\hookrightarrow} \mathcal{S}'(\mathbb{K},E). \tag{4.4.3}$$

Choosing for $\mathfrak{F}$ the symbol $\overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q}$ we so obtain the anisotropic **Besov scales**

$$\big[\, \overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q}(\mathbb{K},E)\ ;\ s \in \mathbb{R} \,\big],$$

for $\mathfrak{F} = H^{s/\boldsymbol{\nu}}_p$ the anisotropic **Bessel potential scales**

$$\big[\, H^{s/\boldsymbol{\nu}}_p(\mathbb{K},E)\ ;\ s \in \mathbb{R} \,\big],$$

for $\mathfrak{F} := \overset{\vartriangle}{N}{}^{s/\boldsymbol{\nu}}_p$ the anisotropic **small Nikol′skiĭ scales**

$$\big[\, \overset{\vartriangle}{N}{}^{s/\boldsymbol{\nu}}_p(\mathbb{K},E)\ ;\ s \in \mathbb{R}^+ \,\big],$$

and for $\mathfrak{F} = C^{s/\boldsymbol{\nu}}_0$ the anisotropic **small Hölder scale**

$$\big[\, C^{s}_0(\mathbb{K},E)\ ;\ s \in \mathbb{R}^+ \,\big]$$

on $\mathbb{K}$.

Similarly as in the case of $\mathbb{R}^n$, the anisotropic **Sobolev–Slobodeckii scale**

$$\big[\, W^{s/\boldsymbol{\nu}}_p(\mathbb{K},E)\ ;\ s \in \mathbb{R} \,\big]$$

on $\mathbb{K}$ is defined by

$$W^{s/\boldsymbol{\nu}}_p(\mathbb{K},E) := \begin{cases} H^{s/\boldsymbol{\nu}}_p(\mathbb{K},E), & s \in \mathbb{Z}, \\ B^{s/\boldsymbol{\nu}}_p(\mathbb{K},E), & s \in \mathbb{R}\backslash\mathbb{Z}. \end{cases}$$

In general, the diagram

$$\begin{tikzcd} \mathfrak{F}(\mathbb{R}^d,E) \arrow[rr, "r_{\mathbb{K}}"] & & \mathfrak{F}(\mathbb{K},E) \\ & \mathfrak{F}(\mathbb{K},E) \arrow[ul, "e_{\mathbb{K}}"] \arrow[ur, "\mathrm{id}"'] & \end{tikzcd} \tag{4.4.4}$$

[5] For simplicity of presentation and since it will suffice for our purposes we consider only UMD spaces and $p \in (1,\infty)$ although some of the following results could be shown for an arbitrary Banach space E and $p \in [1,\infty]$.

is commuting. Thus $r_{\mathbb{K}}$ is a **universal retraction**, and $e_{\mathbb{K}}$ a universal coretraction for it, in the sense that, given a Banach space $\mathfrak{G}(\mathbb{R}^d, E)$ satisfying

$$\mathfrak{F}(\mathbb{R}^d, E) \overset{d}{\hookrightarrow} \mathfrak{G}(\mathbb{R}^d, E) \overset{d}{\hookrightarrow} \mathcal{S}'(\mathbb{R}^d, E),$$

the diagram

$$\begin{tikzcd} \mathfrak{F}(\mathbb{R}^d, E) \arrow[r, shift left, "r_{\mathbb{K}}"] \arrow[d, hook, "d"] & \mathfrak{F}(\mathbb{K}, E) \arrow[l, shift left, "e_{\mathbb{K}}"] \arrow[d, hook, "d"] \\ \mathfrak{G}(\mathbb{R}^d, E) \arrow[r, shift left, "r_{\mathbb{K}}"] & \mathfrak{G}(\mathbb{K}, E) \arrow[l, shift left, "e_{\mathbb{K}}"] \end{tikzcd} \tag{4.4.5}$$

is commuting. Furthermore, $r_{\mathbb{K}}$ is also independent, in an obvious sense, of the parameters p, q, and s subject to indicated restrictions. This parameter-independence is also meant when we use the qualifier 'universal'.

We now collect some important consequences of (4.4.4) of which we shall make frequent use in the following, often without explicitly referring to the theorems below.

4.4.1 Theorem *The Besov, Bessel potential, small Nikol′skiĭ, small Hölder, and Sobolev–Slobodeckii scales on $\mathbb{K}$ possess the same embedding and interpolation properties as the corresponding scales on $\mathbb{R}^d$.*

Proof. This follows from Remarks 2.2.1, Proposition I.2.3.2 in H.Amann [4], and from (4.4.4). □

Lemma 2.3.7 shows that differentiation behaves naturally with respect to the order of the Banach scales on $\mathbb{R}^d$. The same is true for the $\mathbb{K}$-case.

4.4.2 Theorem *Suppose $\alpha \in \mathbb{N}^d$. Then ∂^α is a continuous linear map from*

$$\overset{\scriptscriptstyle\triangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q}(\mathbb{K}, E) \text{ into } \overset{\scriptscriptstyle\triangle}{B}{}^{(s-\alpha\cdot\boldsymbol{\omega})/\boldsymbol{\nu}}_{p,q}(\mathbb{K}, E)$$

and from

$$H^{s/\boldsymbol{\nu}}_p(\mathbb{K}, E) \text{ into } H^{(s-\alpha\cdot\boldsymbol{\omega})/\boldsymbol{\nu}}_p(\mathbb{K}, E).$$

If $k \in \mathbb{N}$ and $k\nu \geq \alpha \cdot \boldsymbol{\omega}$, then ∂^α is also a continuous linear map from $C^{k\nu/\boldsymbol{\nu}}_0(\mathbb{K}, E)$ into $C^{(k\nu-\alpha\cdot\boldsymbol{\omega})/\boldsymbol{\nu}}_0(\mathbb{K}, E)$.

Proof. (1) Suppose $\mathbb{K} = \mathbb{H}^d$. Given $u \in H^{s/\boldsymbol{\nu}}_p(\mathbb{H}^d, E)$ and $\varphi \in \mathcal{D}(\mathring{\mathbb{H}}^d)$,

$$\partial^\alpha(e^+u)(\varphi) = (-1)^{|\alpha|}(e^+u)(\partial^\alpha\varphi) = (-1)^{|\alpha|}u(\partial^\alpha\varphi) = \partial^\alpha u(\varphi).$$

From this, Lemma 2.3.7, Theorem 4.2.4(i), and (4.4.3) we infer that $\partial^\alpha u$ belongs to $H^{(s-\alpha\cdot\boldsymbol{\omega})/\boldsymbol{\nu}}_p(\mathbb{H}^d, E)$. Thus the diagram

$$\begin{tikzcd} H^{s/\boldsymbol{\nu}}_p(\mathbb{H}^d, E) \arrow[r, "e^+"] \arrow[d, "\partial^\alpha"'] & H^{s/\boldsymbol{\nu}}_p(\mathbb{R}^d, E) \arrow[d, "\partial^\alpha"] \\ H^{(s-\alpha\cdot\boldsymbol{\omega})/\boldsymbol{\nu}}_p(\mathbb{H}^d, E) & H^{(s-\alpha\cdot\boldsymbol{\omega})/\boldsymbol{\nu}}_p(\mathbb{R}^d, E) \arrow[l, "r^+"'] \end{tikzcd} \tag{4.4.6}$$

is commuting. This proves the assertion in this case for the Bessel potential scale. The proof for the other cases is identical.

(2) Assume $\mathbb{K} = \mathring{\mathbb{H}}^d$. If $u \in H_p^{s/\boldsymbol{\nu}}(\mathring{\mathbb{H}}^d, E)$, then $e_0^+ u \in H_{p,\mathbb{H}^d}^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E)$ by Theorem 4.2.4(ii), using obvious notation. It is clear that then $\partial^\alpha(e_0^+ u)$ belongs to $H_{p,\mathbb{H}^d}^{(s-\alpha\cdot\boldsymbol{\omega})/\boldsymbol{\nu}}(\mathbb{R}^d, E)$. From this we see that we obtain a commuting diagram by replacing $\mathbb{H}^d$, e^+, and r^+ in (4.4.6) by $\mathring{\mathbb{H}}^d$, e_0^+, and r_0^+, respectively. This way we deduce the assertions for $\mathbb{K} = \mathring{\mathbb{H}}^d$.

(3) Using (1) and (2) the assertions for the general case follow now by an obvious argument from the definition of $e_{\mathbb{K}}$ and $r_{\mathbb{K}}$. □

Our next theorem shows that Sobolev–Slobodeckii and small Hölder scales of positive order on $\mathbb{K}$ can be characterized intrinsically. For this we define

$$\|\cdot\|_{s/\boldsymbol{\nu},p,\mathbb{K}}$$

etc. by restricting integration (respectively the essential supremum) in (3.8.3) to $\mathbb{K}$. For example, suppose $s > 0$ and $d_1 \le k$. Then

$$[u]^p_{s/\nu_1,p,\mathbb{K}} = \int_{(\mathring{\mathbb{R}}^+)^{d_1}} |h|^{-ps/\nu_1} \, \|\triangle_{(h,0)}^{[s/\nu_1]+1} u\|^p_{L_p(\mathbb{K},E)} \, dh/|h|^{d_1}.$$

4.4.3 Theorem

(i) *Suppose $s \ge 0$. Then $\|\cdot\|_{s/\boldsymbol{\nu},p,\mathbb{K}}$ is an equivalent norm for $W_p^{s/\boldsymbol{\nu}}(\mathbb{K}, E)$, and $\|\cdot\|_{s/\boldsymbol{\nu},\infty,\mathbb{K}}$ is one for $C_0^s(\mathbb{K}, E)$.*
Thus if $k \in \mathbb{N}$ and $\alpha \cdot \boldsymbol{\omega} \le k\nu$, then

$$u \in W_p^{k\nu/\boldsymbol{\nu}}(\mathbb{K}, E) \text{ iff } \partial^\alpha u \in L_p(\mathbb{K}, E),$$

and

$$u \in C_0^{k\nu/\boldsymbol{\nu}}(\mathbb{K}, E) \text{ iff } \partial^\alpha u \in C_0(\mathbb{K}, E).$$

(ii) *Suppose $s > 0$. Then*

$$u \in B_p^{s/\boldsymbol{\nu}}(\mathbb{K}, E) \text{ iff } u \in L_p(\mathbb{K}, E) \text{ and } [u]_{s/\boldsymbol{\nu},p,\mathbb{K}} < \infty,$$

and $u \in C_0^{s/\boldsymbol{\nu}}(\mathbb{K}, E)$ iff u belongs to the closure of $\mathcal{S}(\mathbb{K}, E)$ in

$$\{\, u \in BUC(\mathbb{K}, E) \ ; \ [u]_{s/\boldsymbol{\nu},\infty,\mathbb{K}} < \infty \,\}.$$

Proof. (1) Trivially,

$$\|r_{\mathbb{K}} v\|_{p,\mathbb{K}} = \|v\|_{p,\mathbb{K}} \le \|v\|_p, \qquad v \in \mathcal{S}(\mathbb{R}^d, E).$$

Thus, given $u \in L_p(\mathbb{K}, E)$, it follows $\|u\|_{p,\mathbb{K}} \le \|u\|_{r_{\mathbb{K}} L_p}$ by definition of the quotient norm of $r_{\mathbb{K}} L_p$. On the other hand, Lemma 4.1.2 and the definition of e_0^+ and $e_{\mathbb{K}}$ imply

$$\|e_{\mathbb{K}} u\|_p \le c\,\|u\|_{p,\mathbb{K}}, \qquad u \in \mathcal{S}(\mathbb{K}, E).$$

From this we infer $\|\cdot\|_{r_{\mathbb{K}} L_p} \le c\,\|\cdot\|_{p,\mathbb{K}}$. This proves $r_{\mathbb{K}} L_p(\mathbb{R}^d, E) \doteq L_p(\mathbb{K}, E)$. Similarly, $r_{\mathbb{K}} C_0(\mathbb{R}^d, E) \doteq C_0(\mathbb{K}, E)$. Thus (i) is true if $s = 0$.

(2) Assume $s = k\nu$ for some $k \in \mathbb{N}$. Then (i) follows from (1) and the fact, obtained from (4.4.6), that ∂^α commutes with $r_{\mathbb{K}}$ and $e_{\mathbb{K}}$.

(3) Suppose $s > 0$. Given $u : \mathbb{R}^+ \to E$ and $t > 0$, set $\psi_t u(y) := u(ty)$ for $y \geq 0$. Then one verifies $\triangle_\tau^k \circ \psi_t = \psi_t \circ \triangle_{t\tau}^k$ for $\tau, t \geq 0$ and $k \in \mathbb{N}$.

Set $X := L_p(\mathbb{R}^{d-1}, E)$. Then, by substitution of variables and (4.1.2),

$$\begin{aligned}
&\Bigl\| \tau^{-s} \bigl\| \triangle_\tau^{[s]+1} \varepsilon u \bigr\|_{L_p(-\mathbb{R}^+,X)} \Bigr\|_{L_p(\dot{\mathbb{R}}^+,d\tau/\tau)}^p \\
&\quad = \int_0^\infty \tau^{-sp} \int_{-\infty}^0 \Bigl\| \int_0^\infty h(t) \triangle_\tau^{[s]+1} \bigl(\psi_t u(-y)\bigr)\, dt \Bigr\|_X^p \, dy \, \frac{d\tau}{\tau} \\
&\quad = \int_0^\infty \tau^{-sp} \int_{-\infty}^0 \Bigl\| \int_0^\infty h(t) \bigl(\triangle_{t\tau}^{[s]+1} u\bigr)(-ty)\, dt \Bigr\|_X^p \, dy \, \frac{d\tau}{\tau} \\
&\quad \leq \int_0^\infty r^{-sp} \int_0^\infty \Bigl\| \int_0^\infty t^{s-1/p} \, |h(t)| \, \bigl| \triangle_r^{[s]+1} u(z) \bigr| \, dt \Bigr\|_X^p \, dz \, \frac{dr}{r} \\
&\quad \leq c \Bigl\| r^{-s} \bigl\| \triangle_r^{[s]+1} u \bigr\|_{L_p(\mathbb{R}^+,X)} \Bigr\|_{L_p(\dot{\mathbb{R}}^+,dr/r)}^p \\
&\quad = c \, [u]_{s,p,\mathbb{H}^d;1}^p .
\end{aligned}$$

This and (4.1.8) imply

$$\begin{aligned}
[e^+ u]_{s,p;1}^p &= [u]_{s,p,\mathbb{H}^d;1}^p + \Bigl\| \tau^{-s} \bigl\| \triangle_\tau^{[s]+1} \varepsilon u \bigr\|_{L_p(-\mathbb{R}^+,X)} \Bigr\|_{L_p(\dot{\mathbb{R}}^+,d\tau/\tau)}^p \\
&\leq c \, [u]_{s,p,\mathbb{H}^d;1}^p .
\end{aligned}$$

Analogously, $[e^+ u]_{s,\infty;1} \leq c \, [u]_{s,\infty,\mathbb{H}^d;1}$. From this and the definition of the (quotient) norm for $B_p^{s/\boldsymbol{\nu}}(\mathbb{H}^d, E)$ it follows

$$\|u\|_{B_p^{s/\boldsymbol{\nu}}(\mathbb{H}^d,E)} \leq c(\|u\|_{p,\mathbb{H}^d} + [u]_{s/\boldsymbol{\nu},p,\mathbb{H}^d}), \qquad u \in \mathcal{S}(\mathbb{H}^d, E).$$

Similarly,

$$\|u\|_{C_0^{s/\boldsymbol{\nu}}(\mathbb{H}^d,E)} \leq c \, \|u\|_{s/\boldsymbol{\nu},\infty,\mathbb{H}^d}, \qquad u \in \mathcal{S}(\mathbb{H}^d, E).$$

Since the converse estimates are obvious, (ii) is true for $s \neq k\nu$, provided $\mathbb{K} = \mathbb{H}^d$. It is easy to see that it also holds for $\mathbb{K} = \mathring{\mathbb{H}}^d$. Now the extension to an arbitrary standard corner $\mathbb{K}$ is obvious and left to the reader. $\square$

Lastly, there is a duality theorem on $\mathbb{K}$ which is the analogue of Theorems 3.3.3 and 3.7.1(i).

4.4.4 Theorem *Suppose $1 < q < \infty$. Then $B_{p,q}^{s/\boldsymbol{\nu}}(\mathbb{K}, E)$ and $H_p^{s/\boldsymbol{\nu}}(\mathbb{K}, E)$ are reflexive. Moreover,*

$$B_{p,q}^{s/\boldsymbol{\nu}}(\mathbb{K}, E)' \doteq B_{p',q'}^{-s/\boldsymbol{\nu}}(\mathbb{K}^*, E') \tag{4.4.7}$$

and

$$H_p^{s/\boldsymbol{\nu}}(\mathbb{K}, E)' \doteq H_{p'}^{-s/\boldsymbol{\nu}}(\mathbb{K}^*, E'). \tag{4.4.8}$$

Proof. (1) Since, by Lemma 4.1.5, $B_{p,q}^{s/\boldsymbol{\nu}}(\mathbb{K}, E)$ is isomorphic to the closed linear subspace $e_{\mathbb{K}} B_{p,q}^{s/\boldsymbol{\nu}}(\mathbb{K}, E)$ of $B_{p,q}^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E)$ and the latter space is reflexive, $B_{p,q}^{s/\boldsymbol{\nu}}(\mathbb{K}, E)$ is also reflexive.

(2) Set $p_{\mathbb{K}} := e_{\mathbb{K}} r_{\mathbb{K}}$. Then (4.4.4) and Lemma 4.1.5 imply

$$B_{p,q}^{s}(\mathbb{R}^d, E) = e_{\mathbb{K}} B_{p,q}^{s/\boldsymbol{\nu}}(\mathbb{K}, E) \oplus (1 - p_{\mathbb{K}}) B_{p,q}^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E) \tag{4.4.9}$$

and

$$B_{p',q'}^{-s/\boldsymbol{\nu}}(\mathbb{R}^d, E') = e_{\mathbb{K}^*} B_{p',q'}^{-s/\boldsymbol{\nu}}(\mathbb{K}^*, E') \oplus (1 - p_{\mathbb{K}^*}) B_{p',q'}^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E'). \tag{4.4.10}$$

Suppose $f = e_{\mathbb{K}^*} g$ and $u = (1 - p_{\mathbb{K}})v$ with $g \in B_{p',q'}^{-s/\boldsymbol{\nu}}(\mathbb{K}^*, E')$ and $v \in B_{p,q}^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E)$. We claim

$$\langle f, u \rangle = 0. \tag{4.4.11}$$

Since $\mathcal{S}(\mathbb{R}^d) \otimes E$ is dense in $\mathcal{S}(\mathbb{R}^d, E)$, hence in $B_{p,q}^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E)$, it suffices to prove the claim for $u = \varphi \otimes \xi$ with $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $\xi \in E$. In this case

$$\langle f, u \rangle = \big\langle \langle e_{\mathbb{K}^*} g, (1 - e_{\mathbb{K}} r_{\mathbb{K}})\varphi \rangle, \xi \big\rangle_E.$$

Now it follows from Theorem 4.2.2 and definitions (4.3.6) that

$$\big\langle e_{\mathbb{K}^*} g, (1 - e_{\mathbb{K}} r_{\mathbb{K}})\varphi \big\rangle = \big\langle g, r_{\mathbb{K}}(1 - e_{\mathbb{K}} r_{\mathbb{K}})\varphi \big\rangle = 0.$$

This proves (4.4.11).

(3) From (4.4.9)–(4.4.11) we deduce

$$\begin{aligned} \langle w', w \rangle &= \langle p_{\mathbb{K}^*} w', p_{\mathbb{K}} w \rangle + \big\langle (1 - p_{\mathbb{K}^*}) w', (1 - p_{\mathbb{K}}) w \big\rangle \\ &= \langle u', u \rangle_{\mathbb{K}} + \big\langle (1 - p_{\mathbb{K}^*}) w', (1 - p_{\mathbb{K}}) w \big\rangle \end{aligned}$$

for $(w', w) \in B_{p',q'}^{-s/\boldsymbol{\nu}}(\mathbb{R}^d, E') \times B_{p,q}^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E)$ and $(u', u) := (r_{\mathbb{K}^*} w', r_{\mathbb{K}} w)$. Hence, as $r_{\mathbb{K}}$ and $r_{\mathbb{K}^*}$ are isomorphisms on the first summand of (4.4.9) and of (4.4.10), respectively, we see that

$$\langle \cdot, \cdot \rangle_{\mathbb{K}} : B_{p',q'}^{-s/\boldsymbol{\nu}}(\mathbb{K}^*, E') \times B_{p,q}^{s/\boldsymbol{\nu}}(\mathbb{K}, E) \to \mathbb{C}$$

is a separating continuous bilinear form. This implies

$$B_{p',q'}^{-s/\boldsymbol{\nu}}(\mathbb{K}^*, E') \hookrightarrow B_{p,q}^{s/\boldsymbol{\nu}}(\mathbb{K}, E)' \tag{4.4.12}$$

with respect to $\langle \cdot, \cdot \rangle_{\mathbb{K}}$.

(4) Suppose $f \in B_{p,q}^{s/\boldsymbol{\nu}}(\mathbb{K}, E)'$. Then $g := f r_{\mathbb{K}} \in B_{p,q}^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E)'$. Hence Corollary 3.3.4 implies $g \in B_{p',q'}^{-s/\boldsymbol{\nu}}(\mathbb{R}^d, E')$. Thus $r_{\mathbb{K}^*} g \in B_{p',q'}^{-s/\boldsymbol{\nu}}(\mathbb{K}^*, E')$. By the arguments of step (2) we find

$$\langle r_{\mathbb{K}^*} g, u \rangle_{\mathbb{K}} = \langle g, e_{\mathbb{K}} u \rangle = f(u), \qquad u \in B_{p,q}^{s/\boldsymbol{\nu}}(\mathbb{K}, E).$$

Combining this with (4.4.12) we obtain assertion (4.4.7).

(5) The proof for the Bessel potential spaces is literally the same, except that we have to use Theorem 3.7.1(i) instead of Corollary 3.3.4. □

4.4.5 Remark The notation introduced above and used throughout most of this treatise is extremely convenient for handling function spaces on corners. However, it should be observed that *in the particular case of half-spaces it is different from standard usage.* This stems from the fact that we build our theory on the *closed* half-space $\mathbb{H}^n$ whereas it is common practice to consider the open half-space, usually denoted by $\mathbb{R}_+^n$. For example, 'our' Sobolev–Slobodeckii space $W_p^s(\mathbb{H}^n)$ corresponds to the 'usual' $W_p^s(\mathbb{R}_+^n)$, whereas $W_p^s(\mathring{\mathbb{H}}^n)$ is usually written as $\mathring{W}_p^s(\mathbb{R}_+^n)$, etc. □

4.5 Traces on half-spaces

Throughout this section

- *E is a UMD space which has property (α) if $\boldsymbol{\omega} \neq \omega(1,\dots,1)$;*
- *$1 < p < \infty,\ \ 1 \le q \le \infty$;* (4.5.1)
- *$d \ge 2$.*

Recall that $\boldsymbol{\omega}' = (\omega_2,\dots,\omega_d)$ and that Lemmas 3.3.1 and 2.3.5 imply

$$B^{s/\boldsymbol{\nu}}_{p,q}(\mathbb{R}^d,E) \doteq B^{s/\boldsymbol{\omega}}_{p,q}(\mathbb{R}^d,E), \quad H^{s/\boldsymbol{\nu}}_{p}(\mathbb{R}^d,E) \doteq H^{s/\boldsymbol{\omega}}_{p}(\mathbb{R}^d,E).$$

The following proposition is the basis for defining a universal trace operator for all Besov and Bessel potential spaces of sufficiently high order.

4.5.1 Proposition *Suppose $s > \omega_1/p$. Then*

$$B^{s/\boldsymbol{\nu}}_{p,q}(\mathbb{R}^d,E) \hookrightarrow BUC\big(\mathbb{R},L_p(\mathbb{R}^{d-1},E)\big) \tag{4.5.2}$$

and

$$H^{s/\boldsymbol{\nu}}_{p}(\mathbb{R}^d,E) \hookrightarrow BUC\big(\mathbb{R},L_p(\mathbb{R}^{d-1},E)\big). \tag{4.5.3}$$

Proof. (1) In the scalar case (4.5.2) is a special case of the more general Proposition 1 in W. Farkas, J. Johnsen, and W. Sickel [23]. It is easily verified that their proof carries over to the vector-valued situation.[6]

(2) Since $H^{s/\boldsymbol{\nu}}_{p}(\mathbb{R}^d,E) \hookrightarrow B^{s/\boldsymbol{\nu}}_{p,\infty}(\mathbb{R}^d,E)$ by Theorem 3.7.1(iii), the assertion follows from (4.5.2). □

Note that

$$\gamma:\ C\big(\mathbb{R},\mathcal{S}'(\mathbb{R}^{d-1},E)\big) \to \mathcal{S}'(\mathbb{R}^{d-1},E), \quad u \mapsto u(0)$$

is a well-defined linear map, the **trace operator** (with respect to $x^1 = 0$). Given any Banach space $\mathfrak{F}(\mathbb{R}^d,E)$ satisfying (with obvious identifications)

$$\mathcal{S}(\mathbb{R}^d,E) \hookrightarrow \mathfrak{F}(\mathbb{R}^d,E) \hookrightarrow C\big(\mathbb{R},\mathcal{S}'(\mathbb{R}^{d-1},E)\big),$$

the restriction of γ to $\mathfrak{F}(\mathbb{R}^d,E)$ is again denoted by γ and called **trace operator on** $\mathfrak{F}(\mathbb{R}^d,E)$. The image space $\gamma\mathfrak{F}(\mathbb{R}^d,E)$ is the **trace space of** $\mathfrak{F}(\mathbb{R}^d,E)$. Thus

$$\gamma\mathfrak{F}(\mathbb{R}^d,E) \hookrightarrow \mathcal{S}'(\mathbb{R}^{d-1},E),$$

that is, each element in the trace space is a temperate distribution on $\mathbb{R}^{d-1}$.

In the cases of interest for us we have more information. Namely, by Proposition 4.5.1,

$$\gamma:\ B^{s/\boldsymbol{\nu}}_{p,q}(\mathbb{R}^d,E) \to L_p(\mathbb{R}^{d-1},E)$$

is well-defined for $s > \omega_1/p$. Similarly, if $s > \omega_1/p$, then

$$\gamma:\ H^{s/\boldsymbol{\nu}}_{p}(\mathbb{R}^d,E) \to L_p(\mathbb{R}^{d-1},E).$$

Thus, if $s > \omega_1/p$,

$$\gamma B^{s/\boldsymbol{\nu}}_{p,q}(\mathbb{R}^d,E) \hookrightarrow L_p(\mathbb{R}^{d-1},E)$$

---

[6] By Lemma 1.2.1 all quasi-norms are equivalent. Thus, instead of basing our considerations on Λ, we can equally well use the Euclidean ω-quasi-norm E, as these authors do.

and

$$\gamma H_p^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E) \hookrightarrow L_p(\mathbb{R}^{d-1}, E).$$

In fact, in these cases the trace spaces can be explicitly characterized by anisotropic Besov spaces.

4.5.2 Theorem *Suppose $s > \omega_1/p$. Then*

$$\gamma \overset{\vartriangle}{B}{}_{p,q}^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E) \doteq \overset{\vartriangle}{B}{}_{p,q}^{(s-\omega_1/p)/\boldsymbol{\omega}'}(\mathbb{R}^{d-1}, E) \tag{4.5.4}$$

and

$$\gamma H_p^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E) \doteq B_p^{(s-\omega_1/p)/\boldsymbol{\omega}'}(\mathbb{R}^{d-1}, E). \tag{4.5.5}$$

Moreover, γ is a retraction possessing a universal coretraction.

Proof. (1) In the Besov space case the assertion follows by literally transcribing the proof of the corresponding part of Theorem 3 of W. Farkas, J. Johnsen, and W. Sickel [23].

(2) To prove (4.5.5) we need the vector-valued anisotropic Triebel–Lizorkin spaces $F_{p,q}^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E)$ which are defined as in the scalar case (see [23] and, for the vector-valued isotropic case, Section 15 in H. Triebel [67] or H.-J. Schmeißer and W. Sickel [54]). It is not difficult to verify, writing for abbreviation $B_{p,q}^{s/\boldsymbol{\nu}}$ for $B_{p,q}^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E)$ etc.,

$$B_{p,1}^{m/\boldsymbol{\nu}} \hookrightarrow F_{p,1}^{m/\boldsymbol{\nu}} \hookrightarrow W_p^{m/\boldsymbol{\nu}} \hookrightarrow F_{p,\infty}^{m/\boldsymbol{\nu}} \hookrightarrow B_{p,\infty}^{m/\boldsymbol{\nu}}, \qquad m \in \nu\mathbb{N}.$$

Using $J^t \in \mathcal{L}\mathrm{is}(F_{p,q}^{(s+t)/\boldsymbol{\nu}}, F_{p,q}^{s/\boldsymbol{\nu}})$ for $s, t \in \mathbb{R}$ and (3.7.1) it thus follows

$$F_{p,1}^{s/\boldsymbol{\nu}} \hookrightarrow H_p^{s/\boldsymbol{\nu}} \hookrightarrow F_{p,\infty}^{s/\boldsymbol{\nu}}, \qquad s \in \mathbb{R}. \tag{4.5.6}$$

Next one verifies that the proof of Proposition 8 in [23] carries over to the vector-valued situation to give

$$\gamma\big(F_{p,1}^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E)\big) = \gamma\big(F_{p,\infty}^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E)\big)$$

(also see Proposition 10 in [54]). Hence the assertion is obtained from (4.5.6) provided we show that γ is a retraction from

$$F_{p,q}^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E) \text{ onto } B_p^{(s-\omega_1/p)/\boldsymbol{\omega}'}(\mathbb{R}^{d-1}, E)$$

for $q \in \{1, \infty\}$. This is done by modifying appropriately the proof of Theorem 2.7.2 in H. Triebel [66]. $\square$

4.5.3 Remark In the scalar case it is well-known that

$$F_{p,2}^s(\mathbb{R}^d) \doteq H_p^s(\mathbb{R}^d), \qquad 1 < p < \infty. \tag{4.5.7}$$

However, this is not true, in general, in the vector-valued situation. In fact, the vector-valued analogue of (4.5.7) holds iff E is a Hilbert space (cf. Remark 7 in H.-J. Schmeißer and W. Sickel [54]). $\square$

The **trace operator for** $\mathbb{H}^d$ is defined by

$$\gamma_{\partial\mathbb{H}^d} := \gamma \circ e^+.$$

The following theorem is almost evident. To simplify the notation we write $\mathbb{H}$ instead of $\mathbb{H}^d$.

4.5.4 Theorem *Suppose $s > \omega_1/p$. Then $\gamma_{\partial\mathbb{H}}$ is a retraction from*

$$\overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q}(\mathbb{H},E) \text{ onto } \overset{\vartriangle}{B}{}^{(s-\omega_1/p)/\boldsymbol{\omega}'}_{p,q}(\partial\mathbb{H},E)$$

and from

$$H^{s/\boldsymbol{\nu}}_p(\mathbb{H},E) \text{ onto } B^{(s-\omega_1/p)/\boldsymbol{\omega}'}_p(\partial\mathbb{H},E).$$

It possesses a universal coretraction.

Proof. We consider the H-case. The proof for Besov spaces is identical.

First note that $\gamma_{\partial\mathbb{H}}$ maps $\mathfrak{F}^s := H^s_p(\mathbb{H},E)$ continuously into

$$\partial\mathfrak{F}^s := B^{(s-\omega_1/p)/\boldsymbol{\omega}'}_p(\partial\mathbb{H},E).$$

Let γ^c be a universal coretraction for γ. Then

$$\gamma^c_{\partial\mathbb{H}} := r^+ \circ \gamma^c \in \mathcal{L}(\partial\mathfrak{F}^s, \mathfrak{F}^s).$$

Suppose $v \in \mathcal{S}(\partial\mathbb{H},E)$. Then $\gamma^c_{\partial\mathbb{H}}v \in \mathfrak{F}^t$ for all $t > 0$. Thus $\gamma^c_{\partial\mathbb{H}}v$ is smooth by the Sobolev embedding theorem 3.9.1. Hence, given $x' \in \mathbb{R}^{d-1}$,

$$\gamma_{\partial\mathbb{H}}\gamma^c_{\partial\mathbb{H}}v(x') = (r^+\gamma^c v)(0,x') = \gamma^c v(0,x') = \gamma\gamma^c v(x') = v(x').$$

This proves $\gamma_{\partial\mathbb{H}}\gamma^c_{\partial\mathbb{H}}v = v$ for $v \in \mathcal{S}(\partial\mathbb{H},E)$. Thus, by density and continuity, $\gamma^c_{\partial\mathbb{H}}$ is a right inverse for $\gamma_{\partial\mathbb{H}}$. □

4.6 Higher order traces on half-spaces

Unless explicitly stated otherwise, throughout the rest of this part it is assumed

- *E is a UMD space which has property (α) if $\boldsymbol{\omega} \neq \omega(1,\ldots,1)$;*
- *$1 < p < \infty$, $1 \le q \le \infty$.*

We write $\mathbb{H} := \mathbb{H}^d$ and denote by $\boldsymbol{n}$ the **outer** (unit) **normal on** $\partial\mathbb{H}$, that is, $\boldsymbol{n} := (-1,0,\ldots,0)$. Then, given $j \in \mathbb{N}$,

$$\partial^j_{\boldsymbol{n}} := (-1)^j\gamma_{\partial\mathbb{H}} \circ \partial^j_1$$

is the the j-th order **normal derivative** on $\partial\mathbb{H}$. Note $\partial^0_{\boldsymbol{n}} = \gamma_{\partial\mathbb{H}}$.

Generalizing the Trace Theorem 4.5.4 we shall now show that $\partial^j_{\boldsymbol{n}}$ is a retraction onto appropriate boundary spaces. The proof will be based on a well-known characterization of real interpolation spaces by analytic semigroups due to H. Komatsu [43]. For the reader’s convenience we formulate this theorem here since it will also be used later.

Henceforth,

$$L^*_q := L_q(\dot{\mathbb{R}}^+, dt/t)$$

for abbreviation. Recall that $\mathcal{H}_-(X)$ is the set of all negative infinitesimal generators of exponentially decaying strongly continuous analytic semigroups on the Banach space X.

4.6.1 Proposition *Let X be a Banach space and $A \in \mathcal{H}_-(X)$. Then, given $0 < \theta < 1$ and $m \in \dot{\mathbb{N}}$,*

$$x \mapsto \big\| \|t^{m(1-\theta)} A^m e^{-tA} x\|_X \big\|_{L_q^*}$$

is an equivalent norm for $\big(X, D(A^m)\big)_{\theta,q}$.

PROOF. See, for example, Theorem 1.14.5 in H. Triebel [65] for a proof. □

For completeness we include the case $j = 0$ in the following theorem, although it is already covered by Theorem 4.5.4. Note, however, that our construction of a universal coretraction is independent of the latter theorem.

4.6.2 Theorem *Suppose $j \in \mathbb{N}$ and $s > \omega_1(j + 1/p)$. Then $\partial_{\boldsymbol{n}}^j$ is a retraction from*

$$\overset{\scriptscriptstyle\triangle}{B}{}_{p,q}^{s/\boldsymbol{\nu}}(\mathbb{H}, E) \text{ onto } \overset{\scriptscriptstyle\triangle}{B}{}_{p,q}^{(s-\omega_1(j+1/p))/\boldsymbol{\omega}'}(\partial\mathbb{H}, E) \tag{4.6.1}$$

and from

$$H_p^{s/\boldsymbol{\nu}}(\mathbb{H}, E) \text{ onto } B_p^{(s-\omega_1(j+1/p))/\boldsymbol{\omega}'}(\partial\mathbb{H}, E). \tag{4.6.2}$$

There exists a universal coretraction γ_j^c for it. Furthermore, γ_j^c is for every $s \in \mathbb{R}$ *a continuous linear map from the space on the right side of* (4.6.1), *respectively* (4.6.2), *into the one on the left side.*[7]

PROOF. (1) It follows from Lemma 2.3.7 and Theorem 4.5.4 that $\partial_{\boldsymbol{n}}^j$ maps the first space of (4.6.1), resp. (4.6.2), continuously into the second one.

(2) First suppose $s = \omega_1(j + m)$ for some $m \in \dot{\mathbb{N}}$. Then, by Theorems 3.7.1(ii), 3.7.3, and Section 4.4,

$$H_p^{s/\boldsymbol{\nu}}(\mathbb{H}, E) \doteq L_p\big(\mathbb{R}^+, H^{s/\boldsymbol{\omega}'}(\mathbb{R}^{d-1}, E)\big) \cap W_p^{j+m}\big(\mathbb{R}^+, L_p(\mathbb{R}^{d-1}, E)\big). \tag{4.6.3}$$

Denote by ω' the least common multiple of $\omega_2, \ldots, \omega_d$ and set

$$\mathsf{K}(\xi', \eta) := \Big(|\eta|^{2\omega'} + \textstyle\sum_{j=2}^d |\xi^j|^{2\omega'/\omega_j}\Big)^{1/2\omega'}, \qquad \xi' \in \mathbb{R}^{d-1}, \quad \eta \in \mathsf{H}.$$

Then $F := L_p(\mathbb{R}^{d-1}, E)$ is admissible and $\big[\, H_p^{t/\boldsymbol{\omega}'}(\mathbb{R}^{d-1}, E) \,;\, t \in \mathbb{R} \big]$ is the fractional power scale generated by $\big(F, \mathsf{K}_1(D')\big)$.

Put $A := \mathsf{K}_1^{\omega_1}(D')$. Theorem 2.2.4 implies

$$A \in \mathcal{H}_-(F), \quad D(A^k) \doteq H_p^{k\omega_1/\boldsymbol{\omega}'}(\mathbb{R}^{d-1}, E), \qquad k \in \mathbb{N}. \tag{4.6.4}$$

In particular, by (4.6.3)

$$H_p^{s/\boldsymbol{\nu}}(\mathbb{R}^n, E) \doteq L_p\big(\mathbb{R}^+, D(A^{j+m})\big) \cap W_p^{j+m}(\mathbb{R}^+, F). \tag{4.6.5}$$

(3) Set

$$\gamma_j^c v := \Big(t \mapsto (-1)^j \frac{t^j}{j!} e^{-tA} v\Big), \qquad t \geq 0, \quad v \in F. \tag{4.6.6}$$

Then

$$\partial^k \gamma_j^c = \sum_{i=0}^k (-1)^{j+k-i} \binom{k}{i} \frac{t^{j-i}}{(j-i)!} A^{k-i} e^{-tA}, \qquad 0 \leq k \leq j.$$

[7] Observe that this means that any two realizations of γ_j^c on different function spaces coincide on all common elements. Henceforth, we express this by calling such a map again **universal**.

Consequently,

$$(-1)^k \partial^k \gamma_j^c v(0) = \begin{cases} 0, & 0 \le k \le j-1, \\ v, & k = j. \end{cases} \tag{4.6.7}$$

Furthermore,

$$\partial^{j+1} \gamma_j^c = \sum_{i=0}^{j-1} (-1)^i \binom{j}{i} \frac{t^{j-i-1}}{(j-i-1)!} A^{j-i} e^{-tA} - A \partial^j \gamma_j^c.$$

Hence, by induction,

$$\partial^{j+m} \gamma_j^c = \sum_{i=0}^{j} c_i t^i A^{i+m} e^{-tA}, \tag{4.6.8}$$

where $c_i = c_i(j,m)$. Thus

$$\big\| \, \|\partial^{j+m} \gamma_j^c v(t)\|_F \, \big\|_{L_p(\mathbb{R}^+)} \le c \sum_{i=0}^{j} \big\| \, \|t^{i+1/p} A^{i+m} e^{-tA} v\|_F \, \big\|_{L_p^*}. \tag{4.6.9}$$

Fix $i \in \{1, \dots, j\}$ and set $\theta := (m - 1/p)/(i+m)$ so that $i + 1/p = (i+m)(1-\theta)$. Then Proposition 4.6.1 implies

$$\big\| \, \|t^{i+1/p} A^{i+m} e^{-tA} v\|_F \, \big\|_{L_p^*} \le \|v\|_{(F, D(A^{i+m}))_{\theta,p}}. \tag{4.6.10}$$

From (4.6.4) and Theorem 3.7.1(iv) we infer

$$\begin{aligned} (F, D(A^{i+m}))_{\theta,p} &\doteq \big(F, H_p^{(i+m)\omega_1/\boldsymbol{\omega}'}(\mathbb{R}^{d-1}, E)\big)_{\theta,p} \\ &\doteq B_p^{\omega_1(m-1/p)/\boldsymbol{\omega}'}(\mathbb{R}^{d-1}, E). \end{aligned}$$

For abbreviation, we set

$$H_p^{\sigma/\boldsymbol{\nu}} := H_p^{\sigma/\boldsymbol{\nu}}(\mathbb{H}, E), \quad \partial H_p^{\sigma/\boldsymbol{\nu}} := B_p^{(\sigma - \omega_1(j+1/p))/\boldsymbol{\omega}'}(\partial \mathbb{H}, E)$$

for $\sigma \in \mathbb{R}$. Then (4.6.9) implies

$$\|\partial^{j+m} \gamma_j^c v\|_{L_p} \le c \, \|v\|_{\partial H_p^{s/\boldsymbol{\nu}}}. \tag{4.6.11}$$

(4) By the definition of γ_j^c

$$\begin{aligned} \big\| \, \|\gamma_j^c v\|_{D(A^{j+m})} \big\|_{L_p(\mathbb{R}^+)} &\le c \big\| \, \|A^{j+m} \gamma_j^c v\|_F \, \big\|_{L_p(\mathbb{R}^+)} \\ &\le c \big\| \, \|t^{j+1/p} A^{j+m} e^{-tA} v\|_F \, \big\|_{L_p^*}. \end{aligned} \tag{4.6.12}$$

Hence it follows from (4.6.10), as in step (3),

$$\|\gamma_j^c v\|_{L_p(\mathbb{R}^+, D(A^{j+m}))} \le c \, \|v\|_{\partial H_p^{s/\boldsymbol{\nu}}}. \tag{4.6.13}$$

Since F is a UMD space, Theorem 2.3.8, Proposition 3.8.3, and Theorem 4.4.3 imply

$$\|\cdot\|_{L_p(\mathbb{R}^+, F)} + \|\partial_1^{j+m} \cdot \|_{L_p(\mathbb{R}^+, F)}$$

is an equivalent norm for $W_p^{j+m}(\mathbb{R}^+, F)$. Hence we infer from (4.6.5) and estimates (4.6.11) and (4.6.13)

$$\gamma_j^c \in \mathcal{L}(\partial H_p^{s/\boldsymbol{\nu}}, H_p^{s/\boldsymbol{\nu}}).$$

Now (4.6.7) implies that γ_j^c is a coretraction for $\partial_{\boldsymbol{n}}^j$.

(5) Suppose $\omega_1(j+1) < s < \omega_1(j+m)$ for some $m \in \mathbb{N}$. Then, by step (4),

$$\gamma_j^c \in \mathcal{L}(\partial H_p^{\omega_1(j+1)/\boldsymbol{\nu}}, H_p^{\omega_1(j+1)/\boldsymbol{\nu}}) \cap \mathcal{L}(\partial H_p^{\omega_1(j+m)/\boldsymbol{\nu}}, H_p^{\omega_1(j+m)/\boldsymbol{\nu}}).$$

Thus by complex interpolation, using (3.4.1) and Theorems 3.7.1(iv) and 4.4.1, it follows that γ_j^c is a coretraction for $\partial_{\boldsymbol{n}}^j \in \mathcal{L}(H_p^{s/\boldsymbol{\nu}}, \partial H_p^{s/\boldsymbol{\nu}})$.

(6) Assume $s < \omega_1(j+1)$ and set $\rho := \omega_1(j+1) - s$. Then

$$A^{-\rho/\omega_1} = \mathsf{K}_1^{-\rho}(D') \in \mathcal{L}(\partial H_p^{s/\boldsymbol{\nu}}, \partial H_p^{\omega_1(j+1)/\boldsymbol{\nu}})$$

by Lemma 2.3.1. Since e^{-tA} commutes with the powers of A it follows that there exists a unique extension of γ_j^c over $\partial H_p^{s/\boldsymbol{\nu}}$, again denoted by the same symbol, such that the diagram

$$\begin{tikzcd}
\partial H_p^{\omega_1(j+1)/\boldsymbol{\nu}} \arrow[r, "\gamma_j^c"] & H_p^{\omega_1(j+1)/\boldsymbol{\nu}} \arrow[d, "A^{\rho/\omega_1}"] \\
\partial H_p^{s/\boldsymbol{\nu}} \arrow[u, "A^{-\rho/\omega_1}"] \arrow[r, "\gamma_j^c"] & H_p^{s/\boldsymbol{\nu}}
\end{tikzcd}$$

is commuting. Note that the map represented by the right vertical arrow is well-defined on smooth functions as a map acting with respect to the variable x' only. Hence by density and continuity it is well-defined on all of $H_p^{\omega_1(j+1)/\boldsymbol{\nu}}$. It is clear that γ_j^c is a coretraction for $\partial_{\boldsymbol{n}}^j$ if $s > \omega_1(j+1/p)$. This proves (4.6.2).

(7) Statement (4.6.1) follows now by interpolation, due to (3.3.13) and Theorems 3.7.1(iv) and 4.4.1. The universality of γ_j^c is obvious and the last claim is obtained by observing that there is no restriction from below on s in step (6). □

4.6.3 Theorem *Let $j_i \in \mathbb{N}$ satisfy $j_1 < \cdots < j_k$. Then there exists a universal map γ^c from*

$$\prod_{i=1}^k \overset{\triangle}{B}{}_{p,q}^{(s-\omega_1(j_i+1/p))/\boldsymbol{\omega}'}(\partial\mathbb{H}, E) \text{ into } \overset{\triangle}{B}{}_{p,q}^{s/\boldsymbol{\nu}}(\mathbb{H}, E) \tag{4.6.14}$$

and from

$$\prod_{i=1}^k B_p^{(s-\omega_1(j_i+1/p))/\boldsymbol{\omega}'}(\partial\mathbb{H}, E) \text{ into } H_p^{s/\boldsymbol{\nu}}(\mathbb{H}, E) \tag{4.6.15}$$

for $s \in \mathbb{R}$ such that

$$\partial_{\boldsymbol{n}}^{j_i} \gamma^c(g^1, \ldots, g^k) = g^i \text{ if } s > \omega_1(j_i + 1/p).$$

In particular, γ^c is a universal coretraction for $(\partial_{\boldsymbol{n}}^{j_1}, \ldots, \partial_{\boldsymbol{n}}^{j_k})$ if $s > \omega_1(j_k + 1/p)$.

Proof. It is clear that $(\partial_{\boldsymbol{n}}^{j_1}, \ldots, \partial_{\boldsymbol{n}}^{j_k})$ is a continuous linear map from the second space of (4.6.14), respectively (4.6.15), into the first one, provided $s > \omega_1(j_k + 1/p)$.

Denote by γ_j^c the universal map constructed in the preceding proof such that it is a coretraction for $\partial_{\boldsymbol{n}}^j$ if $s > \omega_1(j+1/p)$. Note that (4.6.7) implies $\partial_{\boldsymbol{n}}^j \gamma_k^c = 0$ if $j < k$ and $s > \omega_1(j+1/p)$. Suppose $(g^1, \ldots, g^k)$ belongs to the product space in

(4.6.14), respectively (4.6.15). Set $u_1 := \gamma_{j_1}^c g^1$. If $2 \le i \le k$ and $u_1, \dots, u_{i-1}$ are already defined, then put

$$u_i := u_{i-1} + \begin{cases} \gamma_{j_i}^c (g^i - \partial_{\boldsymbol{n}}^{j_i} u_{i-1}), & \text{if } s > \omega_1(j_i + 1/p), \\ \gamma_{j_i}^c g^i & \text{otherwise.} \end{cases}$$

This induction argument defines u_k in $\overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q}(\mathbb{H}, E)$, respectively in $H_p^{s/\boldsymbol{\nu}}(\mathbb{H}, E)$, and $\partial_{\boldsymbol{n}}^{j_i} u_k = g^i$ if $s > \omega_1(j_i + 1/p)$. Thus, setting $\gamma^c(g^1, \dots, g^k) := u_k$, one verifies by induction that γ^c has the stated continuity properties. $\square$

4.7 Vanishing traces

Our next theorem gives an important characterization of anisotropic Besov and Bessel potential spaces on $\mathring{\mathbb{H}}$ in terms of the corresponding spaces on $\mathbb{H}$ and higher order trace operators.

4.7.1 Theorem

(i) *Suppose $k \in \mathbb{N}$ and $\omega_1(k + 1/p) < s < \omega_1(k + 1 + 1/p)$. Then*

$$\overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q}(\mathring{\mathbb{H}}, E) = \left\{ u \in \overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q}(\mathbb{H}, E) \ ; \ \partial_{\boldsymbol{n}}^j u = 0, \ 0 \le j \le k \right\} \tag{4.7.1}$$

and

$$H_p^{s/\boldsymbol{\nu}}(\mathring{\mathbb{H}}, E) = \left\{ u \in H_p^{s/\boldsymbol{\nu}}(\mathbb{H}, E) \ ; \ \partial_{\boldsymbol{n}}^j u = 0, \ 0 \le j \le k \right\}. \tag{4.7.2}$$

(ii) *If $\omega_1(-1 + 1/p) < s < \omega_1/p$, then*

$$B^{s/\boldsymbol{\nu}}_{p,q}(\mathring{\mathbb{H}}, E) = B^{s/\boldsymbol{\nu}}_{p,q}(\mathbb{H}, E), \qquad q \ne \infty,$$

and

$$H_p^{s/\boldsymbol{\nu}}(\mathring{\mathbb{H}}, E) = H_p^{s/\boldsymbol{\nu}}(\mathbb{H}, E).$$

(iii) *Suppose $0 \le s < \omega_1/p$. Then*

$$\overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,\infty}(\mathring{\mathbb{H}}, E) = \overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,\infty}(\mathbb{H}, E).$$

Proof. (1) Let the hypotheses of (i) be satisfied. Theorem 4.6.3 implies that the second space in (4.7.1), respectively (4.7.2), is a closed linear subspace of $\overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q}(\mathbb{H}, E)$, respectively $H_p^{s/\boldsymbol{\nu}}(\mathbb{H}, E)$, and it is obvious that $\overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q}(\mathring{\mathbb{H}}, E)$, respectively $H_p^{s/\boldsymbol{\nu}}(\mathring{\mathbb{H}}, E)$, is contained in it. Thus it suffices to show that $\mathcal{S}(\mathring{\mathbb{H}}, E)$ is dense in the space characterized by vanishing traces.

(2) Fix any t satisfying $s < t < \omega_1(k + 1 + 1/p)$. Then, by[8] Theorem 3.3.2,

$$B_p^{t/\boldsymbol{\nu}}(\mathbb{H}, E) \overset{d}{\hookrightarrow} \overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q}(\mathbb{H}, E).$$

Furthermore, using also Theorem 3.7.1(iii),

$$B_p^{t/\boldsymbol{\nu}}(\mathbb{H}, E) \overset{d}{\hookrightarrow} H_p^{s/\boldsymbol{\nu}}(\mathbb{H}, E).$$

[8]For simplicity, we refer here and in similar situations only to embedding and interpolation theorems, etc., on $\mathbb{R}^d$. This is justified by the results of Section 4.4.

Suppose $u \in H_p^{s/\boldsymbol{\nu}}(\mathbb{H}, E)$ satisfies $\partial_{\boldsymbol{n}}^j u = 0$ for $0 \leq j \leq k$. Let $\varepsilon > 0$ be given. Then there exists $v \in B_p^{t/\boldsymbol{\nu}}(\mathbb{H}, E)$ satisfying

$$\|u - v\|_{H_p^{s/\boldsymbol{\nu}}(\mathbb{H},E)} < \varepsilon.$$

Hence the norm of $\partial_{\boldsymbol{n}}^j v = -\partial_{\boldsymbol{n}}^j (u - v)$ is estimated in $B_p^{(s-\omega_1(j+1/p))/\boldsymbol{\omega}'}(\partial\mathbb{H}, E)$ by $c\varepsilon$ for $0 \leq j \leq k$. Let γ^c be a coretraction for $(\partial_{\boldsymbol{n}}^0, \dots, \partial_{\boldsymbol{n}}^k)$, guaranteed by Theorem 4.6.3. Set

$$w := v - \gamma^c(\partial_{\boldsymbol{n}}^0 v, \dots, \partial_{\boldsymbol{n}}^k v). \tag{4.7.3}$$

Then w belongs to $B_p^{t/\boldsymbol{\nu}}(\mathbb{H}, E)$, satisfies $\partial_{\boldsymbol{n}}^j w = 0$ for $0 \leq j \leq k$, and

$$\|u - w\|_{H_p^{s/\boldsymbol{\nu}}(\mathbb{H},E)} \leq c\varepsilon.$$

This shows that

$$\left\{\, u \in B_p^{t/\boldsymbol{\nu}}(\mathbb{H}, E) \ ; \ \partial_{\boldsymbol{n}}^j u = 0,\ 0 \leq j \leq k \,\right\} \tag{4.7.4}$$

is dense in the second space in (4.7.2).

Similarly, we see that (4.7.4) is dense in the second space of (4.7.1). Thus it suffices to prove the assertion for

$$B_p^{s/\boldsymbol{\nu}}(\mathring{\mathbb{H}}, E), \qquad \omega_1(k+1) \leq s < \omega_1(k+1+1/p). \tag{4.7.5}$$

(3) Let s be as in (4.7.5). Suppose

$$u \in \left\{\, v \in B_p^{s/\boldsymbol{\nu}}(\mathbb{H}, E) \ ; \ \partial_{\boldsymbol{n}}^j v = 0,\ 0 \leq j \leq k \,\right\}.$$

Since $\mathcal{S}(\mathbb{H}, E)$ is dense in $B_p^{s/\boldsymbol{\nu}}(\mathbb{H}, E)$, given $\varepsilon > 0$ we find $v \in \mathcal{S}(\mathbb{H}, E)$ with

$$\|u - v\|_{B_p^{s/\boldsymbol{\nu}}(\mathbb{H},E)} < \varepsilon.$$

Define w by (4.7.3). Since $v \in B_p^{t/\boldsymbol{\nu}}(\mathbb{H}, E)$ for each $t > 0$, Theorem 4.6.3 and the Sobolev embedding Theorem 3.9.1 imply

$$w \in C_0^\infty(\mathbb{H}, E) := \bigcap_{t>0} C_0^t(\mathbb{H}, E), \qquad \partial_1^j w(0, x') = 0, \quad 0 \leq j \leq k.$$

Thus we can assume

$$u \in C_0^\infty \cap B_p^{s/\boldsymbol{\nu}}(\mathbb{H}, E), \qquad \partial_1^j u(0, x') = 0, \quad 0 \leq j \leq k. \tag{4.7.6}$$

(4) Let F be a Banach space. Choose $\varphi \in \mathcal{D}(\mathbb{R}^+)$ satisfying $\varphi(t) = 1$ for $0 \leq t \leq 1/2$ and $\varphi(t) = 0$ for $t \geq 1$. Put $\varphi_\varepsilon(t) := \varphi(t/\varepsilon)$ for $t \geq 0$ and $\varepsilon > 0$. For $v \in L_p(\mathbb{H}, F)$ put $(\varphi_\varepsilon v)(x) := \varphi_\varepsilon(t) u(t, x')$ for a.a. $x = (t, x') \in \mathbb{H}$. Suppose $m \in \mathbb{N}$. Proposition 1.1.1(i) and (ii) and Leibniz' rule imply

$$\|\varphi_\varepsilon v\|_{m,p} \leq c\, \|v\|_{m,p}, \qquad \varepsilon \geq 1, \quad v \in W_p^m(\mathbb{H}, F). \tag{4.7.7}$$

It is obvious that, given $v \in L_p(\mathbb{H}, F)$,

$$\varphi_\varepsilon v \to v \text{ in } L_p(\mathbb{H}, F) \text{ as } \varepsilon \to \infty. \tag{4.7.8}$$

It follows from Proposition 3.5.3 and Theorem 4.4.1 that

$$B_p^r(\mathbb{H}, F) \doteq \big(L_p(\mathbb{H}, F), W_p^m(\mathbb{H}, F)\big)_{r/m,p}, \qquad 0 < r < m.$$

Thus

$$\|\cdot\|_{B_p^r} \leq c\, \|\cdot\|_p^{1-r/m}\, \|\cdot\|_{m,p}^{r/m}. \tag{4.7.9}$$

Hence we deduce from (4.7.7) and (4.7.8) that, for $r > 0$ and $v \in B_p^r(\mathbb{H}, F)$,

$$\varphi_\varepsilon v \to v \text{ in } B_p^r(\mathbb{H}, F) \text{ as } \varepsilon \to \infty. \tag{4.7.10}$$

Let $X := L_p(\mathbb{R}^{d-1}, E)$ and $Y := B_p^{s/\boldsymbol{\omega}'}(\mathbb{R}^{d-1}, E)$. Then Theorems 3.6.7 and 4.4.3 imply

$$B_p^{s/\boldsymbol{\nu}}(\mathbb{H}, E) \doteq L_p(\mathbb{R}^+, Y) \cap B_p^{s/\omega_1}(\mathbb{R}^+, X). \tag{4.7.11}$$

Suppose u satisfies (4.7.6) and set $u = u(\cdot, x')$ from now on. Then $\varphi_\varepsilon u \in \mathcal{D}(\mathbb{R}^+, Y)$. This shows that we can assume

$$u \in \mathcal{D}(\mathbb{R}^+, Y), \qquad \partial^j u(0) = 0, \quad j = 0, \ldots, k. \tag{4.7.12}$$

Then $(1 - \varphi_\varepsilon)u \in \mathcal{D}(\dot{\mathbb{R}}^+, Y)$ and it is obvious that $(1 - \varphi_\varepsilon)u \to u$ in $L_p(\mathbb{R}^+, Y)$ as $\varepsilon \to 0$. Set $m := k + 2$. Thus we see from (4.7.11) and Theorem 3.6.1 that it remains to show, due to $u - (1 - \varphi_\varepsilon)u = \varphi_\varepsilon u$ and Remark 3.6.2,

$$[\varphi_\varepsilon u]^p_{s/\omega_1,p} = \int_0^\infty \int_0^\infty \frac{\|\triangle_h^m(\varphi_\varepsilon u)(y)\|_X^p}{h^{ps/\omega_1}}\, dy\, \frac{dh}{h} \to 0 \tag{4.7.13}$$

as $\varepsilon \to 0$.

(5) By induction one verifies

$$\triangle_h^m(\varphi_\varepsilon u)(y) = \sum_{j=0}^m \binom{m}{j} \triangle_h^j \varphi_\varepsilon(y) \triangle_h^{m-j} u(y + jh). \tag{4.7.14}$$

Fix $j \in \{1, \ldots, m\}$. Note $\triangle_h \varphi_\varepsilon(y) = 0$ if $y \geq \varepsilon$. Thus, by a change of variables,

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{\|\triangle_h^j \varphi_\varepsilon(y) \triangle_h^{m-j} u(y + jh)\|_X^p}{h^{ps/\omega_1}}\, dy\, \frac{dh}{h} \\ &\qquad = \varepsilon^{1-ps/\omega_1} \int_0^1 \int_0^1 \frac{|\triangle_t^j \varphi(z)|^p}{t^{p(s/\omega_1 - m + j)}} \frac{\|\triangle_{\varepsilon t}^{m-j} u(\varepsilon(z + tj))\|_X^p}{t^{p(m-j)}}\, dz\, \frac{dt}{t}. \end{aligned} \tag{4.7.15}$$

By the mean value theorem we deduce, setting $x := z + tj$,

$$\begin{aligned} &\|\triangle_{\varepsilon t}^{m-j} u(\varepsilon x))\|_X \\ &\qquad \leq (\varepsilon t)^{m-j} \int_0^1 \cdots \int_0^1 \big\|\partial^{m-j} u\big(\varepsilon(x + (\tau_1 + \cdots + \tau_{m-j})t)\big)\big\|_X\, d\tau_1 \cdots d\tau_{m-j}. \end{aligned}$$

From $\partial^i u(0) = 0$ for $0 \leq i \leq k$ and the fact that u has compact support it follows

$$\partial^{m-j} u(\varepsilon x) = (\varepsilon x)^{j-1} v_j(\varepsilon x), \qquad x \in \mathbb{R}^+,$$

for some $v_j \in \mathcal{D}(\mathbb{R}^+, Y)$. Hence

$$\|\triangle_{\varepsilon t}^{m-j} u(\varepsilon z))\|_X \leq c_j \varepsilon^{m-1} t^{m-j}(1 + t^j).$$

Similarly,

$$|\triangle_t^j \varphi(y)| \leq c_j t^j, \qquad 0 \leq y \leq 1, \quad 0 < t \leq 1.$$

Thus we find that the first double integral in (4.7.15) is estimated from above by

$$c\varepsilon^{1+p(m-1-s/\omega_1)} \int_0^1 t^{p(m-s/\omega_1)-1}\, dt.$$

Since $m - s/\omega_1 = k + 2 - s/\omega_1 > 1 - 1/p$ this integral exists. Hence the first integral in (4.7.15) is from estimated above by $c\varepsilon^\sigma$, where $\sigma := 1 + p(k + 1 - s/\omega_1) > 0$.

(6) Fix $\delta > 0$ such that $u(y) = 0$ for $y \geq \delta$. Since $\varphi_\varepsilon(y) = 0$ for $y \geq \varepsilon$ it follows

$$\int_0^\infty \int_0^\infty \frac{\varphi_\varepsilon(y)\,\|\triangle_h^m u(x)\|_X^p}{h^{ps/\omega_1}}\,dy\,\frac{dh}{h} \leq c\int_0^\delta \int_0^\varepsilon \frac{\|\triangle_h^m u(y)\|_X^p}{h^{ps/\omega_1}}\,dy\,\frac{dh}{h} \leq c\varepsilon \int_0^\delta h^{p(m-s/\omega_1)-1}\,dh \leq c\varepsilon.$$

From this and step (5) we infer, due to (4.7.14), that (4.7.13) is true. This proves assertion (i).

(7) Suppose $0 \leq s < \omega_1/p$. It is clear that $\overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q}(\mathring{\mathbb{H}}, E)$ is a linear subspace of $\overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q}(\mathbb{H}, E)$. Similarly, $H_p^{s/\boldsymbol{\nu}}(\mathring{\mathbb{H}}, E) \subset H_p^{s/\boldsymbol{\nu}}(\mathbb{H}, E)$. Thus, as above, it remains to show that $\mathcal{D}(\mathring{\mathbb{H}}, E)$ is dense in $\overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q}(\mathbb{H}, E)$, where $0 < s < \omega_1/p$. But this follows from steps (5) and (6) setting there $m = 1$. This proves (iii) and assertion (ii) for $s \geq 0$.

(8) The missing part of statement (ii) is now obtained by duality, due to Theorem 4.4.4. □

Observe $\mathcal{D}(\mathring{\mathbb{H}}, E) \overset{d}{\hookrightarrow} \mathcal{S}(\mathring{\mathbb{H}}, E)$ and (4.4.3) imply that $\overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q}(\mathring{\mathbb{H}}, E)$, respectively $H_p^{s/\boldsymbol{\nu}}(\mathring{\mathbb{H}}, E)$, is the completion of $\mathcal{D}(\mathring{\mathbb{H}}, E)$ in $B^{s/\boldsymbol{\nu}}_{p,q}(\mathbb{H}, E)$, respectively $H_p^{s/\boldsymbol{\nu}}(\mathbb{H}, E)$. Hence, e.g. Theorem 2.9.3 in H. Triebel [65], Theorem 4.7.1 is well-known in the (scalar) isotropic case. Note, however, that even in this case our proof differs substantially from the one given in Triebel's book.

4.7.2 Corollary *If $s < \omega_1/p$, then $\mathcal{D}(\mathring{\mathbb{H}}, E)$ is dense in $\overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q}(\mathbb{H}, E)$ and in $H_p^{s/\boldsymbol{\nu}}(\mathbb{H}, E)$.*

Proof. Setting $\mathfrak{F}^s(X) := \overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q}(X, E)$, resp. $\mathfrak{F}^s(X) := H_p^{s/\boldsymbol{\nu}}(X, E)$, where X equals either $\mathbb{H}$ or $\mathring{\mathbb{H}}$, the claim follows from

$$\mathfrak{F}^t(\mathring{\mathbb{H}}) = \mathfrak{F}^t(\mathbb{H}) \overset{d}{\hookrightarrow} \mathfrak{F}^s(\mathbb{H})$$

for $t > 0$ with $s < t < \omega_1/p$. □

Note that $-\boldsymbol{n} = (1, 0, \dots, 0)$ is the outer unit normal of $-\mathbb{H}$, thus the **inner** unit **normal** of $\mathbb{H}$. Suppose $j \in \mathbb{N}$ and $s > \omega_1(j + 1/p)$. Then

$$\partial^j_{-\boldsymbol{n}} := \gamma_{\partial\mathbb{H}} \circ \partial_1^j \in \mathcal{L}\big(\overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q}(-\mathbb{H}, E), \overset{\vartriangle}{B}{}^{(s-\omega_1(j+1/p))/\boldsymbol{\omega}'}_{p,q}(\partial\mathbb{H}, E)\big)$$

is the j-th outer normal derivative for $-\mathbb{H}$ on $\partial(-\mathbb{H}) = \partial\mathbb{H}$ and

$$\partial^j_{-\boldsymbol{n}} \in \mathcal{L}\big(\overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q}(\mathbb{H}, E), \overset{\vartriangle}{B}{}^{(s-\omega_1(j+1/p))/\boldsymbol{\omega}'}_{p,q}(\partial\mathbb{H}, E)\big)$$

is the j-th **inner normal derivative** for $\mathbb{H}$ on $\partial\mathbb{H}$. For $u \in \overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q}(\mathbb{H}, E)$ it follows

$$\partial^j_{\boldsymbol{n}} u = (-1)^j \partial^j_{-\boldsymbol{n}} u. \tag{4.7.16}$$

Similar results are valid for $u \in H_p^{s/\boldsymbol{\nu}}(\mathbb{H}, E)$.

The following *patching theorem* shows that there is a converse result. It is an easy consequence of Theorems 4.6.3 and 4.7.1.

4.7.3 Theorem *Suppose either $s < \omega_1/p$ or*

$$\omega_1(k+1/p) < s < \omega_1(k+1+1/p) \tag{4.7.17}$$

for some $k \in \mathbb{N}$. Let

$$u^\pm \in \overset{\scriptscriptstyle\triangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q}(\pm\mathbb{H}, E), \quad \textit{resp. } u^\pm \in H_p^{s/\boldsymbol{\nu}}(\pm\mathbb{H}, E),$$

and suppose

$$\partial_{\boldsymbol{n}}^j u^+ = (-1)^j \partial_{-\boldsymbol{n}}^j u^-, \qquad 0 \le j \le k, \tag{4.7.18}$$

if (4.7.17) is true. Set

$$u := \begin{cases} u^+ & \textit{on} \quad \mathbb{H}, \\ u^- & \textit{on} \ -\mathbb{H}. \end{cases}$$

Then u belongs to $\overset{\scriptscriptstyle\triangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q}(\mathbb{R}^d, E)$, respectively $H_p^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E)$.

Proof. (1) Let (4.7.17) be satisfied. Set $\mathfrak{F}^s(X) := \overset{\scriptscriptstyle\triangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q}(X, E)$, respectively $\mathfrak{F}^s(X) := H_p^{s/\boldsymbol{\nu}}(X, E)$ for $X \in \{\mathbb{R}^d, \pm\mathbb{H}\}$. Let $v := u^+ - r^+e^-u^-$. Then $v \in \mathfrak{F}^s(\mathbb{H})$. Hence (4.7.16) and (4.7.18) imply

$$\partial_{\boldsymbol{n}}^j v = \partial_{\boldsymbol{n}}^j u^+ - \partial_{\boldsymbol{n}}^j e^- u^- = \partial_{\boldsymbol{n}}^j u^+ - (-1)^j \partial_{-\boldsymbol{n}}^j u^- = 0.$$

Thus $v \in \mathfrak{F}^s(\mathring{\mathbb{H}})$ by Theorem 4.7.1. Therefore $w := e^-u^- + e_0^+ v \in \mathfrak{F}^s(\mathbb{R}^d)$ with $r^-w = u^-$ and $r^+w = r^+e^-u^- + r^+e_0^+v = u^+$. Consequently, $u = w \in \mathfrak{F}^s(\mathbb{R}^d)$.

(2) Suppose $s < \omega_1/p$. Then $\mathcal{D}(\pm\mathring{\mathbb{H}}, E)$ is dense in $\mathfrak{F}^s(\pm\mathbb{H})$ by Corollary 4.7.2. It is obvious that the claim is true for $u^\pm \in \mathcal{D}(\pm\mathring{\mathbb{H}}, E)$. Hence the assertion holds in this case as well. □

4.8 Normal boundary operators on half-spaces

In this section we assume in addition

- *F is a finite-dimensional Banach space and $F_0, \ldots, F_n$ are nontrivial linear subspaces thereof;* (4.8.1)
- *$0 \le m_0 < m_1 < \cdots < m_n$ are integers,*

where 'nontrivial' means $\ne \{0\}$. Then $\mathbb{F} := (F_0, \ldots, F_n)$ is said to be a **sequence of range spaces** and $\vec{m} := (m_0, \ldots, m_n)$ an **order sequence** of length $n+1$.

We set

$$\mathfrak{B} = \mathfrak{B}(\mathbb{F}) := \left\{ b = (b_0, \ldots, b_n) \in \mathcal{L}\left(F, \textstyle\prod_{i=0}^n F_i\right) ;\ b_i \text{ is surjective} \right\}. \tag{4.8.2}$$

It is easily verified, by introducing a basis in F and identifying it with $\mathbb{C}^N$, for example, that

$$\mathfrak{B} \text{ is open in } \mathcal{L}\left(F, \textstyle\prod_{i=0}^n F_i\right). \tag{4.8.3}$$

Given a finite-dimensional Banach space G, we identify $a \in \mathcal{L}(F, G)$ with

$$1 \otimes a : \ E \otimes F \to E \otimes G, \quad e \otimes f \mapsto e \otimes af$$

so that $a \in \mathcal{L}(E \otimes F, E \otimes G)$.

Keeping this identification in mind we define, for $b \in \mathfrak{B}$, a **normal boundary operator** $\mathcal{B}$ of order m_n for the half-space $\mathbb{H} = \mathbb{H}^d$ by

$$\mathcal{B} = \mathcal{B}(b) := (\mathcal{B}^0, \dots, \mathcal{B}^n), \quad \mathcal{B}^i := b_i \partial_{\boldsymbol{n}}^{m_i},$$

where 'normal' refers to the fact that the b_i are surjective.

We assume

$$\mathfrak{F}^s \in \{H_p^{s/\boldsymbol{\nu}}, \overset{\vartriangle}{B}{}_{p,q}^{s/\boldsymbol{\nu}}\}$$

and often write $\mathfrak{F}^s(\mathbb{H})$ for $\mathfrak{F}^s(\mathbb{H}, E \otimes F)$. We also define

$$\partial_{\mathcal{B}}\mathfrak{F}^s(\partial\mathbb{H}) = \partial_{\mathcal{B}}\mathfrak{F}^s(\partial\mathbb{H}, E \otimes F) := \prod_{i=0}^{n} \overset{\vartriangle}{B}{}_{p,q}^{(s-\omega_1(m_i+1/p))/\boldsymbol{\omega}'}(\partial\mathbb{H}, E \otimes F_i)$$

for $s \in \mathbb{R}$, where it is understood that $q = p$ if $\mathfrak{F}^s = H_p^{s/\boldsymbol{\nu}}$. Note $\partial_{\mathcal{B}}\mathfrak{F}^s(\partial\mathbb{H}, E \otimes F)$ is independent of $b \in \mathfrak{B}$. It depends only on the order sequence $\vec{m}$ and on $\mathbb{F}$. Moreover, given $s > \omega_1(m_n + 1/p)$,

$$\mathfrak{B} \to \mathcal{L}\big(\mathfrak{F}^s(\mathbb{H}), \partial_{\mathcal{B}}\mathfrak{F}^s(\partial\mathbb{H})\big), \quad b \mapsto \mathcal{B}(b)$$

is a well-defined analytic map. In fact, it is the restriction to $\mathfrak{B}$ of a continuous linear map.

The following theorem is an easy generalization of Theorem 4.6.3. It is of great importance in the weak theory of elliptic and parabolic boundary value problems developed in Part 2.

4.8.1 Theorem *Suppose $b \in \mathfrak{B}$. There exists a universal map $\mathcal{B}^c = \mathcal{B}^c(b)$ from*

$$\prod_{i=0}^{n} B_p^{(s-\omega_1(m_i+1/p))/\boldsymbol{\omega}'}(\partial\mathbb{H}, E \otimes F_i) \ \text{ into } \ H_p^{s/\boldsymbol{\nu}}(\mathbb{H}, E \otimes F)$$

and from

$$\prod_{i=0}^{n} \overset{\vartriangle}{B}{}_{p,q}^{(s-\omega_1(m_i+1/p))/\boldsymbol{\omega}'}(\partial\mathbb{H}, E \otimes F_i) \ \text{ into } \ \overset{\vartriangle}{B}{}_{p,q}^{s/\boldsymbol{\nu}}(\mathbb{H}, E \otimes F)$$

for $s \in \mathbb{R}$, satisfying

$$\mathcal{B}^i\mathcal{B}^c(g^0, \dots, g^n) = g^i \ \text{ if } \ s > \omega_1(m_i + 1/p). \tag{4.8.4}$$

In particular, $\mathcal{B}^c$ is a universal coretraction for $\mathcal{B}$ if $s > \omega_1(m_n + 1/p)$. The map $b \mapsto \mathcal{B}^c(b)$ is analytic on $\mathfrak{B}$, uniformly with respect to $s \in \mathbb{R}$.

Proof. Suppose $b \in \mathfrak{B}$. Since $b_i : F \to F_i$ is surjective, we can find a right inverse $b_i^c : F_i \to F$ for it. Set

$$\mathfrak{G}^s(\partial\mathbb{H}) := \prod_{i=0}^{n} \overset{\vartriangle}{B}{}_{p,q}^{(s-\omega_1(m_i+1/p))/\boldsymbol{\omega}'}(\partial\mathbb{H}, E \otimes F), \qquad s \in \mathbb{R}. \tag{4.8.5}$$

Theorem 4.6.3 guarantees[9] the existence of a universal $\mathcal{R} \in \mathcal{L}\big(\mathfrak{G}^s(\partial\mathbb{H}), \mathfrak{F}^s(\mathbb{H})\big)$ satisfying

$$\partial_{\boldsymbol{n}}^{m_i}\mathcal{R}(h^0, \dots, h^n) = h^i, \qquad s > \omega_1(m_i + 1/p).$$

[9] Recall $E \otimes F \cong E^N$ with $N := \dim(F)$, and $E^N \cong L_2(X, \mu, E)$ with $X := \{1, \dots, N\}$ and μ the counting measure. Thus $E \otimes F$ is a UMD space which has property (α) if E has it.

Then

$$g = (g^0, \dots, g^n) \mapsto \mathcal{B}^c g := \mathcal{R}(b_0^c g^0, \dots, b_n^c g^n)$$

defines a universal $\mathcal{B}^c \in \mathcal{L}\big(\partial_{\mathcal{B}} \mathfrak{F}^s(\partial\mathbb{H}), \mathfrak{F}^s(\mathbb{H})\big)$ for $s \in \mathbb{R}$ such that

$$\mathcal{B}^i \mathcal{B}^c g = b_i \partial_{\boldsymbol{n}}^{m_i} \mathcal{R}(b_0^c g^0, \dots, b_n^c g^n) = b_i b_i^c g^i = g^i, \qquad s > \omega_1(m_i + 1/p).$$

Hence $\mathcal{B}^c$ is a universal coretraction for $\mathcal{B}$ if $s > \omega_1(m_n + 1/p)$.

It is not difficult to verify that we can choose the right inverse b_i^c such that the map $\mathfrak{B} \to \mathcal{L}(F_i, F)$, $b \mapsto b_i^c$ is analytic. Now the last assertion is obvious. □

We define

$$\mathfrak{F}_{\mathcal{B}}^s = \mathfrak{F}_{\mathcal{B}}^s(\mathbb{H}) := \big\{\, u \in \mathfrak{F}^s(\mathbb{H}) \ ; \ \mathcal{B}^i u = 0, \ s > \omega_1(m_i + 1/p) \,\big\} \tag{4.8.6}$$

for $s \neq \omega_1(m_j + 1/p)$ and $0 \leq j \leq n$. In other words, $\mathfrak{F}_{\mathcal{B}}^s$ is the kernel of $\mathcal{B}\,|\,\mathfrak{F}^s$, where $\mathcal{B}\,|\,\mathfrak{F}^s$ contains only those parts of $\mathcal{B}$ which are well-defined. In particular, $\mathfrak{F}_{\mathcal{B}}^s = \mathfrak{F}^s$ if $s < \omega_1(m_0 + 1/p)$. Note that $\mathfrak{F}_{\mathcal{B}}^s$ is not defined if s is one of the **singular values** $\omega_1(m_j + 1/p)$, $0 \leq j \leq n$.

4.8.2 Corollary *Suppose $s > \omega_1(m_n + 1/p)$ and $b \in \mathfrak{B}$. Fix a coretraction $\mathcal{B}^c$ for $\mathcal{B}$. Then*

$$\mathfrak{F}^s(\mathbb{H}) = \mathfrak{F}_{\mathcal{B}}^s(\mathbb{H}) \oplus \mathcal{B}^c \partial_{\mathcal{B}} \mathfrak{F}^s(\partial\mathbb{H}).$$

Proof. Lemma 4.1.5. □

Suppose $m \in \mathbb{N}$ with $m \geq m_n$ and

- either $m_n < m$ or $F_i \neq F$ for at least one $i \in \{0, \dots, n\}$. (4.8.7)

Let $\widetilde{\mathbb{F}} = \big(\widetilde{F}_0, \dots, \widetilde{F}_{\widetilde{n}}\big)$ and $\vec{\widetilde{m}} = (\widetilde{m}_0, \dots, \widetilde{m}_{\widetilde{n}})$ be a sequence of range spaces and an order sequence of length $\widetilde{n}$, respectively. Then $\big(\widetilde{\mathbb{F}}, \vec{\widetilde{m}}\big)$ is **complementary to** $(\mathbb{F}, \vec{m})$ **to order** m, provided:

(i) $\{m_0, \dots, m_n\} \cup \{\widetilde{m}_0, \dots, \widetilde{m}_{\widetilde{n}}\} = \{0, \dots, m\}$;

(ii) If $m_i \notin \{\widetilde{m}_0, \dots, \widetilde{m}_{\widetilde{n}}\}$, then $F_i = F$;

(iii) If $\widetilde{m}_k \notin \{m_0, \dots, m_n\}$, then $\widetilde{F}_k = F$;

(iv) If $m_i = \widetilde{m}_k$ for some $k \in \{0, \dots, \widetilde{n}\}$, then $F = F_i \oplus \widetilde{F}_k$.

In the 'scalar case' where $\dim(F) = 1$ (then we identify $E \otimes F \cong E \otimes \mathbb{C}$ with E, of course) these conditions reduce to (i), that is,

$$\{\widetilde{m}_0, \dots, \widetilde{m}_{\widetilde{n}}\} = \{0, \dots, m\} \backslash \{m_0, \dots, m_n\}.$$

Let $\big(\widetilde{\mathbb{F}}, \vec{\widetilde{m}}\big)$ be complementary to $(\mathbb{F}, \vec{m})$ to order m. Denote by π_{ik} the projection of F onto F_i parallel to $\widetilde{F}_k$ if (iv) is satisfied. Suppose $b \in \mathfrak{B}(\mathbb{F})$ and $\widetilde{b} \in \mathfrak{B}\big(\widetilde{\mathbb{F}}\big)$. Then

$$\widetilde{\mathcal{B}} = \widetilde{\mathcal{B}}\big(\widetilde{b}\big) = \big(\widetilde{\mathcal{B}}^0, \dots, \widetilde{\mathcal{B}}^{\widetilde{n}}\big), \quad \widetilde{\mathcal{B}}^j = \widetilde{b}_j \partial_{\boldsymbol{n}}^{\widetilde{m}_j}$$

is **complementary to** $\mathcal{B}$ **to order** m, provided

$$b_i \pi_{ik} \oplus b_k (1 - \pi_{ik}) \in \mathcal{L}\mathrm{aut}(F)$$

if $m_i = \widetilde{m}_k$ with $(i, k) \in \{0, \dots, n\} \times \{0, \dots, \widetilde{n}\}$.

Note that (4.8.7) implies $\widetilde{\mathcal{B}} \neq 0$, and $\widetilde{\mathcal{B}}$ is a normal boundary operator. Furthermore,

$$\big(\mathcal{B}, \widetilde{\mathcal{B}}\big) \in \mathcal{L}\big(\mathfrak{F}^s(\mathbb{H}), \partial_{\mathcal{B}}\mathfrak{F}^s(\partial\mathbb{H}) \times \partial_{\widetilde{\mathcal{B}}}\mathfrak{F}^s(\partial\mathbb{H})\big), \qquad s > \omega_1(m+1/p).$$

4.8.3 Remarks (a) Let $m \geq m_n$ satisfy (4.8.7). Then there exists a boundary operator $\widetilde{\mathcal{B}}$ complementary to $\mathcal{B}$ to order m.

Proof. (1) Let G be a proper linear subspace of F and suppose $b \in \mathcal{L}(F, G)$ is surjective. Choose an ordered basis $\{g_1, \ldots, g_M\}$ of G and extend it to an ordered basis $\{f_1, \ldots, f_N\}$ of F. Denote by $[b^\alpha_\beta] \in \mathbb{C}^{M\times N}$ the corresponding matrix representation of b. Since b is surjective the M rows of $[b^\alpha_\beta]$ are linearly independent. Hence there exists an ordered subset $\{h_1, \ldots, h_M\}$ of $\{f_1, \ldots, f_N\}$ such that the $(M \times M)$-submatrix of $[b^\alpha_\beta]$ containing only the corresponding columns is nonsingular. Denote by χ the projection onto the subspace H of F spanned by $\{h_1, \ldots, h_M\}$, parallel to the subspace $\widetilde{H}$ spanned by the remaining elements of $\{f_1, \ldots, f_N\}$. Then $F = H \oplus \widetilde{H}$ and

$$c := \begin{bmatrix} b\chi & b(1-\chi) \\ 0 & 1-\chi \end{bmatrix} \in \mathcal{L}\mathrm{is}(H \oplus \widetilde{H}, G \oplus \widetilde{H}).$$

Hence $b\chi \oplus (1-\chi) \in \mathcal{L}\mathrm{aut}(F)$.

(2) Write

$$\{\widetilde{m}_0, \ldots, \widetilde{m}_{\widetilde{n}}\} := \{0, \ldots, m\} \backslash \{\, m_i \ ; \ F_i = F \,\}.$$

If $\widetilde{m}_k \notin \{\, m_i \ ; \ 0 \leq i \leq n \,\}$ set $\widetilde{F}_k := F$ and $\widetilde{b}_k := 1_F$. Otherwise we see from step (1) that there exist a complement $\widetilde{F}_k$ of F_i in F and a surjective $\widetilde{b}_k \in \mathcal{L}\big(F, \widetilde{F}_k\big)$ such that $b_i\pi_{ik} \oplus \widetilde{b}_k(1-\pi_{ik})$ is an automorphism of F. This defines a normal boundary operator $\widetilde{\mathcal{B}}$ complementary to $\mathcal{B}$ to order m. □

(b) For $s \in \mathbb{R}$ set

$$\partial_{\mathcal{C}}\mathfrak{F}^s(\partial\mathbb{H}) := \prod_{j=0}^{m} \overset{\vartriangle}{B}{}^{(s-\omega_1(j+1/p))/\boldsymbol{\omega}'}_{p,q}(\partial\mathbb{H}, E \otimes F).$$

Given

$$(g, \widetilde{g}) \in \partial_{\mathcal{B}}\mathfrak{F}^s(\partial\mathbb{H}) \times \partial_{\widetilde{\mathcal{B}}}\mathfrak{F}^s(\partial\mathbb{H}),$$

define

$$\varphi^j(g, \widetilde{g}) \in \overset{\vartriangle}{B}{}^{(s-\omega_1(j+1/p))/\boldsymbol{\omega}'}_{p,q}(\partial\mathbb{H}, E \otimes F), \qquad 0 \leq j \leq m,$$

by

$$\varphi^j(g, \widetilde{g}) := \begin{cases} g^i, & \text{if } j = m_i \notin \{\widetilde{m}_0, \ldots, \widetilde{m}_{\widetilde{n}}\},\ 0 \leq i \leq n, \\ \widetilde{g}^k, & \text{if } j = \widetilde{m}_k \notin \{m_0, \ldots, m_n\},\ 0 \leq k \leq \widetilde{n}, \\ g^i \oplus \widetilde{g}^k, & \text{if } j = m_i = \widetilde{m}_k,\ (i,k) \in \{0, \ldots, n\} \times \{0, \ldots, \widetilde{n}\}. \end{cases}$$

Set $\varphi(g, \widetilde{g}) := \big(\varphi^0(g, \widetilde{g}), \ldots, \varphi^m(g, \widetilde{g})\big)$. Then

$$\varphi \in \mathcal{L}\mathrm{is}\big(\partial_{\mathcal{B}}\mathfrak{F}^s(\partial\mathbb{H}) \times \partial_{\widetilde{\mathcal{B}}}\mathfrak{F}^s(\partial\mathbb{H}), \partial_{\mathcal{C}}\mathfrak{F}^s(\partial\mathbb{H})\big).$$

PROOF. It is obvious that φ is a continuous linear map. Given $f \in \partial_{\mathcal{C}}\mathfrak{F}^s(\partial\mathbb{H})$, set $g^i := \pi_{ik} f$ and $\widetilde{g}^k := (1-\pi_{ik})f$ if $m_i = \widetilde{m}_k$. It is now clear how to define g^i and $\widetilde{g}^k$ for the remaining values of i and k so that $f \mapsto (g, \widetilde{g})$ is a continuous inverse of φ. □

(c) Let $\widetilde{\mathcal{B}}$ be complementary to $\mathcal{B}$ to order m. Define $c_j \in \mathcal{L}\mathrm{aut}(F)$ for $0 \le j \le m$ by

$$c_j := \begin{cases} b_j, & \text{if } j = m_i \notin \{\widetilde{m}_0, \dots, \widetilde{m}_{\widetilde{n}}\}, \\ \widetilde{b}_k, & \text{if } j = \widetilde{m}_k \notin \{m_0, \dots, m_n\}, \\ b_i \pi_{ik} \oplus \widetilde{b}_k (1-\pi_{ik}), & \text{if } j = m_i = \widetilde{m}_k. \end{cases}$$

Set

$$\Phi\big(\mathcal{B}, \widetilde{\mathcal{B}}\big) = \mathcal{C} = (\mathcal{C}^0, \dots, \mathcal{C}^m), \quad \mathcal{C}^j := c_j \partial^j_{\boldsymbol{n}}.$$

Then the diagram

$$\begin{tikzcd} \mathfrak{F}^s(\mathbb{H}) \arrow[rr, "(\mathcal{B},\widetilde{\mathcal{B}})"] \arrow[dr, "\mathcal{C}"'] & & \partial_{\mathcal{B}}\mathfrak{F}^s(\partial\mathbb{H}) \times \partial_{\widetilde{\mathcal{B}}}\mathfrak{F}^s(\partial\mathbb{H}) \arrow[dl, "\cong", "\varphi"'] \\ & \partial_{\mathcal{C}}\mathfrak{F}^s(\partial\mathbb{H}) & \end{tikzcd}$$

is commuting if $s > \omega_1(m + 1/p)$. In particular,

$$\big(\mathcal{B}, \widetilde{\mathcal{B}}\big)u = 0 \iff \partial^j_{\boldsymbol{n}} u = 0, \ 0 \le j \le m \tag{4.8.8}$$

if $s > \omega_1(m + 1/p)$.

PROOF. Obvious. □

In many important situations a boundary operator complementary to $\mathcal{B}$ is given rather naturally as, for instance, in Example 4.8.6 below.

4.8.4 Theorem *Let* (4.8.7) *be satisfied and let* $\widetilde{\mathcal{B}}$ *be complementary to* $\mathcal{B}$ *to order* m.

There exists a universal

$$\widetilde{\mathcal{B}}^c_{\mathcal{B}} \in \mathcal{L}\big(\partial_{\widetilde{\mathcal{B}}}\mathfrak{F}^s(\partial\mathbb{H}), \mathfrak{F}^s_{\mathcal{B}}(\mathbb{H})\big)$$

such that it is a coretraction for $\widetilde{\mathcal{B}}_{\mathcal{B}} := \widetilde{\mathcal{B}} \,|\, \mathfrak{F}^s_{\mathcal{B}}(\mathbb{H})$ *if* $s > \omega_1(m+1/p)$. *If*

$$\omega_1(m+1/p) < s < \omega_1(m+1+1/p), \tag{4.8.9}$$

then $\big(1 - \widetilde{\mathcal{B}}^c_{\mathcal{B}}\widetilde{\mathcal{B}}_{\mathcal{B}}, \widetilde{\mathcal{B}}^c_{\mathcal{B}}\big)$ *is a toplinear isomorphism from*

$$\mathfrak{F}^s_{\mathcal{B}}(\mathbb{H}) \text{ onto } \mathfrak{F}^s(\mathring{\mathbb{H}}) \times \partial_{\widetilde{\mathcal{B}}}\mathfrak{F}^s(\partial\mathbb{H}).$$

PROOF. Set $\mathcal{C} := \Phi\big(\mathcal{B}, \widetilde{\mathcal{B}}\big)$. Let $\mathcal{C}^c$ be a universal continuous linear map from $\partial_{\mathcal{C}}\mathfrak{F}^s(\partial\mathbb{H})$ into $\mathfrak{F}^s(\mathbb{H})$ such that it is a coretraction for $\mathcal{C}$ if $s > \omega_1(m+1/p)$. The existence of $\mathcal{C}^c$ follows from Theorem 4.8.1.

Put

$$\widetilde{\mathcal{B}}^c_{\mathcal{B}} := \mathcal{C}^c \circ \varphi(0, \cdot) \in \mathcal{L}\big(\partial_{\widetilde{\mathcal{B}}}\mathfrak{F}^s(\partial\mathbb{H}), \mathfrak{F}^s(\mathbb{H})\big).$$

Then $\widetilde{\mathcal{B}}^c_{\mathcal{B}}$ is universal.

If $s > \omega_1(m + 1/p)$, then it follows from Remark 4.8.3(c) that

$$\big(\mathcal{B}, \widetilde{\mathcal{B}}\big)\widetilde{\mathcal{B}}^c_{\mathcal{B}}\widetilde{g} = \varphi^{-1} \circ \mathcal{C} \circ \mathcal{C}^c \circ \varphi(0, \widetilde{g}) = (0, \widetilde{g}), \qquad \widetilde{g} \in \partial_{\widetilde{\mathcal{B}}}\mathfrak{F}^s(\partial\mathbb{H}).$$

Thus the image of $\widetilde{\mathcal{B}}^c_{\mathcal{B}}$ is contained in $\mathfrak{F}^s_{\mathcal{B}}$ and $\widetilde{\mathcal{B}}_{\mathcal{B}}\widetilde{\mathcal{B}}^c_{\mathcal{B}}\widetilde{g} = \widetilde{g}$ for $\widetilde{g} \in \partial_{\widetilde{\mathcal{B}}}\mathfrak{F}^s(\partial\mathbb{H})$. Hence $\widetilde{\mathcal{B}}^c_{\mathcal{B}}$ is a universal coretraction for $\widetilde{\mathcal{B}}_{\mathcal{B}}$ if $s > \omega_1(m + 1/p)$.

Note $(\mathfrak{F}^s_{\mathcal{B}})_{\widetilde{\mathcal{B}}}(\mathbb{H}) = \mathfrak{F}^s_{(\mathcal{B},\widetilde{\mathcal{B}})}(\mathbb{H})$. Thus, by (4.8.8) and Theorem 4.7.1, $(\mathfrak{F}^s_{\mathcal{B}})_{\widetilde{\mathcal{B}}}(\mathbb{H})$ equals $\mathfrak{F}^s(\mathring{\mathbb{H}})$ if (4.8.9) is satisfied. Now Lemma 4.1.5 implies the assertion. □

The importance of this theorem lies in the fact that it allows to represent the dual of $\mathfrak{F}^s_{\mathcal{B}}(\mathbb{H})$ in terms of distributions on $\mathbb{H}$ and on $\partial\mathbb{H}$. This fact, basic for the weak theory of parabolic equations, is made precise in the following theorem.

4.8.5 Theorem *Let* (4.8.7) *be satisfied and let $\widetilde{\mathcal{B}}$ be a normal boundary operator on $\partial\mathbb{H}$ complementary to $\mathcal{B}$ to order m. Suppose*

$$\omega_1(m + 1/p') < s < \omega_1(m + 1 + 1/p').$$

Choose a universal coretraction $\widetilde{\mathcal{B}}^c_{\mathcal{B}}$ for $\widetilde{\mathcal{B}}_{\mathcal{B}}$. Then $\big(1 - \widetilde{\mathcal{B}}^c_{\mathcal{B}}\widetilde{\mathcal{B}}_{\mathcal{B}}, \widetilde{\mathcal{B}}_{\mathcal{B}}\big)'$ is a toplinear isomorphism from $H^{s/\boldsymbol{\nu}}_{p',\mathcal{B}}(\mathbb{H}, E \otimes F)'$ onto

$$H^{-s/\boldsymbol{\nu}}_p(\mathbb{H}, E \otimes F) \times \prod_{i=0}^{\widetilde{n}} B^{-(s-\omega_1(\widetilde{m}_i+1/p'))/\boldsymbol{\omega}'}_p(\partial\mathbb{H}, E \otimes \widetilde{F}_i)$$

and from $B^{s/\boldsymbol{\nu}}_{p',\mathcal{B}}(\mathbb{H}, E \otimes F)'$ onto

$$B^{-s/\boldsymbol{\nu}}_p(\mathbb{H}, E \otimes F) \times \prod_{i=0}^{\widetilde{n}} B^{-(s-\omega_1(\widetilde{m}_i+1/p'))/\boldsymbol{\omega}'}_p(\partial\mathbb{H}, E \otimes \widetilde{F}_i).$$

PROOF. This follows from the preceding theorem, and Theorems 3.3.3, 3.7.1(i), and 4.4.4. □

We illustrate this theorem by first order normal boundary operators which are of particular relevance for the weak theory of reaction-diffusion systems.

4.8.6 Examples (Reaction-diffusion boundary operators)
Let $\chi \in \mathcal{L}(F)$ be a projection and set $F_0 := (1 - \chi)F$ and $F_1 := \chi F$. Suppose $a \in \mathcal{L}(F)$ satisfies

$$\chi a \chi \in \mathcal{L}\mathrm{aut}(F_1), \quad (1 - \chi)a(1 - \chi) \in \mathcal{L}\mathrm{aut}(F_0). \tag{4.8.10}$$

Set

$$\mathcal{B} := \chi a \partial_{\boldsymbol{n}} + (1 - \chi)\gamma_{\partial\mathbb{H}}.$$

Since $F = F_0 \oplus F_1 \cong F_0 \times F_1$ we write, accordingly,

$$\mathcal{B} = (\mathcal{B}^0, \mathcal{B}^1) := \big((1 - \chi)\gamma_{\partial\mathbb{H}}, \chi a \partial_{\boldsymbol{n}}\big).$$

Then, given $g = (g^0, g^1) \in F_0 \times F_1$,

$$\mathcal{B}u = g \quad \text{iff} \quad \mathcal{B}^0 u = g^0 \text{ and } \mathcal{B}^1 u = g^1.$$

Note that $\mathcal{B}^0$ is a (zero order) Dirichlet and $\mathcal{B}^1$ a (first order) Neumann boundary operator. It is obvious from (4.8.10) that $\mathcal{B}$ is a normal boundary operator on $\partial\mathbb{H}$.

(a) It is clear that $\widetilde{\mathcal{B}} := (1-\chi)a\partial_{\boldsymbol{n}} + \chi\gamma_{\partial\mathbb{H}}$ is complementary to $\mathcal{B}$ to order 1. Thus, if

$$\omega_1(1+1/p) < s < \omega_1(2+1/p),$$

then $H^{s/\boldsymbol{\nu}}_{p',\mathcal{B}}(\mathbb{H}, E\otimes F)'$ is toplinearly isomorphic to

$$H^{-s/\boldsymbol{\nu}}_{p}(\mathbb{H}, E\otimes F) \times B^{-(s-\omega_1/p')/\boldsymbol{\omega}'}_{p}(\partial\mathbb{H}, E\otimes F_1) \times B^{-(s-\omega_1(1+1/p'))/\boldsymbol{\omega}'}_{p}(\partial\mathbb{H}, E\otimes F_0).$$

In particular,

$$H^{s/\boldsymbol{\nu}}_{p',\gamma_{\partial\mathbb{H}}}(\mathbb{H}, E\otimes F)' \cong H^{-s/\boldsymbol{\nu}}_{p}(\mathbb{H}, E\otimes F) \times B^{-(s-\omega_1(2-1/p))/\boldsymbol{\omega}'}_{p}(\partial\mathbb{H}, E\otimes F)$$

and

$$H^{s/\boldsymbol{\nu}}_{p',\partial_{\boldsymbol{n}}}(\mathbb{H}, E\otimes F)' \cong H^{-s/\boldsymbol{\nu}}_{p}(\mathbb{H}, E\otimes F) \times B^{-(s-\omega_1(1-1/p))/\boldsymbol{\omega}'}_{p}(\partial\mathbb{H}, E\otimes F).$$

(b) Suppose $\chi \notin \{0,1\}$. Then $\widetilde{\mathcal{B}}^0 := \chi\gamma_{\partial\mathbb{H}}$ is complementary to $\mathcal{B}^0$ to order 0. Thus, if

$$\omega_1/p < s < \omega_1(1+1/p),$$

then

$$H^{s/\boldsymbol{\nu}}_{p',\mathcal{B}^0}(\mathbb{H}, E\otimes F)' \cong H^{-s/\boldsymbol{\nu}}_{p}(\mathbb{H}, E\otimes F) \times B^{-(s-\omega_1(1-1/p))/\boldsymbol{\omega}'}_{p}(\partial\mathbb{H}, E\otimes F_1).$$

(c) Obviously, analogous results hold for $B^{s/\boldsymbol{\nu}}_{p}(\mathbb{H}, E\otimes F)$. □

4.9 Interpolation with boundary conditions

The following 'interpolation theorem with boundary conditions' plays a fundamental role in the weak theory of elliptic and parabolic boundary value problems.

We use the assumptions and notation of the preceding section. To simplify the writing we omit $(\mathbb{H}, E\otimes F)$ in the notation of function spaces so that

$$H^{s/\boldsymbol{\nu}}_{p} = H^{s/\boldsymbol{\nu}}_{p}(\mathbb{H}, E\otimes F) \text{ etc.}$$

4.9.1 Theorem *Suppose $s_0, s_1 \in \mathbb{R}$ and $\theta \in (0,1)$ satisfy*

$$\omega_1(-1+1/p) < s_0 < \omega_1/p, \quad \omega_1(m_1+1/p) < s_1, \qquad s_0 < s_1, \tag{4.9.1}$$

with $s_0 \geq 0$ if $q=\infty$, and

$$s_\theta, s_1 \neq \omega_1(m_i+1/p), \qquad 0 \leq i \leq n. \tag{4.9.2}$$

Then

$$[H^{s_0/\boldsymbol{\nu}}_{p}, H^{s_1/\boldsymbol{\nu}}_{p,\mathcal{B}}]_\theta \doteq H^{s_\theta/\boldsymbol{\nu}}_{p,\mathcal{B}}, \quad [\overset{\vartriangle}{B}{}^{s_0/\boldsymbol{\nu}}_{p,q}, \overset{\vartriangle}{B}{}^{s_1/\boldsymbol{\nu}}_{p,q,\mathcal{B}}]_\theta \doteq \overset{\vartriangle}{B}{}^{s_\theta/\boldsymbol{\nu}}_{p,q,\mathcal{B}},$$

and

$$(H^{s_0/\boldsymbol{\nu}}_{p}, H^{s_1/\boldsymbol{\nu}}_{p,\mathcal{B}})^0_{\theta,q} \doteq (\overset{\vartriangle}{B}{}^{s_0/\boldsymbol{\nu}}_{p,q}, \overset{\vartriangle}{B}{}^{s_1/\boldsymbol{\nu}}_{p,q,\mathcal{B}})^0_{\theta,q} \doteq \overset{\vartriangle}{B}{}^{s_\theta/\boldsymbol{\nu}}_{p,q,\mathcal{B}}.$$

Before we prove this theorem we recall some facts from interpolation theory.

Denote by S the strip $[0 \leq \operatorname{Re} z \leq 1]$ in the complex plane and by S_ϑ the complex line $[\operatorname{Re} z = \vartheta]$ for $0 \leq \vartheta \leq 1$.

Let E_0 and E_1 be Banach spaces with $E_1 \hookrightarrow E_0$. Write $\mathcal{F}(E_0, E_1)$ for the Banach space of all bounded continuous $f : S \to E_0$ satisfying $f \,|\, S_j \in C_0(S_j, E_j)$ for $j = 0, 1$ and being holomorphic in $\mathring{S}$, endowed with the norm

$$f \mapsto \|f\|_{\mathcal{F}(E_0,E_1)} := \max_{j=0,1} \sup_{z\in S_j} |f(z)|_{E_j}.$$

Then $[E_0, E_1]_\theta$ is for $0 < \theta < 1$ the image space of the evaluation map

$$\mathcal{F}(E_0, E_1) \to E_0, \quad f \mapsto f(\theta)$$

(see A.P. Calderón [18]; also cf. J. Bergh and J. Löfström [16] or H. Triebel [65]).

Suppose $f \in \mathcal{F}(E_0, E_1)$ and $0 < \vartheta < 1$. Then, cf. Remark 2.2.1(a),

$$\|f(\vartheta + it)\|_{[E_0,E_1]_\vartheta} \le c\,\|f\|_{\mathcal{F}(E_0,E_1)}, \qquad t \in \mathbb{R}. \tag{4.9.3}$$

Indeed, given $t \in \mathbb{R}$, set $g_t(z) := f(z + it)$ for $z \in S$. Then $g_t \in \mathcal{F}(E_0, E_1)$ and $g_t(\vartheta) = f(\vartheta + it)$.

For $t \in \mathbb{R}$ set

$$J(t, e) := \max\{\|e\|_0, t\,\|e\|_1\}, \qquad e \in E_1.$$

Then the J-method of real interpolation theory guarantees $e \in (E_0, E_1)_{\theta,q}$ iff there exists $f \in C\big((0,\infty), E_1\big)$ satisfying

$$\big\|t^{-\theta} J\big(t, f(t)\big)\big\|_{L_q^*} < \infty, \quad e = \int_0^\infty f(t)\,dt/t, \tag{4.9.4}$$

the integral converging in E_0 (e.g., Chapter 3 in [16] or Section I.6 in [65]).

Proof of Theorem 4.9.1. (1) Note that $\mathfrak{F}_{\mathcal{B}}^{s_1} \hookrightarrow \mathfrak{F}^{s_1}$ implies

$$(\mathfrak{F}^{s_0}, \mathfrak{F}_{\mathcal{B}}^{s_1})_\theta \hookrightarrow (\mathfrak{F}^{s_0}, \mathfrak{F}^{s_1})_\theta,$$

where $(\cdot,\cdot)_\theta$ equals either $[\cdot,\cdot]_\theta$ or $(\cdot,\cdot)_{\theta,q}^0$.

(2) From (1) and Theorems 3.7.1(iv) and 4.4.1 we obtain

$$[H_p^{s_0/\boldsymbol{\nu}}, H_{p,\mathcal{B}}^{s_1/\boldsymbol{\nu}}]_\theta \hookrightarrow H_p^{s_\theta/\boldsymbol{\nu}}.$$

Similarly, by invoking (3.4.1),

$$[\overset{\scriptscriptstyle\triangle}{B}{}_{p,q}^{s_0/\boldsymbol{\nu}}, \overset{\scriptscriptstyle\triangle}{B}{}_{p,q,\mathcal{B}}^{s_1/\boldsymbol{\nu}}]_\theta \hookrightarrow \overset{\scriptscriptstyle\triangle}{B}{}_{p,q}^{s_\theta/\boldsymbol{\nu}}.$$

(3) Denote by $(E_j, E_{j,\mathcal{B}})$ either the pair $(H_p^{s_j/\boldsymbol{\nu}}, H_{p,\mathcal{B}}^{s_j/\boldsymbol{\nu}})$ or $(\overset{\scriptscriptstyle\triangle}{B}{}_{p,q}^{s_j/\boldsymbol{\nu}}, \overset{\scriptscriptstyle\triangle}{B}{}_{p,q,\mathcal{B}}^{s_j/\boldsymbol{\nu}})$. Suppose $u \in [E_0, E_{1,\mathcal{B}}]_\theta$. Then there exists a function f in $\mathcal{F}(E_0, E_{1,\mathcal{B}})$ with $f(\theta) = u$.

Suppose m_i satisfies $\omega_1(m_i + 1/p) < s_\theta$. It follows from (4.9.3) and step (1) that the restriction of f to $[\theta \le \operatorname{Re} z \le 1]$ is a bounded continuous map into $\mathfrak{F}^{s_\theta}$ being holomorphic on $[\theta \le \operatorname{Re} z < 1]$. Thus $h := \mathcal{B}^i f$ is a bounded continuous map from $[\theta \le \operatorname{Re} z \le 1]$ into $\overset{\scriptscriptstyle\triangle}{B}{}_{p,q}^{(s_\theta - \omega_1(m_i+1/p))/\boldsymbol{\omega}'}(\partial\mathbb{H}, E \otimes F_i)$ which is holomorphic on $[\theta \le \operatorname{Re} z < 1]$ and vanishes on S_1. Thus $u \in \mathfrak{F}_{\mathcal{B}}^{s_\theta}$ by the Three Lines Theorem (cf. Theorem VI.10.3 in [22]). This proves

$$[H_p^{s_0/\boldsymbol{\nu}}, H_{p,\mathcal{B}}^{s_1/\boldsymbol{\nu}}]_\theta \hookrightarrow H_{p,\mathcal{B}}^{s_\theta/\boldsymbol{\nu}} \tag{4.9.5}$$

and

$$[\overset{\scriptscriptstyle\triangle}{B}{}_{p,q}^{s_0/\boldsymbol{\nu}}, \overset{\scriptscriptstyle\triangle}{B}{}_{p,q,\mathcal{B}}^{s_1/\boldsymbol{\nu}}]_\theta \hookrightarrow \overset{\scriptscriptstyle\triangle}{B}{}_{p,q,\mathcal{B}}^{s_\theta/\boldsymbol{\nu}}. \tag{4.9.6}$$

(4) Suppose

$$s_1 > \omega_1(m_n + 1/p). \tag{4.9.7}$$

Denote by m the largest integer i satisfying $\omega_1(i+1/p) < s_1$. Let $b_i^c : F_i \to F$ be a right inverse for b_i. Then $\pi_i := b_i^c b_i$ is the projection of F onto F_i parallel to $G_i := \ker(b_i)$, and $F = F_i \oplus G_i$ (cf. Lemma 4.1.5). Thus

$$\mathcal{B}^i u = b_i \partial_{\boldsymbol{n}}^{m_i} u = 0 \Longleftrightarrow \pi_i \partial_{\boldsymbol{n}}^{m_i} u = 0. \tag{4.9.8}$$

Consider the normal boundary operator $\mathcal{C} = (\mathcal{C}^0, \dots, \mathcal{C}^m)$ with $\mathcal{C}^j = \partial_{\boldsymbol{n}}^j$. Set

$$\partial_{\mathcal{C}^j} \mathfrak{F}^s := \overset{\scriptscriptstyle\Delta}{B}{}_{p,q}^{(s-\omega_1(j+1/p))/\boldsymbol{\omega}'}(\partial\mathbb{H}, E \otimes F), \quad \partial_{\mathcal{C}} \mathfrak{F}^s := \prod_{j=0}^m \partial_{\mathcal{C}^j} \mathfrak{F}^s.$$

Then

$$\mathcal{C}^j \in \mathcal{L}(\mathfrak{F}^s, \partial_{\mathcal{C}^j} \mathfrak{F}^s) \text{ if } s > \omega_1(j+1/p), \tag{4.9.9}$$

and Theorem 4.8.1 guarantees the existence of a universal $\mathcal{C}^c \in \mathcal{L}(\partial_{\mathcal{C}} \mathfrak{F}^s, \mathfrak{F}^s)$ for $s \in \mathbb{R}$, satisfying

$$\mathcal{C}^j \mathcal{C}^c(g^0, \dots, g^m) = g^j \text{ if } s > \omega_1(j+1/p). \tag{4.9.10}$$

From (3.4.1) we infer

$$[\partial_{\mathcal{C}} \mathfrak{F}^{s_0}, \partial_{\mathcal{C}} \mathfrak{F}^{s_1}]_\theta \doteq \partial_{\mathcal{C}} \mathfrak{F}^{s_\theta}. \tag{4.9.11}$$

(5) Suppose $u \in \mathfrak{F}_{\mathcal{B}}^{s_\theta}$. Choose $h^j \in \mathcal{F}(\partial_{\mathcal{C}^i} \mathfrak{F}^{s_0}, \partial_{\mathcal{C}^i} \mathfrak{F}^{s_1})$ as follows:

$$h^j(\theta) = \begin{cases} \partial_{\boldsymbol{n}}^j u, & \text{if } s_\theta > \omega_1(j+1/p),\ j \notin \{m_0, \dots, m_n\}, \\ (1-\pi_i)\partial_{\boldsymbol{n}}^{m_i} u, & \text{if } s_\theta > \omega_1(j+1/p),\ j = m_i,\ 0 \le i \le n, \\ 0 & \text{otherwise.} \end{cases} \tag{4.9.12}$$

Due to (4.9.9) and (4.9.11) this is possible. Then (4.9.8) and (4.9.9) imply

$$f := \mathcal{C}^c(h^0, \dots, h^m) \in \mathcal{F}(\mathfrak{F}^{s_0}, \mathfrak{F}_{\mathcal{B}}^{s_1}). \tag{4.9.13}$$

Thus

$$v := f(\theta) \in [\mathfrak{F}^{s_0}, \mathfrak{F}_{\mathcal{B}}^{s_1}]_\theta. \tag{4.9.14}$$

Moreover, by step (1), $u - v$ belongs to $\mathfrak{F}^{s_\theta}$ and satisfies

$$\mathcal{C}^j(u-v) = \mathcal{C}^j u - \mathcal{C}^j \mathcal{C}^c\big(h^0(\theta), \dots, h^m(\theta)\big) = 0, \qquad s_\theta > \omega_1(j+1/p),$$

due to (4.9.10) and (4.9.14). Hence Theorem 4.7.1 and $s_\theta > s_0 > \omega_1(-1+1/p)$, with $s_0 \ge 0$ if $\mathfrak{F}^s = \overset{\scriptscriptstyle\Delta}{B}{}_{p,\infty}^{s/\boldsymbol{\nu}}$, show that $u - v \in \mathfrak{F}^{s_\theta}(\mathring{\mathbb{H}}, E \otimes F)$.

From Theorem 4.4.1 we infer

$$\mathfrak{F}^{s_\theta}(\mathring{\mathbb{H}}, E \otimes F) \doteq \big[\mathfrak{F}^{s_0}(\mathring{\mathbb{H}}, E \otimes F), \mathfrak{F}^{s_1}(\mathring{\mathbb{H}}, E \otimes F)\big]_\theta.$$

Since (4.9.1) and Theorem 4.7.1 imply $\mathfrak{F}^{s_0}(\mathring{\mathbb{H}}, E \otimes F) = \mathfrak{F}^{s_0}$ and since $\mathfrak{F}^{s_1}(\mathring{\mathbb{H}}, E \otimes F)$ is contained in $\mathfrak{F}_{\mathcal{B}}^{s_1}$ it thus follows $\mathfrak{F}^{s_\theta}(\mathring{\mathbb{H}}, E \otimes F) \subset [\mathfrak{F}^{s_0}, \mathfrak{F}_{\mathcal{B}}^{s_1}]_\theta$. Consequently, $u - v$ belongs to $[\mathfrak{F}^{s_0}, \mathfrak{F}_{\mathcal{B}}^{s_1}]_\theta$ which, together with (4.9.14), implies $u \in [\mathfrak{F}^{s_0}, \mathfrak{F}_{\mathcal{B}}^{s_1}]_\theta$. Hence the inclusions converse to (4.9.5) and (4.9.6) are valid. This proves the assertions for the complex interpolation functor, provided (4.9.7) is satisfied.

(6) Now suppose $s_1 < \omega_1(m_n + 1/p)$. Fix any t bigger than $\omega_1(m_n + 1/p)$ and set $\eta := (s_1 - s_0)/(t - s_0)$. Then $\mathfrak{F}_{\mathcal{B}}^{s_1} \doteq [\mathfrak{F}^{s_0}, \mathfrak{F}_{\mathcal{B}}^t]_\eta$ by what has just been shown. Hence by the reiteration theorem

$$[\mathfrak{F}^{s_0}, \mathfrak{F}_{\mathcal{B}}^{s_1}]_\theta \doteq \big[\mathfrak{F}^{s_0}, [\mathfrak{F}^{s_0}, \mathfrak{F}_{\mathcal{B}}^t]_\eta\big]_\theta \doteq [\mathfrak{F}^{s_0}, \mathfrak{F}_{\mathcal{B}}^t]_{\eta\theta} \doteq \mathfrak{F}_{\mathcal{B}}^{s_\theta},$$

the last equality following again from the results of step (5). This proves the assertions for the complex interpolation functor.

(7) Suppose $\omega_1(m_n+1/p)<t_0<t_1$. Assume

$$u\in(\mathfrak{F}_{\mathcal{B}}^{t_0},\mathfrak{F}_{\mathcal{B}}^{t_1})_{\theta,q}^0\hookrightarrow(\mathfrak{F}^{t_0},\mathfrak{F}^{t_1})_{\theta,q}^0=\overset{\vartriangle}{B}{}_{p,q}^{t_\theta/\boldsymbol{\nu}}.$$

Choose f such that (4.9.4) is satisfied with $e=u$ and $(E_0,E_1)=(\mathfrak{F}_{\mathcal{B}}^{t_0},\mathfrak{F}_{\mathcal{B}}^{t_1})$. For $0<\varepsilon<1$ it is obvious that

$$\mathcal{B}\int_\varepsilon^{1/\varepsilon}f(t)\,dt/t=\int_\varepsilon^{1/\varepsilon}\mathcal{B}f(t)\,dt/t=0$$

since the integrals exist in $\mathfrak{F}^{t_0}$. This implies $\mathcal{B}u=0$, that is, $u\in\overset{\vartriangle}{B}{}_{p,q,\mathcal{B}}^{t_\theta/\boldsymbol{\nu}}$.

(8) Conversely, suppose

$$u\in\overset{\vartriangle}{B}{}_{p,q}^{t_\theta/\boldsymbol{\nu}}\doteq(\mathfrak{F}^{t_0},\mathfrak{F}^{t_1})_{\theta,q}^0.$$

By Theorem 4.8.1, $\mathcal{B}$ is a retraction from $\mathfrak{F}^s$ onto $\partial_{\mathcal{B}}\mathfrak{F}^s$ for $s>\omega_1(m_n+1/p)$ possessing a universal coretraction $\mathcal{B}^c$. Thus $\mathcal{P}:=1-\mathcal{B}^c\mathcal{B}$ is a projection from $(\mathfrak{F}^{t_0},\mathfrak{F}^{t_1})_{\theta,q}^0$ onto $(\mathfrak{F}_{\mathcal{B}}^{t_0},\mathfrak{F}_{\mathcal{B}}^{t_1})_{\theta,q}^0$ (cf. Lemma 4.1.5 and Propositions I.2.3.2 and I.2.3.3 in [4]). As above,

$$\mathcal{P}u=\int_0^\infty\mathcal{P}f(t)\,dt/t=\int_0^\infty f(t)\,dt/t=u$$

since $\mathcal{P}f(t)=f(t)\in\mathfrak{F}_{\mathcal{B}}^{t_1}$ for $t>0$. Thus $u\in(\mathfrak{F}_{\mathcal{B}}^{t_0},\mathfrak{F}_{\mathcal{B}}^{t_1})_{\theta,q}^0$. Combining this with the result of step (7) we obtain

$$(\mathfrak{F}_{\mathcal{B}}^{t_0},\mathfrak{F}_{\mathcal{B}}^{t_1})_{\theta,q}^0\doteq\overset{\vartriangle}{B}{}_{p,q,\mathcal{B}}^{t_\theta/\boldsymbol{\nu}}. \tag{4.9.15}$$

(9) Choose $\varepsilon>0$ such that

$$s_{\theta\pm\varepsilon}\in\big(\omega_1(m_i+1/p),\omega_1(m_{i+1}+1/p)\big)$$

if s_θ belongs to this interval, where $m_{n+1}:=\infty$. Then, by the validity of the theorem for the complex interpolation functor, $[\mathfrak{F}^{s_0},\mathfrak{F}_{\mathcal{B}}^{s_1}]_{\theta\pm\varepsilon}\doteq\mathfrak{F}_{\mathcal{B}}^{s_\theta\pm\varepsilon}$. Thus the reiteration theorem implies

$$\begin{aligned}(\mathfrak{F}^{s_0},\mathfrak{F}_{\mathcal{B}}^{s_1})_{\theta,q}^0&\doteq\big([\mathfrak{F}^{s_0},\mathfrak{F}_{\mathcal{B}}^{s_1}]_{\theta-\varepsilon},[\mathfrak{F}^{s_0},\mathfrak{F}_{\mathcal{B}}^{s_1}]_{\theta+\varepsilon}\big)_{1/2,q}^0\\&\doteq(\mathfrak{F}_{\mathcal{B}}^{s_{\theta-\varepsilon}},\mathfrak{F}_{\mathcal{B}}^{s_{\theta+\varepsilon}})_{\theta,q}^0\doteq\overset{\vartriangle}{B}{}_{p,q,\mathcal{B}}^{s_\theta/\boldsymbol{\nu}}\end{aligned}$$

where the last equality follows by applying the result of the preceding step (with m_n replaced by m_i), provided $s_\theta>\omega_1(m_1+1/p)$. If

$$\omega_1(-1+1/p)<s<\omega_1(m_1+1/p),$$

then $\mathfrak{F}_{\mathcal{B}}^s=\mathfrak{F}^s$ by Theorem 4.7.1 and the definition of $\mathfrak{F}_{\mathcal{B}}^s$. Thus (4.9.15) holds in this case also. This proves the theorem. □

4.9.2 Corollary

(i) *Suppose $\omega_1(-1+1/p)<s_0<s_1$ with $s_0,s_1,s_\theta\notin\omega_1(m_i+1/p)$ for $0\le i\le n$. Then*

$$[H_{p,\mathcal{B}}^{s_0/\boldsymbol{\nu}},H_{p,\mathcal{B}}^{s_1/\boldsymbol{\nu}}]_\theta\doteq H_{p,\mathcal{B}}^{s_\theta/\boldsymbol{\nu}},\quad[\overset{\vartriangle}{B}{}_{p,q,\mathcal{B}}^{s_0/\boldsymbol{\nu}},\overset{\vartriangle}{B}{}_{p,q,\mathcal{B}}^{s_1/\boldsymbol{\nu}}]_\theta\doteq\overset{\vartriangle}{B}{}_{p,q,\mathcal{B}}^{s_\theta/\boldsymbol{\nu}}$$

and

$$(H^{s_0/\boldsymbol{\nu}}_{p,\mathcal{B}}, H^{s_1/\boldsymbol{\nu}}_{p,\mathcal{B}})^0_{\theta,q} \doteq (\overset{\vartriangle}{B}{}^{s_0/\boldsymbol{\nu}}_{p,q,\mathcal{B}}, \overset{\vartriangle}{B}{}^{s_1/\boldsymbol{\nu}}_{p,q,\mathcal{B}})^0_{\theta,q} \doteq \overset{\vartriangle}{B}{}^{s_\theta/\boldsymbol{\nu}}_{p,q,\mathcal{B}}.$$

(ii) *Suppose* $s, \theta s \neq \omega_1(m_i + 1/p)$ *for* $0 \leq i \leq n$*. Then*

$$(L_p, \overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q,\mathcal{B}})^0_{\theta,r} \doteq \overset{\vartriangle}{B}{}^{\theta s/\boldsymbol{\nu}}_{p,r,\mathcal{B}}, \qquad 1 \leq r \leq \infty.$$

Proof. (i) follows easily by reiteration from Theorem 4.9.1.

(ii) Fix $s_1 > s$ with $s_1 > \omega_1(m_n + 1/p)$. Then, choosing $s_0 := 0$, it follows from (i)

$$\overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q,\mathcal{B}} \doteq (L_p, H^{s_1/\boldsymbol{\nu}}_{p,\mathcal{B}})^0_{s/s_1,q}$$

Hence, by Theorem 4.9.1,

$$(L_p, H^{s_1/\boldsymbol{\nu}}_{p,\mathcal{B}})_{s/s_1,1} \doteq B^{s/\boldsymbol{\nu}}_{p,1,\mathcal{B}} \hookrightarrow \overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q,\mathcal{B}} \hookrightarrow \overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,\infty,\mathcal{B}} = (L_p, H^{s_1/\boldsymbol{\nu}}_{p,\mathcal{B}})^0_{s/s_1,\infty},$$

and the reiteration theorem implies

$$(L_p, \overset{\vartriangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q,\mathcal{B}})^0_{\theta,r} \doteq (L_p, H^{s_1/\boldsymbol{\nu}}_{p,\mathcal{B}})^0_{\theta s/s_1,r} \doteq \overset{\vartriangle}{B}{}^{\theta s/\boldsymbol{\nu}}_{p,r,\mathcal{B}},$$

that is, the claim. □

The interpolation result for the complex interpolation functor and Bessel potential spaces is due, in the isotropic scalar case (that is, for $E = \mathbb{C}$), to R. Seeley [61] and, for $p = 2$, to P. Grisvard [29]. In the latter paper P. Grisvard characterized the real interpolation spaces between L_p and the Sobolev–Slobodeckii spaces $W^s_{p,\mathcal{B}}$. The extensions of those results to include general (isotropic scalar) Besov spaces is due to D. Guidetti [35]. None of these authors imposed condition (4.9.2). If $s_\theta = m_i + 1/p$ for some i, then the interpolation space of exponent θ is not a closed subspace of $\mathfrak{F}^{s_\theta}$ but carries a strictly stronger topology which can be described by a non-local norm. For simplicity, we do not consider singular values in this work.

Our proof for the complex interpolation functor follows essentially R. Seeley. The simple and elegant method to deduce the statement for the real interpolation method from the result for the complex one is due to D. Guidetti.

We conclude this section by considering two special instances which are of particular importance in the weak theory of elliptic and parabolic equations.

4.9.3 Examples (isotropic spaces) We suppose $\boldsymbol{\omega} = (1, \ldots, 1)$, that is, we consider isotropic spaces. We also assume $m \in \dot{\mathbb{N}} \backslash \{m_0, \ldots, m_n\}$.

(a) (Sobolev–Slobodeckii scales) For $0 < \theta < 1$ we set

$$(\cdot,\cdot)_\theta := \begin{cases} [\cdot,\cdot]_\theta, & \text{if } \theta m \in \mathbb{N}, \\ (\cdot,\cdot)_{\theta,p} & \text{otherwise.} \end{cases}$$

Then

$$(L_p, W^m_{p,\mathcal{B}})_{s/m} \doteq W^s_{p,\mathcal{B}}$$

for $0 < s < m$ with $s \notin \{m_0, \ldots, m_n\}$.

Proof. This follows from (3.8.1) and Theorems 4.4.3 and 4.9.1. □

(b) (small Nikol′skiĭ scales) For $s \in (0,\infty)\backslash\mathbb{N}$ and $m \in \mathbb{N}$ with $m > s$,

$$(L_p, W^m_{p,\mathcal{B}})^0_{s/m,\infty} \doteq \overset{\vartriangle}{N}{}^s_{p,\mathcal{B}}.$$

Furthermore,

$$(\overset{\vartriangle}{N}{}^{s_0}_{p,\mathcal{B}}, \overset{\vartriangle}{N}{}^{s_1}_{p,\mathcal{B}})^0_{\theta,\infty} \doteq [\overset{\vartriangle}{N}{}^{s_0}_{p,\mathcal{B}}, \overset{\vartriangle}{N}{}^{s_1}_{p,\mathcal{B}}]_\theta \doteq \overset{\vartriangle}{N}{}^{s_\theta}_{p,\mathcal{B}}$$

for $0 < s_0 < s_1$ and $0 < \theta < 1$ with $s_0, s_1, s_\theta \notin \mathbb{N}$.

Proof. Obvious by Corollary 4.9.2 and the definition of the small Nikol′skiĭ scales. □

4.9.4 Examples ($2m$-parabolic weight vector) Let $\boldsymbol{\omega} = (1,\dots,1,2m)$ for some $m \in \dot{\mathbb{N}}$ and write $\mathbb{H}^d = \mathbb{H}^r \times \mathbb{R}$. Suppose $m_n < 2m$ and

$$s \in (0,2m)\backslash\{\, m_i + 1/p \;;\; i = 0,\dots,n \,\}.$$

(a) (Sobolev–Slobodeckii scales) If $s \notin \mathbb{N}$, then

$$\begin{aligned}
&\Bigl(L_p(\mathbb{H}^r\times\mathbb{R}), L_p\bigl(\mathbb{R}, W^{2m}_{p,\mathcal{B}}(\mathbb{H}^r)\bigr) \cap W^1_p\bigl(\mathbb{R}, L_p(\mathbb{H}^r)\bigr)\Bigr)_{s/2m,p}\\
&\qquad \doteq \bigl(L_p(\mathbb{H}^r\times\mathbb{R}), W^{(2m,1)}_{p,\mathcal{B}}(\mathbb{H}^r\times\mathbb{R})\bigr)_{s/2m,p}\\
&\qquad \doteq W^{(s,s/2m)}_{p,\mathcal{B}}(\mathbb{H}^r\times\mathbb{R}) \doteq W^{s/2m}_{p,\mathcal{B}}\bigl(\mathbb{R}, L_p(\mathbb{H}^r)\bigr) \cap L_p\bigl(\mathbb{R}, W^s_p(\mathbb{H}^r)\bigr).
\end{aligned}$$

Otherwise

$$\begin{aligned}
&\Bigl[L_p(\mathbb{H}^r\times\mathbb{R}), L_p\bigl(\mathbb{R}, W^{2m}_{p,\mathcal{B}}(\mathbb{H}^r)\bigr) \cap W^1_p\bigl(\mathbb{R}, L_p(\mathbb{H}^r)\bigr)\Bigr]_{s/2m}\\
&\qquad \doteq H^{s/2m}_p\bigl(\mathbb{R}, L_p(\mathbb{H}^r)\bigr) \cap L_p\bigl(\mathbb{R}, W^s_{p,\mathcal{B}}(\mathbb{H}^r)\bigr).
\end{aligned}$$

Proof. Suppose $s \notin \mathbb{N}$. From Theorem 4.9.1 we know

$$(L_p, W^{(2m,1)}_{p,\mathcal{B}})_{\theta,p} \doteq (L_p, H^{(2m,1)}_{p,\mathcal{B}})_{\theta,p} \doteq B^{(2m\theta,\theta)}_{p,\mathcal{B}},$$

provided $2m\theta \neq m_i + 1/p$ for $0 \le i \le n$. Since $\theta = s/2m \notin \mathbb{Z}$ we find, by Theorems 3.8.1, 3.8.5, and 4.4.3, $B^{(2m\theta,\theta)}_{p,\mathcal{B}} \doteq W^{(2m\theta,\theta)}_{p,\mathcal{B}}$. Together with Example 3.8.6 this implies the first assertion.

If $s = 2m\theta \in \mathbb{N}$, then, by Theorem 4.9.1,

$$[L_p, W^{(2m,1)}_{p,\mathcal{B}}]_\theta \doteq [L_p, H^{(2m,1)}_{p,\mathcal{B}}]_\theta \doteq H^{(2m\theta,\theta)}_{p,\mathcal{B}}.$$

Similarly as above, but invoking Theorem 3.7.3,

$$H^{(2m\theta,\theta)}_{p,\mathcal{B}} \doteq H^\theta_p\bigl(\mathbb{R}, L_p(\mathbb{H}^r)\bigr) \cap L_p\bigl(\mathbb{R}, H^{2m\theta}_{p,\mathcal{B}}(\mathbb{H}^r)\bigr).$$

Since $H^{2m\theta}_{p,\mathcal{B}}(\mathbb{H}^r) \doteq W^s_{p,\mathcal{B}}(\mathbb{H}^r)$ the second claim is also valid. □

(b) (small Nikol′skiĭ scales) If $s \notin \mathbb{N}$, then

$$\begin{aligned}
&\Bigl(L_p(\mathbb{H}^r\times\mathbb{R}), W^1_p\bigl(\mathbb{R}, L_p(\mathbb{H}^r)\bigr) \cap L_p\bigl(\mathbb{R}, W^{2m}_{p,\mathcal{B}}(\mathbb{H}^r)\bigr)\Bigr)^0_{s/2m,\infty}\\
&\qquad \doteq \bigl(L_p(\mathbb{H}^r\times\mathbb{R}), W^{(2m,1)}_{p,\mathcal{B}}(\mathbb{H}^r\times\mathbb{R})\bigr)^0_{s/2m,\infty} \doteq \overset{\vartriangle}{N}{}^{(s,s/2m)}_{p,\mathcal{B}}(\mathbb{H}^r\times\mathbb{R})\bigr)\\
&\qquad \doteq \overset{\vartriangle}{N}{}^{s/2m}_p\bigl(\mathbb{R}, L_p(\mathbb{H}^r)\bigr) \cap L_p\bigl(\mathbb{R}, \overset{\vartriangle}{N}{}^s_{p,\mathcal{B}}(\mathbb{H}^r)\bigr).
\end{aligned}$$

Proof. This follows by arguments similar to the ones in (a). □

4.10 Traces on half-open wedges

In this and the following two sections we prove analogues of the boundary retraction Theorem 4.8.1 and some of its consequences for wedges. First we consider the case of partially open wedges to prepare ourselves for the proofs of the next section. For this we need some background material which we discuss first.

Given Banach spaces $X_1 \overset{d}{\hookrightarrow} X_0$, we put

$$\mathcal{H}_-(X_1, X_0) := \mathcal{H}_-(X_0) \cap \mathcal{L}(X_1, X_0).$$

Thus $\mathcal{H}_-(X_1, X_0)$ consists of all negative generators A of exponentially decaying analytic semigroups on X_0 satisfying $D(A) \doteq X_1$ (cf. H. Amann [4, Lemma I.1.1.2]). We also set $\mathbb{H} = \mathbb{H}^d$.

The proof of the next lemma is postponed to Part 2.

4.10.1 Lemma *Fix $k \in (\omega/\omega_1)\mathbb{N}$ and set*

$$\mathcal{B} := (\partial^0_{\boldsymbol{n}}, \ldots, \partial^{k-1}_{\boldsymbol{n}}), \quad \mathcal{C} := (\partial_{\boldsymbol{n}}, \ldots, \partial^k_{\boldsymbol{n}}), \tag{4.10.1}$$

and

$$\mathcal{A} := \Lambda_1^{2k\omega_1}(D) \in \mathcal{L}\big(\mathcal{S}'(\mathbb{R}^d, E)\big). \tag{4.10.2}$$

For $\mathcal{D} \in \{\mathcal{B}, \mathcal{C}\}$ denote by $\mathcal{A}_\mathcal{D}$ the restriction of $\mathcal{A}$ to $H^{2k\nu/\boldsymbol{\nu}}_{p,\mathcal{D}}(\mathbb{H}, E)$. Then

$$\mathcal{A}_\mathcal{D} \in \mathcal{H}_-\big(H^{2k\omega_1/\boldsymbol{\nu}}_{p,\mathcal{D}}(\mathbb{H}, E), L_p(\mathbb{H}, E)\big) \cap \mathcal{BIP}\big(L_p(\mathbb{H}, E)\big).$$

Observe that $k = \kappa\omega/\omega_1$ for some $\kappa \in \mathbb{N}$ and $\omega = \nu$ imply $2\kappa\nu = 2k\omega_1$. Hence

$$\Lambda_1^{2k\omega_1}(D) = \Big(1 + \sum_{i=1}^{\ell} (-\Delta_{x_i})^{\nu/\nu_i}\Big)^\kappa$$

which, due to $\nu/\nu_i \in \dot{\mathbb{N}}$, shows that $\mathcal{A}_\mathcal{D}$ is well-defined.

In the following proposition and in similar situations we do not notationally distinguish between a linear operator and its various uniquely determined restrictions and extensions in a given scale of Banach spaces.

4.10.2 Proposition *Suppose $k \in (\omega/\omega_1)\dot{\mathbb{N}}$. There exists*

$$A \in \mathcal{H}_-\big(H_p^{\omega_1/\boldsymbol{\nu}}(\mathring{\mathbb{H}}, E), L_p(\mathbb{H}, E)\big)$$

satisfying

$$A^r \in \mathcal{L}\mathrm{is}\big(H_p^{(r+s)\omega_1/\boldsymbol{\nu}}(\mathring{\mathbb{H}}, E), H_p^{s\omega_1/\boldsymbol{\nu}}(\mathring{\mathbb{H}}, E)\big) \tag{4.10.3}$$

and

$$A^r \in \mathcal{L}\mathrm{is}\big(\overset{\vartriangle}{B}{}^{(r+s)\omega_1/\boldsymbol{\nu}}_{p,q}(\mathring{\mathbb{H}}, E), \overset{\vartriangle}{B}{}^{s\omega_1/\boldsymbol{\nu}}_{p,q}(\mathring{\mathbb{H}}, E)\big) \tag{4.10.4}$$

for $r, s \in \mathbb{R}$ with

$$-2 + 1/p < \min\{s, r+s\} \le \max\{s, r+s\} < k + 1/p. \tag{4.10.5}$$

Moreover,

$$A^\rho \in \mathcal{H}_-\big(H_p^{\rho\omega_1/\boldsymbol{\nu}}(\mathring{\mathbb{H}}, E), L_p(\mathbb{H}, E)\big) \tag{4.10.6}$$

for $0 \le \rho < k + 1/p$.

Proof. (1) Define $\mathcal{B}$ by (4.10.1) and $\mathcal{A}$ by (4.10.2). Then, by Lemma 4.10.1,

$$\mathcal{A}_{\mathcal{B}} \in \mathcal{H}_-\big(H^{2k\omega_1/\boldsymbol{\nu}}_{p,\mathcal{B}}(\mathbb{H},E), L_p(\mathbb{H},E)\big) \cap \mathcal{BIP}\big(L_p(\mathbb{H},E)\big). \tag{4.10.7}$$

It follows from Theorems 4.9.1 and 4.7.1

$$\big[L_p(\mathbb{H},E), H^{2k\omega_1/\boldsymbol{\nu}}_{p,\mathcal{B}}(\mathbb{H},E)\big]_{s/2k} \doteq H^{s\omega_1/\boldsymbol{\nu}}_p(\mathring{\mathbb{H}},E) \tag{4.10.8}$$

for $0 < s < k+1/p$ with $s \neq j+1/p$, $0 \leq j \leq k-1$. From this and Theorems 3.7.1(iv) and 4.4.1 we deduce by reiteration that (4.10.8) holds in fact for all $s \in (0, k+1/p)$.

Set $A := (\mathcal{A}_{\mathcal{B}})^{1/2k}$. Then (4.10.7), R. Seeley's theorem [60], and (4.10.8) imply

$$D(A^s) \doteq \big[L_p(\mathbb{H},E), H^{2k\omega_1/\boldsymbol{\nu}}_{p,\mathcal{B}}(\mathbb{H},E)\big]_{s/2k} \doteq H^{s\omega_1/\boldsymbol{\nu}}_p(\mathring{\mathbb{H}},E) \tag{4.10.9}$$

for $0 < s < k+1/p$. From this, (4.10.7), and Corollary III.4.6.11 in H. Amann [4] we infer (4.10.6).

(2) Denote by $\big[\,(E_\alpha, A_\alpha)\ ;\ \alpha \geq -1\,\big]$ the interpolation extrapolation scale of order 1 generated by $\big(L_p(\mathbb{H},E), A\big)$ and $[\cdot,\cdot]_\theta$, $0<\theta<1$, in the sense of Section V.1.5 of [4]. It follows from (4.10.9) and Theorem V.1.5.4 in [4]

$$E_\alpha \doteq H^{\alpha\omega_1/\boldsymbol{\nu}}_p(\mathring{\mathbb{H}},E), \qquad 0 \leq \alpha < k+1/p.$$

The latter theorem also implies (4.10.3) for $0 \leq r, s, r+s < k+1/p$.

(3) Define $\mathcal{C}$ as in (4.10.1) and set

$$\mathcal{A}^\sharp := \Lambda^{2k\omega_1}_1(D) \in \mathcal{L}\big(\mathcal{S}'(\mathbb{H},E')\big).$$

Denote by $\mathcal{A}^\sharp_{\mathcal{C}}$ the restriction of $\mathcal{A}^\sharp$ to $H^{2k\omega_1/\boldsymbol{\nu}}_{p',\mathcal{C}}(\mathbb{H},E')$. Then

$$\mathcal{A}^\sharp_{\mathcal{C}} \in \mathcal{H}_-\big(H^{2k\omega_1/\boldsymbol{\nu}}_{p',\mathcal{C}}(\mathbb{H},E'), L_{p'}(\mathbb{H},E')\big) \cap \mathcal{BIP}\big(L_{p'}(\mathbb{H},E')\big)$$

by Lemma 4.10.1. From Theorem 4.9.1 and an additional reiteration we find, similarly as in step (1),

$$H^{s\omega_1/\boldsymbol{\nu}}_{p',\mathcal{C}}(\mathbb{H},E') \doteq H^{s\omega_1/\boldsymbol{\nu}}_{p'}(\mathbb{H},E'), \qquad 0 \leq s < 1+1/p'.$$

Put $A^\sharp := (\mathcal{A}^\sharp_{\mathcal{C}})^{1/2k}$. Let $\big[\,(E^\sharp_\alpha, A^\sharp_\alpha)\ ;\ \alpha \geq -1\,\big]$ be the interpolation extrapolation scale of order 1 generated by $\big(L_{p'}(\mathbb{H},E'), A^\sharp\big)$. Then, as in step (2),

$$E^\sharp_\alpha \doteq H^{\alpha\omega_1/\boldsymbol{\nu}}_{p',\mathcal{C}}(\mathbb{H},E'), \qquad 0 \leq \alpha \leq 2k, \quad \alpha \notin \dot{\mathbb{N}} + 1/p,$$

and

$$(A^\sharp)^r \in \mathcal{L}\mathrm{is}\big(H^{(r+s)\omega_1/\boldsymbol{\nu}}_{p'}(\mathbb{H},E'), H^{s\omega_1/\boldsymbol{\nu}}_{p'}(\mathbb{H},E')\big) \tag{4.10.10}$$

for $r, s \in \mathbb{R}$ with $0 \leq r, r+s \leq 1+1/p'$. By Theorem 4.4.4

$$\big(H^{s\omega_1/\boldsymbol{\nu}}_{p'}(\mathbb{H},E')\big)' \doteq H^{-s\omega_1/\boldsymbol{\nu}}_p(\mathring{\mathbb{H}},E), \qquad s \in \mathbb{R}. \tag{4.10.11}$$

It is an easy consequence of (4.10.6), (4.10.11), and Theorem V.1.5.12 in [4] that $A^\sharp = A'$ in the sense of unbounded linear operators with respect to the $L_p(\mathbb{H},E)$-duality pairing. Hence $A_{-\alpha} = (A^\sharp_\alpha)'$ for $\alpha \geq 0$ by Theorem V.1.5.12 in [4] and reflexivity. Since

$$E_{-\alpha} \doteq \big(H^{\alpha\omega_1/\boldsymbol{\nu}}_{p'}(\mathbb{H},E')\big)' \doteq H^{-\alpha\omega_1/\boldsymbol{\nu}}_p(\mathring{\mathbb{H}},E), \qquad 0 < \alpha < 1+1/p',$$

we thus obtain the truth of claim (4.10.3) for all r and s satisfying (4.10.5) from Theorem V.1.5.4 in H. Amann [4] by taking into consideration $A^r = (\mathcal{A}_{\mathcal{B}})^{r/2k}$ and $(A^\sharp)^r = (\mathcal{A}_{\mathcal{C}}^\sharp)^{r/2k}$.

(4) Assertion (4.10.4) follows now from (4.10.3) by interpolation due to Theorems 3.7.1(iv) and 4.4.1. □

Suppose $\mathbb{K}$ is a standard wedge in $\mathbb{R}^d$. We denote by γ_ρ the trace operator for the variable x^ρ (defined in the obvious way). Then

$$\gamma_{\partial_\rho \mathbb{K}} := r_{\partial_\rho \mathbb{K}} \circ \gamma_\rho \circ e_{\mathbb{K}}$$

is the **trace operator for the face** $\partial_\rho \mathbb{K}$. Let[10]

$$\mathfrak{F}^s \in \big\{ H_p^{s/\boldsymbol{\nu}}, B_p^{s/\boldsymbol{\nu}} \big\}, \qquad s \in \mathbb{R}.$$

Then $\gamma_{\partial_\rho \mathbb{K}}$ is well-defined on $\mathfrak{F}^s$ for $s > \omega_\rho/p$.

We write $\boldsymbol{n}_\rho = -(\delta_\rho^1, \ldots, \delta_\rho^d)$ for the outer normal on the ρ-face $\partial_\rho \mathbb{K}$ of $\mathbb{K}$. Then, given $m \in \mathbb{N}$,

$$\partial_{\boldsymbol{n}_\rho}^m := (-1)^m \gamma_{\partial_\rho \mathbb{K}} \partial_\rho^m$$

is the ***m*-th order normal derivative** on $\partial_\rho \mathbb{K}$, and

$$\partial_{\boldsymbol{n}_\rho}^m \in \mathcal{L}\big(\mathfrak{F}^s(\mathbb{K}, E), B_p^{(s-\omega_\rho(m+1/p))/\boldsymbol{\omega}_{\widehat{\rho}}}(\partial_\rho \mathbb{K}, E) \big) \tag{4.10.12}$$

for $s > \omega_\rho(m + 1/p)$, where[11]

$$\boldsymbol{\omega}_{\widehat{1}} = \boldsymbol{\omega}' = (\omega_2, \ldots, \omega_d), \quad \boldsymbol{\omega}_{\widehat{2}} = (\omega_1, \omega_3, \ldots, \omega_d).$$

Observe that, using the notation of (3.6.3) and (3.6.4),

$$\partial_{\boldsymbol{n}_\rho}^m u(x_{\widehat{\rho}}) = (-1)^m \partial_\rho^m u(x_{\widehat{\rho}}, 0), \qquad u \in \mathcal{S}(\mathbb{K}, E), \quad x \in \partial \mathbb{K}.$$

Now we assume that $\mathbb{K}$ is a **half-open** (that is, partially open) wedge. Thus, without loss of generality,

$$\mathbb{K} = \mathbb{R}^+ \times \mathring{\mathbb{H}}^{d-1}$$

and, consequently, $\partial_1 \mathbb{K} \cong \mathring{\mathbb{H}}^{d-1}$ is the only essential face of $\mathbb{K}$.

We also assume, as in Section 4.8,

- *F is a finite-dimensional Banach space and $F_0, \ldots, F_n$ are nontrivial linear subspaces thereof;* (4.10.13)
- *$0 \le m_0 < m_1 < \cdots < m_n$ are integers.*

Again

$$\mathfrak{B} = \mathfrak{B}(\mathbb{F}) := \big\{ b = (b_0, \ldots, b_n) \in \mathcal{L}\big(F, \textstyle\prod_{i=0}^n F_i\big) \ ; \ b_i \text{ is surjective} \big\}.$$

Given $b \in \mathfrak{B}$, we define a normal boundary operator for the face $\partial_1 \mathbb{K}$ by

$$\mathcal{B}_1 = \mathcal{B}_1(b) := (\mathcal{B}_1^0, \ldots, \mathcal{B}_1^n), \quad \mathcal{B}_1^i := b_i \partial_{\boldsymbol{n}_1}^{m_i}.$$

[10] For simplicity, we restrict ourselves from now on to the most important case $p = q$, although some of the following results can also be shown if $p \neq q$.

[11] Here and below, we formulate all results for the case $d > 2$ and leave it to the reader to carry out the obvious modifications if $d = 2$. Recall $\mathfrak{F}(\mathbb{R}^0, F) = F$.

We set

$$\partial_{\mathcal{B}_1}\mathfrak{F}^s(\partial_1\mathbb{K}, E\otimes F) := \prod_{i=0}^{n} B_p^{(s-\omega_1(m_i+1/p))/\boldsymbol{\omega}'}(\partial_1\mathbb{K}, E\otimes F_i).$$

Note that this space is independent of $b\in\mathfrak{B}$. It depends only on the weight vector $\vec{m}$ and on $\mathbb{F}$.

The following theorem is an analogue of Theorem 4.8.1 for half-open wedges. Here we denote by ω' the least common multiple of $\{\omega_2,\dots,\omega_d\}$.

4.10.3 Theorem *Fix any $\kappa\in\dot{\mathbb{N}}$ satisfying $\kappa\omega'\geq\max\{\omega_1 m_n,\omega_1\}$. Assume*

$$\omega_1(m_n+1/p)+\omega_2(-2+1/p) < s < \omega_1(k+1/p), \tag{4.10.14}$$

where $k:=\kappa\omega'/\omega_2$. Let $b\in\mathfrak{B}$. Then there exists a universal map

$$\mathcal{B}_1^c=\mathcal{B}_1^c(b)\in\mathcal{L}\big(\partial_{\mathcal{B}_1}\mathfrak{F}^s(\partial_1\mathbb{K},E\otimes F),\mathfrak{F}^s(\mathbb{K},E\otimes F)\big)$$

satisfying

$$\mathcal{B}_1^i\mathcal{B}_1^c(g^0,\dots,g^n)=g^i \ \text{ if } \ s>\omega_1(m_i+1/p).$$

In particular, $\mathcal{B}_1^c$ is a universal coretraction for $\mathcal{B}_1$ if $s>\omega_1(m_n+1/p)$. The map $b\mapsto\mathcal{B}_1^c(b)$ is analytic on $\mathfrak{B}$, uniformly with respect to s in the indicated range.

Proof. For notational simplicity we omit $E\otimes F$ in this proof.

(1) By Proposition 4.10.2 we can choose

$$B\in\mathcal{H}_-\big(H_p^{\omega_2/\boldsymbol{\omega}'}(\mathring{\mathbb{H}}^{d-1}),L_p(\mathbb{H}^{d-1})\big)$$

satisfying

$$B^r\in\mathcal{L}\mathrm{is}\big(H_p^{(s+r)\omega_2/\boldsymbol{\omega}'}(\mathring{\mathbb{H}}^{d-1}),H_p^{s\omega_2/\boldsymbol{\omega}'}(\mathring{\mathbb{H}}^{d-1})\big)$$

and

$$B^r\in\mathcal{L}\mathrm{is}\big(B_p^{(s+r)\omega_2/\boldsymbol{\omega}'}(\mathring{\mathbb{H}}^{d-1}),B_p^{s\omega_2/\boldsymbol{\omega}'}(\mathring{\mathbb{H}}^{d-1})\big) \tag{4.10.15}$$

for

$$-2+1/p<\min\{s,r+s\}\leq\max\{s,r+s\}<k+1/p.$$

Set $A:=B^{\omega_1/\omega_2}$. Since $\omega_1/\omega_2\leq\kappa\omega'/\omega_2=k$ it follows from (4.10.6) and (4.10.3) that

$$A\in\mathcal{H}_-\big(H_p^{\omega_1/\boldsymbol{\omega}'}(\mathring{\mathbb{H}}^{d-1}),L_p(\mathbb{H}^{d-1})\big),\quad D(A^j)\doteq H_p^{j\omega_1/\boldsymbol{\omega}'}(\mathring{\mathbb{H}}^{d-1})$$

for $j\in\mathbb{N}$ with $j\leq m_n$, due to $m_n\omega_1/\omega_2\leq\kappa\omega'/\omega_2=k<k+1/p$. Hence, setting $F:=L_p(\mathbb{H}^{d-1})$,

$$H_p^{j\omega_1/\boldsymbol{\nu}}(\mathbb{K})\doteq L_p\big(\mathbb{R}^+,D(A^j)\big)\cap W_p^j(\mathbb{R}^+,F),\qquad 0\leq j\leq m_n,$$

due to Theorems 3.7.1(ii) and 3.7.2, and Section 4.4. Hence we are in the same situation as in (4.6.4) and (4.6.5) with $\mathbb{R}^{d-1}$ replaced by $\mathring{\mathbb{H}}^{d-1}$ (and E by $E\otimes F$). Thus steps (3)–(5) of the proof of Theorem 4.6.2 guarantee that, given $j\in\mathbb{N}$ with $j\leq m_n$,

$$\gamma_j^c v := \Big(t\mapsto(-1)^j\,\frac{t^j}{j!}\,e^{-tA}v\Big),\qquad t\geq 0,\quad v\in F,$$

is a coretraction for

$$\partial_{\boldsymbol{n}_1}^j\in\mathcal{L}\big(H_p^{s/\boldsymbol{\nu}}(\mathbb{K}),B_p^{(s-\omega_1(j+1/p))/\boldsymbol{\omega}'}(\partial_1\mathbb{K})\big),$$

provided $\omega_1(j+1)\leq s\leq\omega_1 k$.

(2) Suppose $s < \omega_1(j+1)$. Since $A^t = B^{t\omega_1/\omega_2}$ we infer from (4.10.15) that

$$A^{j+1-\sigma} \in \mathcal{L}\mathrm{is}\big(B_p^{\omega_1(1-1/p)/\boldsymbol{\omega}'}(\mathring{\mathbb{H}}^{d-1}), B_p^{\omega_1(\sigma-j-1/p))/\boldsymbol{\omega}'}(\mathring{\mathbb{H}}^{d-1})\big), \tag{4.10.16}$$

provided $\omega_1(\sigma - j - 1/p) > \omega_2(-2+1/p)$. Hence (4.10.14) implies that (4.10.16) holds for all $j \in \mathbb{N}$ with $j \le m_n$, provided $\omega_1\sigma = s$. From this it follows, similarly as in step (6) of the proof of Theorem 4.6.2, that

$$\gamma_j^c \in \mathcal{L}\mathrm{is}\big(B_p^{(s-\omega_1(1-1/p))/\boldsymbol{\omega}'}(\partial_1\mathbb{K}), H_p^{s/\boldsymbol{\nu}}(\mathbb{K})\big)$$

for $j \in \mathbb{N}$ with $j \le m_n$ and $\omega_1(m_n + 1/p) + \omega_2(-2+1/p) < s < \omega_1(j+1)$. Clearly, γ_j^c is a coretraction for $\partial_{\boldsymbol{n}_1}^j$ if $s > \omega_1(j+1/p)$.

(3) Steps (1) and (2) guarantee that, given $j \in \mathbb{N}$ with $j \le m_n$,

$$\gamma_j^c \in \mathcal{L}\big(B_p^{(s-\omega_1(j+1/p))/\boldsymbol{\omega}'}(\partial_1\mathbb{K}), H_p^{s/\boldsymbol{\nu}}(\mathbb{K})\big),$$

and γ_j^c is a universal coretraction for $\partial_{\boldsymbol{n}_1}^j$ if $s > \omega_1(j+1/p)$. From this we obtain by interpolation

$$\gamma_j^c \in \mathcal{L}\big(B_p^{(s-\omega_1(j+1/p))/\boldsymbol{\omega}'}(\partial_1\mathbb{K}), B_p^{s/\boldsymbol{\nu}}(\mathbb{K})\big).$$

Now the proofs of Theorems 4.6.3 and 4.8.1 apply to give the assertion. □

As a first application of this boundary retraction theorem we prove an analogue to Theorem 4.7.1.

4.10.4 Theorem

(i) *Suppose* $m \in \mathbb{N}$ *and* $\omega_1(m+1/p) < s < \omega_1(m+1+1/p)$. *Then*

$$\mathfrak{F}^s(\mathring{\mathbb{K}}, E) = \big\{\, u \in \mathfrak{F}^s(\mathbb{K}, E)\ ;\ \partial_{\boldsymbol{n}_1}^j u = 0,\ 0 \le j \le m \,\big\}.$$

(ii) *If* $\omega_1(-1+1/p) < s < \omega_1/p$, *then*

$$\mathfrak{F}^s(\mathring{\mathbb{K}}, E) = \mathfrak{F}^s(\mathbb{K}, E).$$

Proof. (1) Let the assumptions of (i) be satisfied. As in step (1) of the proof of Theorem 4.7.1 we see that it suffices to show that $\mathcal{S}(\mathring{\mathbb{K}}, E)$ is dense in the spaces characterized by vanishing traces. Fix $\kappa \in \mathbb{N}$ satisfying $k := \kappa\omega'/\omega_2 > m+1$. Then Theorem 4.10.3 guarantees the existence of a coretraction γ^c for $(\partial_{\boldsymbol{n}_1}^0, \dots, \partial_{\boldsymbol{n}_1}^m)$. Hence steps (2) and (3) of the proof of Theorem 4.7.1 show that each

$$u \in C_0^\infty \cap B_p^{s/\boldsymbol{\nu}}(\mathbb{K}, E) \quad \text{with} \quad \partial_{\boldsymbol{n}_1}^j u = 0, \qquad 0 \le j \le m, \tag{4.10.17}$$

can be arbitrarily closely approximated by elements of $\mathcal{S}(\mathring{\mathbb{K}}, E)$.

(2) Set $X := L_p(\mathring{\mathbb{H}}^{d-1}, E)$ and $Y := B_p^{s/\boldsymbol{\omega}'}(\mathring{\mathbb{H}}^{d-1}, E)$. Then we see by (1) and steps (4)–(6) of the proof of Theorem 4.7.1 that, given u satisfying (4.10.17) and $\varepsilon > 0$, there exists $v \in \mathcal{D}(\dot{\mathbb{R}}^+, Y)$ satisfying

$$\|u - v\|_{B_p^{s/\boldsymbol{\nu}}(\mathbb{K},E)} < \varepsilon/2.$$

Since $\mathcal{D}(\mathring{\mathbb{H}}^{d-1}, E)$ is dense in Y by Lemma 4.1.4 and (4.4.3), it is not difficult to verify that we can find $w \in \mathcal{D}(\mathring{\mathbb{K}}, E)$ satisfying

$$\|v - w\|_{B_p^{s/\boldsymbol{\nu}}(\mathbb{K},E)} < \varepsilon/2.$$

This proves (i).

(3) Let the hypotheses of (ii) be satisfied. Then Theorem 4.7.1 guarantees

$$\mathfrak{F}^s(\dot{X}^+ \times Y \times Z, E) = \mathfrak{F}^s(X^+ \times Y \times Z, E).$$

Hence the assertion follows by restricting the elements of $\mathfrak{F}^s(\dot{X}^+ \times Y \times Z, E)$ further to $\mathbb{K}$ by means of r_2^0. □

4.11 Traces on closed wedges

After the preceding preparations we are now in a position to prove extension theorems for data given on the boundary of closed wedges. They are fundamental for the study of nonhomogeneous parabolic problems.

Throughout this section we retain the assumptions and notation of Section 4.10, except that we now assume

- $\mathbb{K}$ *is the standard closed wedge in* $\mathbb{R}^d$.

As before, we write $\mathbb{K} = X^+ \times Y^+ \times Z$.

The main objective in this section is to prove boundary retraction theorems for $\mathbb{K}$. First we consider the case where s is small.

4.11.1 Theorem *Suppose* $\omega_1(-1 + 1/p) < s < \omega_1/p$. *Then*

$$\mathfrak{F}^s(\mathbb{K}, E) = \mathfrak{F}^s(\dot{X}^+ \times Y^+ \times Z, E).$$

Proof. Theorem 4.7.1 guarantees

$$\mathfrak{F}^s(X^+ \times Y \times Z, E) = \mathfrak{F}^s(\dot{X}^+ \times Y \times Z, E).$$

Now the assertion follows from the definition of $r_{\mathbb{K}}$, that is, by restricting further from the open half-space $\dot{X}^+ \times Y \times Z$ by means of r_2 to the half-open wedge $\dot{X}^+ \times Y^+ \times Z$. □

4.11.2 Corollary *Suppose*

$$\max\{\omega_1, \omega_2\}(-1 + 1/p) < s < \min\{\omega_1, \omega_2\}/p. \tag{4.11.1}$$

Then $\mathfrak{F}^s(\mathbb{K}, E) = \mathfrak{F}^s(\mathring{\mathbb{K}}, E)$.

Proof. This is now implied by part (ii) of Theorem 4.10.4. □

The above results show that the elements of $\mathfrak{F}^s(\mathbb{K}, E)$ do not possess traces on $\partial\mathbb{K}$ if (4.11.1) is satisfied. Similarly, if the hypotheses of Theorem 4.11.1 are fulfilled, then there are no traces on $\partial_1\mathbb{K}$. However, if $s > \omega_1/p$, then $\gamma_{\partial_1\mathbb{K}}$ is well-defined. Now we investigate this case more closely.

4.11.3 Theorem *Suppose* $\omega_1(m_n + 1/p) < \omega_2/p$ *and*

$$\omega_1(m_n + 1/p) + \omega_2(-2 + 1/p) < s < \omega_2/p.$$

Then there exists a universal map

$$\mathcal{B}_1^c = \mathcal{B}_1^c(b) \in \mathcal{L}\big(\partial_{\mathcal{B}_1}\mathfrak{F}^s(\partial_1\mathbb{K}, E \otimes F), \mathfrak{F}^s(\mathbb{K}, E \otimes F)\big)$$

satisfying

$$\mathcal{B}_1^i \mathcal{B}_1^c(g) = g^i \ \text{ if } \ s > \omega_1(m_i + 1/p)$$

and depending analytically on $b \in \mathfrak{B}$. *In particular,* $\mathcal{B}_1^c$ *is a universal coretraction for* $\mathcal{B}_1$ *if* $s > \omega_1(m_n + 1/p)$.

PROOF. It follows from Theorem 4.11.1 that

$$\mathfrak{F}^s(\mathbb{K}, E \otimes F) = \mathfrak{F}^s(X^+ \times \mathring{\mathbb{H}}^{d-1}, E \otimes F).$$

Fix $\kappa \in \mathbb{N}$ satisfying $\kappa\omega' \geq \max\{\omega_1 m_n, \omega_1\}$ and $\omega_1(\kappa\omega'/\omega_2 + 1/p) \geq \omega_2/p$. Then the assertion is a consequence of Theorem 4.10.3. □

Taking relabeling of coordinates into account the boundary trace behavior for $\mathbb{K}$ has thus been clarified if

$$\min\{\omega_1, \omega_2\}/p < s < \max\{\omega_1, \omega_2\}/p.$$

Now we turn to the case where $s > \max\{\omega_1, \omega_2\}/p$. In this case $\gamma_{\partial_1\mathbb{K}}$ and $\gamma_{\partial_2\mathbb{K}}$ are both well-defined.

If $u \in \mathcal{S}(\mathbb{K}, E)$, then

$$\partial^i_{\boldsymbol{n}_1}\partial^j_{\boldsymbol{n}_2} u = (-1)^{i+j}\gamma_{\partial_{12}\mathbb{K}}\partial^i_1\partial^j_2 u = \partial^j_{\boldsymbol{n}_2}\partial^i_{\boldsymbol{n}_1} u. \tag{4.11.2}$$

It follows that u has to satisfy compatibility conditions on the $(d-2)$-dimensional face $\partial_{12}\mathbb{K}$ of $\mathbb{K}$ if $u \in \mathfrak{F}^s$ and s is sufficiently large. This is made precise in the next proposition where $\boldsymbol{\omega}'' := (\omega_3, \ldots, \omega_d)$.

4.11.4 Proposition *If $s > \omega_1(i + 1/p) + \omega_2(j + 1/p)$, then $\partial^i_{\boldsymbol{n}_1}\partial^j_{\boldsymbol{n}_2}$ is a continuous linear map from $\mathfrak{F}^s$ into*

$$B_p^{(s-\omega_1(i+1/p)-\omega_2(j+1/p))/\boldsymbol{\omega}''}(\partial_{12}\mathbb{K}, E),$$

and $\partial^i_{\boldsymbol{n}_1}\partial^j_{\boldsymbol{n}_2} = \partial^j_{\boldsymbol{n}_2}\partial^i_{\boldsymbol{n}_1}$.

PROOF. This follows from (4.10.12) and (4.11.2) by density. □

4.11.5 Corollary *If $0 \leq i \leq n$, $j \in \mathbb{N}$, and*

$$s > \omega_1(m_i + 1/p) + \omega_2(j + 1/p),$$

then $\partial^j_{\boldsymbol{n}_2}\mathcal{B}^i_1 u = \mathcal{B}^i_1\partial^j_{\boldsymbol{n}_2} u$ for $u \in \mathfrak{F}^s(\mathbb{K}, E \otimes F)$.

In the remainder of this section we assume, in addition to (4.10.13),

$$\begin{aligned} &\bullet\quad k \in \mathbb{N};\\ &\bullet\quad \mathcal{B}_2 = (\partial^0_{\boldsymbol{n}_2}, \ldots, .\partial^k_{\boldsymbol{n}_2}). \end{aligned} \tag{4.11.3}$$

We set, for $s \in \mathbb{R}$,

$$\partial_{\mathcal{B}_2}\mathfrak{F}^s(\partial_2\mathbb{K}, E \otimes F) := \prod_{j=0}^{k} B_p^{(s-\omega_2(j+1/p))/\boldsymbol{\omega}_{\hat{2}}}(\partial_2\mathbb{K}, E \otimes F).$$

Moreover,

$$\partial_{\mathcal{B}}\mathfrak{F}^s(\partial\mathbb{K}, E \otimes F) = \partial_{\mathcal{B}_1}\mathfrak{F}^s(\partial_1\mathbb{K}, E \otimes F) \times \partial_{\mathcal{B}_2}\mathfrak{F}^s(\partial_2\mathbb{K}, E \otimes F)$$

for $s \in \mathbb{R}$. Then

$$\mathcal{B} := (\mathcal{B}_1, \mathcal{B}_2) \in \mathcal{L}\big(\mathfrak{F}^s(\mathbb{K}, E \otimes F), \partial_{\mathcal{B}}\mathfrak{F}^s(\partial\mathbb{K}, E \otimes F)\big),$$

provided $s > \max\big\{\omega_1(m_n + 1/p), \omega_2(k + 1/p)\big\}$. Corollary 4.11.5 shows that $\mathcal{B}$ is not surjective, in general. In fact, $\mathrm{im}(\mathcal{B})$ is contained in

$$\partial^{cc}_{\mathcal{B}}\mathfrak{F}^s(\partial\mathbb{K}, E \otimes F),$$

the closed linear subspace of $\partial_{\mathcal{B}}\mathfrak{F}^s(\partial\mathbb{K}, E\otimes F)$ consisting of all (g_1, g_2) satisfying the **compatibility conditions**

$$\partial^j_{\boldsymbol{n}_2} g^i_1 = \mathcal{B}^i g^j_2 \text{ if } s > \omega_1(m_i + 1/p) + \omega_2(j + 1/p). \tag{4.11.4}$$

Our next theorem shows that, in fact, $\mathrm{im}(\mathcal{B}) = \partial^{cc}_{\mathcal{B}}\mathfrak{F}^s(\partial\mathbb{K}, E\otimes F)$, provided s satisfies suitable restrictions.

4.11.6 Theorem *Suppose*

$$\max\bigl\{\omega_1(m_n + 1/p), \omega_2(k + 1/p)\bigr\} < s < \omega_2(k + 1 + 1/p) \tag{4.11.5}$$

and

$$s \notin \bigl\{\, \omega_1(m_i + 1/p) + \omega_2(j + 1/p)\ ;\ 0 \le i \le n,\ 0 \le j \le k \,\bigr\}. \tag{4.11.6}$$

Then $\mathcal{B}$ is a retraction from

$$\mathfrak{F}^s(\mathbb{K}, E\otimes F) \text{ onto } \partial^{cc}_{\mathcal{B}}\mathfrak{F}^s(\partial\mathbb{K}, E\otimes F).$$

It possesses a universal coretraction depending analytically on $b \in \mathfrak{B}$.

Proof. (1) Denote by r_1 the point-wise restriction from $\mathbb{R}^{d-1} = X \times Z$ onto the half-space $\mathbb{H}^{d-1} = X^+ \times Z \cong \partial_2\mathbb{K}$ (see (4.3.1)). Let

$$e_1 \in \mathcal{L}\bigl(B^{t/\boldsymbol{\omega}'}_p(\partial_2\mathbb{K}, E\otimes F), B^{t/\boldsymbol{\omega}'}_p(X\times Z, E\otimes F)\bigr), \qquad t \in \mathbb{R},$$

be a universal coretraction for r_1. Its existence is guaranteed by (4.4.4). Note $X \times Z \cong \partial(X \times Y^+ \times Z)$. Hence we infer from Theorem 4.6.3 the existence of

$$\gamma^c \in \mathcal{L}\Bigl(\prod_{j=0}^{k} B^{(s-\omega_2(j+1/p))/\boldsymbol{\omega}_{\hat{2}}}_p(X\times Z, E\otimes F), \mathfrak{F}^s(X\times Y^+\times Z, E\otimes F)\Bigr)$$

being a universal coretraction for $(\partial^0_2, \dots, \partial^k_2)$ on $X \times Y^+ \times Z$.

Lastly, let r be the point-wise restriction from $X \times Y^+ \times Z$ onto $\mathbb{K}$. Then

$$\mathcal{B}^c_2 := r \circ \gamma^c \circ e_1 \in \mathcal{L}\bigl(\partial_{\mathcal{B}_2}\mathfrak{F}^s(\partial_2\mathbb{K}, E\otimes F), \mathfrak{F}^s(\mathbb{K}, E\otimes F)\bigr).$$

Given $g \in \mathcal{S}(\partial_2\mathbb{K}, E\otimes F)^k$, the construction of γ^c easily implies $\mathcal{B}_2\mathcal{B}^c_2 g = g$. By density and continuity this holds then for all $g \in \partial_{\mathcal{B}_2}\mathfrak{F}^s(\partial_2\mathbb{K}, E\otimes F)$. Consequently, $\mathcal{B}^c_2$ is a universal coretraction for $\mathcal{B}_2$.

(2) Let $(g_1, g_2) \in \partial^{cc}_{\mathcal{B}}\mathfrak{F}^s(\partial\mathbb{K}, E\otimes F)$ be given. Set $h := g_1 - \mathcal{B}_1\mathcal{B}^c_2 g_2$. Then

$$h^i \in B^{(s-\omega_1(m_i+1/p))/\boldsymbol{\omega}'}_p(\partial_1\mathbb{K}, E\otimes F_i), \qquad 0 \le i \le n.$$

Denote by m be the largest integer j satisfying

$$\omega_2(j + 1/p) < s - \omega_1(m_i + 1/p) < \omega_2(j + 1 + 1/p).$$

It follows from (4.11.5) that m is well-defined and $m \ge -1$. By the commutativity of $\mathcal{B}_1$ and $\mathcal{B}_2$ (see Proposition 4.11.4) and compatibility condition (4.11.4),

$$\partial^j_{\boldsymbol{n}_2} h^i = \partial^j_{\boldsymbol{n}_2} g^i_1 - \mathcal{B}^i_1 \partial^j_{\boldsymbol{n}_2}\mathcal{B}^c_2 g_2 = \partial^j_{\boldsymbol{n}_2} g^i_1 - \mathcal{B}^i_1 g^j_2 = 0$$

if $0 \le j \le m$. Hence $\partial_1\mathbb{K} \cong Y^+ \times Z$ and Theorem 4.7.1 imply

$$h^i \in B^{(s-\omega_1(m_i+1/p))/\boldsymbol{\omega}'}_p(\mathring{Y}^+ \times Z, E\otimes F), \qquad 0 \le i \le n.$$

By Theorem 4.10.3 there exists a universal map

$$\mathring{\mathcal{B}}_1^c \in \mathcal{L}\big(\partial_{\mathcal{B}_1}\mathfrak{F}^s(\dot{Y}^+ \times Z, E \otimes F), \mathfrak{F}^s(X^+ \times \dot{Y}^+ \times Z, E \otimes F)\big)$$

being a coretraction for $\mathcal{B}_1$ and depending analytically on $b \in \mathfrak{B}$. Set

$$\mathcal{B}^c g := \mathcal{B}_2^c g_2 + \mathring{\mathcal{B}}_1^c(g_1 - \mathcal{B}_1\mathcal{B}_2^c g_2) \tag{4.11.7}$$

for $g = (g_1, g_2) \in \partial_{\mathcal{B}}^{cc}\mathfrak{F}^s(\partial\mathbb{K}, E \otimes F)$. Then

$$\mathcal{B}^c \in \mathcal{L}\big(\partial_{\mathcal{B}}^{cc}\mathfrak{F}^s(\partial\mathbb{K}, E \otimes F), \mathfrak{F}^s(\mathbb{K}, E \otimes F)\big)$$

is universal and depends analytically on $b \in \mathfrak{B}$. Furthermore, $\mathcal{B}_1\mathcal{B}^c g = g_1$ and $\mathcal{B}_2\mathcal{B}^c g = g_2$ since

$$f := \mathring{\mathcal{B}}_1^c(g_1 - \mathcal{B}_1\mathcal{B}_2^c g_2) = \mathring{\mathcal{B}}_1^c h \in \mathfrak{F}^s(X^+ \times \dot{Y}^+ \times Z, E \otimes F)$$

implies $\mathcal{B}_2 f = 0$. Thus $\mathcal{B}^c$ is a coretraction for $\mathcal{B}$. □

4.11.7 Example Suppose $m, n \in \dot{\mathbb{N}}$ and consider the closed wedge $\mathbb{H}^m \times \mathbb{H}^n$. Denote the first coordinate of $\mathbb{H}^m$ by ξ and the one of $\mathbb{H}^n$ by η. Let $r, s \in \dot{\mathbb{R}}^+$ with $r > 1/p$ and $s \notin \mathbb{N} + 1/p$ be given.

For $i \in \mathbb{N}$ with $i + 1/p < r$ define ρ_i and σ_i by

$$\frac{\rho_i}{r} = \frac{\sigma_i}{s} = \frac{r - i - 1/p}{r}.$$

Similarly, if $j \in \mathbb{N}$ satisfies $j + 1/p < s$, let λ_j and μ_j be given by

$$\frac{\lambda_j}{r} = \frac{\mu_j}{s} = \frac{s - j - 1/p}{s}.$$

Finally, suppose

$$\frac{i}{r} + \frac{j}{s} \neq 1 - \frac{1}{p}\Big(\frac{1}{r} + \frac{1}{s}\Big) \quad \text{for} \quad i + 1/p < r \text{ and } j + 1/p < s. \tag{4.11.8}$$

Then the map

$$u \mapsto \big((\partial_\xi^i u|_{\xi=0})_{i+1/p<r}, (\partial_\eta^j u|_{\eta=0})_{j+1/p<s}\big)$$

is a retraction from the anisotropic Bessel potential space

$$H_p^{(r,s)}(\mathbb{H}^m \times \mathbb{H}^n) = L_p\big(\mathbb{H}^n, H_p^r(\mathbb{H}^m)\big) \cap H_p^s\big(\mathbb{H}^n, L_p(\mathbb{H}^m)\big)$$

onto the closed linear subspace of

$$\prod_{i+1/p<r} B_p^{(\rho_i,\sigma_i)}(\mathbb{R}^{m-1} \times \mathbb{H}^n) \times \prod_{j+1/p<s} B_p^{(\lambda_j,\mu_j)}(\mathbb{H}^m \times \mathbb{R}^{n-1})$$

consisting of all (g_1, g_2) satisfying the compatibility conditions

$$\partial_\eta^j g_1^i|_{\eta=0} = \partial_\xi^i g_2^j|_{\xi=0} \quad \text{if} \quad \frac{i}{r} + \frac{j}{s} < 1 - \frac{1}{p}\Big(\frac{1}{r} + \frac{1}{s}\Big).$$

Note

$$B_p^{(\rho_i,\sigma_i)}(\mathbb{R}^{m-1} \times \mathbb{H}^n) = L_p\big(\mathbb{H}^n, B_p^{\rho_i}(\mathbb{R}^{m-1})\big) \cap B_p^{\sigma_i}\big(\mathbb{H}^n, L_p(\mathbb{R}^{m-1})\big),$$

and an analogous representation holds for $B_p^{(\lambda_j,\mu_j)}(\mathbb{H}^m \times \mathbb{R}^{n-1})$.

PROOF. Fix $\nu_1, \nu_2 \in \dot{\mathbb{N}}$ satisfying $\nu_1 r = \nu_2 s$, which is possible. Set

$$\boldsymbol{\omega} = (\nu_1, \dots, \nu_1, \nu_2, \dots, \nu_2)$$

with ν_1 occurring m-times and ν_2 n-times. Then the reduced weight system $(\boldsymbol{d}, \boldsymbol{\nu})$ associated with $\boldsymbol{\omega}$ is given by $\boldsymbol{d} = (m, n)$ and $\boldsymbol{\nu} = (\nu_1, \nu_2)$. Set $t := \nu_1 r = \nu_2 s$. Then $(r, s) = t/\boldsymbol{\nu}$. One verifies

$$\big(t - \nu_1(i + 1/p)\big)/\boldsymbol{\omega}' = (\underbrace{\rho_i, \dots, \rho_i}_{m-1}, \underbrace{\sigma_i, \dots, \sigma_i}_{n})$$

and

$$\big(t - \nu_2(j + 1/p)\big)/\boldsymbol{\omega}_{\widehat{m+1}} = (\underbrace{\lambda_j, \dots, \lambda_j}_{m}, \underbrace{\mu_j, \dots, \mu_j}_{n-1}).$$

Hence

$$B_p^{(\rho_i, \sigma_i)}(\mathbb{R}^{m-1} \times \mathbb{H}^n) = B_p^{(t - \nu_1(i+1/p))/\boldsymbol{\omega}'}\big(\partial_1(\mathbb{H}^m \times \mathbb{H}^n)\big)$$

and

$$B_p^{(\lambda_j, \mu_j)}(\mathbb{H}^m \times \mathbb{R}^{n-1}) = B_p^{(t - \nu_2(j+1/p))/\boldsymbol{\omega}_{\widehat{m+1}}}\big(\partial_{m+1}(\mathbb{H}^m \times \mathbb{H}^n)\big).$$

Let k be the largest integer j satisfying $j + 1/p < s$. Then $i + 1/p < r$ implies

$$\max\big\{\nu_1(i + 1/p), \nu_2(k + 1/p)\big\} < \nu_1 r = t = \nu_2 s < \nu_2(k + 1 + 1/p)$$

since $s \notin \mathbb{N} + 1/p$. Thus condition (4.11.5) is satisfied (with s replaced by t). Moreover,

$$t \ge \nu_1(i + 1/p) + \nu_2(j + 1/p) \quad \text{iff} \quad \frac{i}{r} + \frac{j}{s} \le 1 - \frac{1}{p}\Big(\frac{1}{r} + \frac{1}{s}\Big).$$

Now the assertion follows from Theorem 4.11.6. □

If $p = 2$, then the assertion of this example has been proved in P. Grisvard [28], where the assumptions (4.11.8) and $s \notin \mathbb{N} + 1/2$ have not been imposed (also see Theorems IV.2.1 and IV.2.3 in J.-L. Lions and E. Magenes [47]). If $p \in (1, \infty)$ and r and s are *integers*, Example 4.11.7 coincides with Théorème 4.2 in P. Grisvard [27] where, again, conditions (4.11.8) and $s \notin \mathbb{N} + 1/p$ are not assumed. Needless to say that our proof is completely different from the ones in those works. P. Grisvard's theorems are the only general extension theorems for anisotropic Sobolev-type spaces on corners known to the author and taking data on all of $\partial\mathbb{K}$ into account.

Intermediate results, that is, extension theorems for $\partial_1\mathbb{K}$ and $\partial_2\mathbb{K}$ separately, $2m$-parabolic weight vectors, and Sobolev spaces $W_p^{k/\boldsymbol{\nu}}(\mathbb{K}, E)$ with $k \in \dot{\mathbb{N}}$ can be found in R. Denk, M. Hieber, and J. Prüss [21].

4.12 Vanishing traces on closed wedges

Similarly as in the case of half-spaces and half-open wedges we can characterize $\mathring{\mathfrak{F}}^s(\mathbb{K}, E)$ on closed wedges by vanishing traces.

4.12.1 Theorem *Let $\mathbb{K}$ be the closed standard wedge in $\mathbb{R}^d$. Suppose $k, m \in \mathbb{N}$ and $s \in \mathbb{R}$ satisfy*

$$\omega_1(m+1/p) < s < \omega_1(m+1+1/p), \quad \omega_2(k+1/p) < s < \omega_2(k+1+1/p),$$

and

$$s \notin \{ \omega_1(i+1/p) + \omega_2(j+1/p) \ ; \ i,j \in \mathbb{N} \}.$$

Then, given $\mathfrak{F}^s \in \{H_p^{s/\boldsymbol{\nu}}, B_p^{s/\boldsymbol{\nu}}\}$,

$$\mathfrak{F}^s(\mathring{\mathbb{K}}, E) = \{ u \in \mathfrak{F}^s(\mathbb{K}, E) \ ; \ \partial^i_{\boldsymbol{n}_1} u = 0, \ \partial^j_{\boldsymbol{n}_2} u = 0, \ i \le m, \ j \le k \}.$$

PROOF. (1) Set $\mathcal{B}_1 := (\partial^0_{\boldsymbol{n}_1}, \ldots, \partial^m_{\boldsymbol{n}_1})$ and $\mathcal{B}_2 := (\partial^0_{\boldsymbol{n}_2}, \ldots, \partial^k_{\boldsymbol{n}_2})$. Then Theorem 4.11.6 guarantees that

$$\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2) : \ \mathfrak{F}^s(\mathbb{K}, E) \to \partial^{cc}_{\mathcal{B}} \mathfrak{F}^s(\partial\mathbb{K}, E).$$

is a retraction. Let $\mathcal{B}^c$ be a coretraction for it. Then, replacing γ^c in step (2) of the proof of Theorem 4.7.1 by $\mathcal{B}^c$ (and $\mathbb{H}$ by $\mathbb{K}$), we see that we can assume $\mathfrak{F}^s = B_p^{s/\boldsymbol{\nu}}$.

(2) The arguments of step (3) of the proof of Theorem 4.7.1, using again $\mathcal{B}^c$ instead of γ^c, show that

$$\{ u \in C_0^\infty \cap B_p^{s/\boldsymbol{\nu}}(\mathbb{K}, E) \ ; \ \partial^i_{\boldsymbol{n}_1} u = 0, \ \partial^j_{\boldsymbol{n}_2} u = 0, \ i \le m, \ j \le k \} \tag{4.12.1}$$

is dense in

$$\{ v \in B_p^{s/\boldsymbol{\nu}}(\mathbb{K}, E) \ ; \ \partial^i_{\boldsymbol{n}_1} v = 0, \ \partial^j_{\boldsymbol{n}_2} v = 0, \ i \le m, \ j \le k \}. \tag{4.12.2}$$

(3) Let G be a Banach space. Choose $\varphi \in \mathcal{D}(\mathbb{R}^+)$ satisfying $\varphi(t) = 1$ for $0 \le t \le 1/2$ and $\varphi(t) = 0$ for $t \ge 1$. Put $\varphi_\varepsilon(t) := \varphi(t/\varepsilon)$ for $t \ge 0$ and $\varepsilon > 0$. Given $v \in L_p(X^+ \times Y^+, G)$, set

$$\psi_\varepsilon v(x,y) := \varphi_\varepsilon(x)\varphi_\varepsilon(y)v(x,y), \qquad \text{a.a. } (x,y) \in X^+ \times Y^+.$$

Denote by ω_{12} the least common multiple of ω_1 and ω_2 and set $1/\widetilde{\boldsymbol{\omega}} := (1/\omega_1, 1/\omega_2)$. Fix $n \in \dot{\mathbb{N}}$. Theorem 4.4.3(i) and Leibniz' rule imply

$$\|\psi_\varepsilon v\|_{W_p^{n\omega_{12}/\widetilde{\boldsymbol{\omega}}}(X^+ \times Y^+, G)} \le c \, \|v\|_{W_p^{n\omega_{12}/\widetilde{\boldsymbol{\omega}}}(X^+ \times Y^+, G)} \tag{4.12.3}$$

for $\varepsilon \ge 1$. It is obvious that, given $v \in L_p(X^+ \times Y^+, G)$,

$$\psi_\varepsilon v \to v \text{ in } L_p(X^+ \times Y^+, G) \text{ as } \varepsilon \to \infty. \tag{4.12.4}$$

From Proposition 3.5.3 and Theorem 4.4.1 we know

$$B_p^{r/\widetilde{\boldsymbol{\omega}}}(X^+ \times Y^+, G) \doteq \big(L_p(X^+ \times Y^+, G), B_p^{n\omega_{12}/\widetilde{\boldsymbol{\omega}}}(X^+ \times Y^+, G) \big)_{r/n\omega_{12}}$$

for $0 < r < n\omega_{12}$. Thus (cf. (4.7.9)) we deduce from (4.12.3) and (4.12.4) that, given $r > 0$ and $v \in B_p^{r/\widetilde{\boldsymbol{\omega}}}(X^+ \times Y^+, G)$,

$$\psi_\varepsilon v \to v \text{ in } B_p^{r/\widetilde{\boldsymbol{\omega}}}(X^+ \times Y^+, G) \text{ as } \varepsilon \to \infty. \tag{4.12.5}$$

(4) Set $\mathcal{X} := L_p(\mathbb{R}^{d-2}, E)$ and $\mathcal{Y} := B_p^{s/\boldsymbol{\omega}''}(\mathbb{R}^{d-2}, E)$. It is a consequence of Theorems 3.6.1 and 4.4.3(ii) that

$$B_p^{s/\boldsymbol{\nu}}(\mathbb{K}, E) \doteq L_p(X^+ \times Y^+, \mathcal{Y}) \cap B_p^{s/\widetilde{\boldsymbol{\omega}}}, \mathcal{X}). \tag{4.12.6}$$

Hence it follows from (4.12.4) and (4.12.5) that, given $v \in B_p^{s/\boldsymbol{\nu}}(\mathbb{K}, E)$,

$$\psi_\varepsilon v \to v \text{ in } B_p^{s/\boldsymbol{\nu}}(\mathbb{K}, E) \text{ as } \varepsilon \to \infty.$$

Suppose u belongs to (4.12.1). Then $\psi_\varepsilon u \in \mathcal{D}(X^+ \times Y^+, \mathcal{Y})$. From this we infer that

$$\big\{\, u \in \mathcal{D}(X^+ \times Y^+, \mathcal{Y}) \ ; \ \partial_{\boldsymbol{n}_1}^i u = 0,\ \partial_{\boldsymbol{n}_2}^j u = 0,\ i \le m,\ j \le k \,\big\} \tag{4.12.7}$$

is dense in (4.12.2).

(5) Put

$$\chi_\varepsilon(x,y) := \varphi_\varepsilon(x) + \varphi_\varepsilon(y) - \varphi_\varepsilon(x)\varphi_\varepsilon(y), \quad \chi_\varepsilon v(x,y) := \chi_\varepsilon(x,y)v(x,y).$$

Let u belong to (4.12.7). Then $(1-\chi_\varepsilon)u \in \mathcal{D}(\dot{X}^+ \times \dot{Y}^+, \mathcal{Y})$ and $(1-\chi_\varepsilon)u$ converges in $L_p(X^+ \times Y^+, \mathcal{Y})$ towards u as $\varepsilon \to 0$. Hence (4.12.6) shows that $(1-\chi_\varepsilon)u$ converges in $B_p^{s/\boldsymbol{\nu}}(\mathbb{K}, E)$ towards u, provided we prove, due to $u - (1-\chi_\varepsilon)u = \chi_\varepsilon u$, that

$$\chi_\varepsilon u \to 0 \text{ in } B_p^{s/\widetilde{\boldsymbol{\omega}}}(X^+ \times Y^+, \mathcal{X}) \text{ as } \varepsilon \to 0. \tag{4.12.8}$$

Then it follows that $\mathcal{D}(\dot{X}^+ \times \dot{Y}^+, \mathcal{Y})$ is dense in (4.12.2). This implies easily that $B_p^{s/\boldsymbol{\nu}}(\mathring{\mathbb{K}}, E)$ is dense in (4.12.2). Thus the theorem will be proved.

(6) Set $n := [s/\omega_1] + 2$. Note

$$\triangle_{(h,0)}^n(\psi_\varepsilon u)(x,y) = \varphi_\varepsilon(y)\triangle_{(h,0)}^n\big(\varphi_\varepsilon(x)u(x,y)\big).$$

Since $0 \le \varphi_\varepsilon(y) \le 1$ we deduce from Fubini's theorem, setting $\mathcal{Z} := L_p(\mathbb{H}^{d-1}, E)$,

$$\begin{aligned}\int_0^\infty \int_{X^+}\int_{Y^+} &\frac{\|\triangle_{(h,0)}^n(\psi_\varepsilon u)(x,y)\|_{\mathcal{X}}^p}{h^{ps/\omega_1}}\,dx\,dy\,\frac{dh}{h}\\ &\le \int_0^\infty\int_0^\infty \frac{\|\triangle_h^n(\varphi_\varepsilon v)(x)\|_{\mathcal{Z}}^p}{h^{ps/\omega_1}}\,dx\,\frac{dh}{h},\end{aligned}$$

where $v(x) := u(x,\cdot)$ and $(\varphi_\varepsilon v)(x) := \varphi_\varepsilon(x)v(x)$. Now it follows from steps (5) and (6) of the proof of Theorem 4.7.1 that

$$[\varphi_\varepsilon u]_{s/\omega_1,p,1} \to 0 \text{ as } \varepsilon \to 0,$$

using the notation of (3.8.2) with $d_1 = d_2 = 1$. Hence

$$[\psi_\varepsilon u]_{s/\omega_1,p,1} \to 0 \text{ as } \varepsilon \to 0.$$

By interchanging the rôles of ω_1 and ω_2 we also obtain

$$[\widetilde{\varphi}_\varepsilon u]_{s/\omega_2,p,2} + [\psi_\varepsilon u]_{s/\omega_2,p,2} \to 0 \text{ as } \varepsilon \to 0,$$

where $\widetilde{\varphi}_\varepsilon u := \varphi_\varepsilon(y)u(x,y)$. From this and Theorems 3.6.1 and 4.4.3 it thus follows that (4.12.8) is true. $\square$

4.13 The structure of wedge spaces

In this section we prove isomorphism theorems for Bessel potential and Besov spaces on closed wedges. They are of particular relevance for the weak theory of parabolic boundary value problems.

We assume throughout

- $\mathbb{K}$ *is the closed standard wedge in* $\mathbb{R}^d$*;*
- *Assumptions* (4.10.13) *and* (4.11.3) *are satisfied;* (4.13.1)
- $\mathfrak{F}^s \in \{H_p^{s/\boldsymbol{\nu}}, B_p^{s/\boldsymbol{\nu}}\}$.

We continue to use the notation and conventions of the preceding sections. For the ease of writing and reading we omit the range space $E \otimes F$ in the following so that

$$\mathfrak{F}^s(\mathbb{K}) = \mathfrak{F}^s(\mathbb{K}, E \otimes F), \text{ etc.}$$

We set

$$\mathcal{B}_{02} := \mathcal{B}_2 \,|\, \partial_1\mathbb{K}, \quad \mathcal{B}_{10} := \mathcal{B}_1 \,|\, \partial_2\mathbb{K},$$

where $\mathcal{B}_2 \,|\, \partial_1\mathbb{K}$ means, of course, that $\mathcal{B}_2$ is restricted to those distributions which are defined on $\partial_1\mathbb{K}$ and for which $\mathcal{B}_2$ is well-defined.

Let $s \in \mathbb{R}$ satisfy

$$\omega_1(m_n + 1/p) + \omega_2(k + 1/p) < s < \omega_2(k + 1 + 1/p). \tag{4.13.2}$$

Note that this implies

$$\omega_1(m_n + 1/p) < \omega_2. \tag{4.13.3}$$

Set

$$\partial_{\mathcal{B}}\mathfrak{F}^s(\partial_{12}\mathbb{K}) := \prod_{i=0}^{n}\prod_{j=0}^{k} B_p^{(s-\omega_1(i+1/p)-\omega_2(j+1/p))/\boldsymbol{\omega}''}(\partial_{12}\mathbb{K}, E \otimes F_i).$$

Since $\partial_{12}\mathbb{K} = Z \cong \partial(Y^+ \times Z) \cong \partial(\partial_1\mathbb{K})$ it follows from Theorem 4.8.1 that

$$\mathcal{B}_{02} \text{ is a retraction from } \partial_{\mathcal{B}_1}\mathfrak{F}^s(\partial_1\mathbb{K}) \text{ onto } \partial_{\mathcal{B}}\mathfrak{F}^s(\partial_{12}\mathbb{K})$$

possessing a universal coretraction $\mathcal{B}_{02}^c$. Similarly,

$$\mathcal{B}_{10} \text{ is a retraction from } \partial_{\mathcal{B}_2}\mathfrak{F}^s(\partial_2\mathbb{K}) \text{ onto } \partial_{\mathcal{B}}\mathfrak{F}^s(\partial_{12}\mathbb{K})$$

having a universal coretraction $\mathcal{B}_{10}^c$ which depends analytically on $b \in \mathfrak{B}$. Hence

$$\pi_{\mathcal{B}_{02}} := 1 - \mathcal{B}_{02}^c\mathcal{B}_{02}, \quad \pi_{\mathcal{B}_{10}} := 1 - \mathcal{B}_{10}^c\mathcal{B}_{10}$$

are projections, and Lemma 4.1.5 implies

$$\partial_{\mathcal{B}_1}\mathfrak{F}^s(\partial_1\mathbb{K}) = \partial_{\mathcal{B}_1}\mathfrak{F}^s_{\mathcal{B}_{02}}(\partial_1\mathbb{K}) \oplus \mathcal{B}_{02}^c\partial_{\mathcal{B}}\mathfrak{F}^s(\partial_{12}\mathbb{K}) \tag{4.13.4}$$

and

$$\partial_{\mathcal{B}_2}\mathfrak{F}^s(\partial_2\mathbb{K}) = \partial_{\mathcal{B}_2}\mathfrak{F}^s_{\mathcal{B}_{10}}(\partial_2\mathbb{K}) \oplus \mathcal{B}_{10}^c\partial_{\mathcal{B}}\mathfrak{F}^s(\partial_{12}\mathbb{K}), \tag{4.13.5}$$

where

$$\partial_{\mathcal{B}_1}\mathfrak{F}^s_{\mathcal{B}_{02}}(\partial_1\mathbb{K}) = \ker(\mathcal{B}_{02}), \text{ etc.}$$

Lastly, we put

$$\mathcal{B}_{12} := \frac{1}{2}\,(\mathcal{B}_{02}\mathcal{B}_1 + \mathcal{B}_{10}\mathcal{B}_2). \tag{4.13.6}$$

Note $\mathcal{B}_{12} \in \mathcal{L}\big(\mathfrak{F}^s(\mathbb{K}), \partial_{\mathcal{B}}\mathfrak{F}^s(\partial_{12}\mathbb{K})\big)$.

4.13.1 Theorem *Let assumptions* (4.13.1) *and* (4.13.2) *be satisfied. Then*

$$\mathcal{R} := (\pi_{\mathcal{B}_{02}}\mathcal{B}_1, \pi_{\mathcal{B}_{10}}\mathcal{B}_2, \mathcal{B}_{12})$$

is a retraction from $\mathfrak{F}^s(\mathbb{K})$ *onto*

$$\partial_{\mathcal{B}_1}\mathfrak{F}^s_{\mathcal{B}_{02}}(\partial_1\mathbb{K}) \times \partial_{\mathcal{B}_2}\mathfrak{F}^s_{\mathcal{B}_{10}}(\partial_2\mathbb{K}) \times \partial_{\mathcal{B}}\mathfrak{F}^s(\partial_{12}\mathbb{K}). \tag{4.13.7}$$

It possesses a universal coretraction depending analytically on $b \in \mathfrak{B}$.

Proof. Clearly, $\mathcal{R}$ is a continuous linear map from $\mathfrak{F}^s(\mathbb{K})$ into (4.13.7).

It follows from (4.13.4) and (4.13.5) that each $g = (g_1, g_2) \in \partial_{\mathcal{B}}\mathfrak{F}^s(\partial\mathbb{K})$ has a unique representation

$$g_1 = v + \mathcal{B}^c_{02}h_1, \quad g_2 = w + \mathcal{B}^c_{10}h_2 \tag{4.13.8}$$

with

$$v = \pi_{\mathcal{B}_{02}}g_1, \quad w = \pi_{\mathcal{B}_{10}}g_2, \quad (h_1, h_2) = (\mathcal{B}_{02}g_1, \mathcal{B}_{10}g_2) \in \partial_{\mathcal{B}}\mathfrak{F}^s(\partial_{12}\mathbb{K})^2.$$

Consequently, $g \in \partial^{cc}_{\mathcal{B}}\mathfrak{F}^s(\partial\mathbb{K})$ iff $h_1 = h_2$. Thus, given

$$(v, w, h) \in \partial_{\mathcal{B}_1}\mathfrak{F}^s_{\mathcal{B}_{02}}(\partial_1\mathbb{K}) \times \partial_{\mathcal{B}_2}\mathfrak{F}^s_{\mathcal{B}_{10}}(\partial_2\mathbb{K}) \times \partial_{\mathcal{B}}\mathfrak{F}^s(\partial_{12}\mathbb{K}),$$

define g by (4.13.8) with $h_1 := h_2 := h$. Then we find, by inserting g into (4.11.7) and rearranging terms,

$$\mathcal{B}^c g = \mathcal{R}^c_1 v + \mathcal{R}^c_2 w + \mathcal{R}^c_{12}h =: \mathcal{R}^c(v, w, h), \tag{4.13.9}$$

where

$$\mathcal{R}^c_1 := \mathring{\mathcal{B}}^c_1, \quad \mathcal{R}^c_2 := \mathcal{B}^c_2 - \mathring{\mathcal{B}}^c_1\mathcal{B}_1\mathcal{B}^c_2 \tag{4.13.10}$$

and

$$\mathcal{R}^c_{12} := \mathcal{B}^c_2\mathcal{B}^c_{10} + \mathring{\mathcal{B}}^c_1(\mathcal{B}^c_{02} - \mathcal{B}_1\mathcal{B}^c_2\mathcal{B}^c_{10}). \tag{4.13.11}$$

Hence $\mathcal{R}^c$ is a continuous linear map from (4.13.7) into $\mathfrak{F}^s(\mathbb{K})$. It depends analytically on $b \in \mathfrak{B}$ and one verifies that it satisfies $\mathcal{R}\mathcal{R}^c(v, w, h) = (v, w, h)$. This proves the theorem. □

4.13.2 Corollary *Set* $\pi_{\mathcal{B}} := 1 - \mathcal{B}^c\mathcal{B} \in \mathcal{L}\big(\mathfrak{F}^s(\mathbb{K})\big)$. *Then* $(\pi_{\mathcal{B}}, \mathcal{R})$ *is a toplinear isomorphism from* $\mathfrak{F}^s(\mathbb{K})$ *onto*

$$\mathfrak{F}^s_{\mathcal{B}}(\mathbb{K}) \times \partial_{\mathcal{B}_1}\mathfrak{F}^s_{\mathcal{B}_{02}}(\partial_1\mathbb{K}) \times \partial_{\mathcal{B}_2}\mathfrak{F}^s_{\mathcal{B}_{10}}(\partial_2\mathbb{K}) \times \partial_{\mathcal{B}}\mathfrak{F}^s(\partial_{12}\mathbb{K}).$$

It depends analytically on $b \in \mathfrak{B}$.

Proof. This is now a consequence of Lemma 4.1.5. □

Finally, we prove now an isomorphism theorem for $\mathfrak{F}^s_{\mathcal{B}_1}(\mathbb{K})$. For this we impose an additional assumption, using the concepts and notation introduced in Section 4.8:

- $m \in \mathbb{N}$ *satisfies* $m \geq m_n$ *and*
$$\omega_1(m + 1/p) + \omega_2(k + 1/p) < s < \omega_1(m + 1 + 1/p) + \omega_2/p; \tag{4.13.12}$$
- $\widetilde{\mathcal{B}}_1$ *is complementary to* $\mathcal{B}_1$ *to order* m *on* $\partial_1\mathbb{K}$.

Note that this assumption implies

$$\omega_2 k < \omega_1. \tag{4.13.13}$$

In particular, $k = 0$ if $\omega_1 \leq \omega_2$.

4.13.3 Theorem *Let assumptions* (4.13.1) *and* (4.13.12) *be satisfied and suppose* $s < \omega_2(k+1+1/p)$. *Set* $\widetilde{\mathcal{B}} := (\widetilde{\mathcal{B}}_1, \mathcal{B}_2)$. *Then there exists a toplinear isomorphism from* $\mathfrak{F}^s_{\mathcal{B}_1}(\mathbb{K})$ *onto*

$$\mathfrak{F}^s(\mathring{\mathbb{K}}) \times \partial_{\widetilde{\mathcal{B}}_1}\mathfrak{F}^s\big((\partial_1\mathbb{K})^0\big) \times \partial_{\mathcal{B}_2}\mathfrak{F}^s\big((\partial_2\mathbb{K})^0\big) \times \partial_{\widetilde{\mathcal{B}}}\mathfrak{F}^s(\partial_{12}\mathbb{K}) \tag{4.13.14}$$

depending analytically on the coefficients of $\widetilde{\mathcal{B}}_1$.

Proof. Define

$$\varphi \in \mathcal{L}\mathrm{is}\big(\partial_{\mathcal{B}_1}\mathfrak{F}^s(\partial_1\mathbb{K}) \times \partial_{\widetilde{\mathcal{B}}_1}\mathfrak{F}^s(\partial_1\mathbb{K}), \partial_{\mathcal{C}_1}\mathfrak{F}^s(\partial_1\mathbb{K})\big)$$

and

$$\mathcal{C}_1 = \varphi\big(\mathcal{B}_1, \widetilde{\mathcal{B}}_1\big) \in \mathcal{L}\big(\mathfrak{F}^s(\mathbb{K}), \partial_{\mathcal{C}_1}\mathfrak{F}^s(\partial_1\mathbb{K})\big)$$

in analogy to Remarks 4.8.3(b) and (c) so that

$$\partial_{\mathcal{C}_1}\mathfrak{F}^s(\partial_1\mathbb{K}) = \prod_{i=0}^{m} B_p^{(s-\omega_1(i+1/p))/\boldsymbol{\omega}'}(\partial_1\mathbb{K}).$$

Also set $\mathcal{C} := (\mathcal{C}_1, \mathcal{B}_2)$. Theorem 4.13.1 guarantees that $\mathcal{R} := (\pi_{\mathcal{B}_{02}}\mathcal{C}_1, \pi_{\mathcal{C}_{10}}\mathcal{B}_2, \mathcal{C}_{12})$ is a retraction from $\mathfrak{F}^s(\mathbb{K})$ onto

$$\partial_{\mathcal{C}_1}\mathfrak{F}^s_{\mathcal{B}_{02}}(\partial_1\mathbb{K}) \times \partial_{\mathcal{B}_2}\mathfrak{F}^s_{\mathcal{C}_{10}}(\partial_2\mathbb{K}) \times \partial_{\mathcal{C}}\mathfrak{F}^s(\partial_{12}\mathbb{K}).$$

It follows from assumption (4.13.12) that

$$\omega_2(k+1/p) < s - \omega_1(i+1/p) < \omega_2(k+1+1/p), \qquad 0 \le i \le m.$$

Hence, since $\partial_1\mathbb{K} = Y^+ \times Z \cong \mathbb{H}^{d-1}$, we deduce from Theorem 4.7.1

$$\partial_{\mathcal{C}_1}\mathfrak{F}^s_{\mathcal{B}_{02}}(\partial_1\mathbb{K}) = \partial_{\mathcal{C}_1}\mathfrak{F}^s\big((\partial_1\mathbb{K})^0\big). \tag{4.13.15}$$

Similarly, since (4.13.12) implies

$$\omega_1(m+1/p) < s - \omega_2(j+1/p) < \omega_1(m+1+1/p), \qquad 0 \le j \le k,$$

we infer from (4.8.8)

$$\partial_{\mathcal{B}_2}\mathfrak{F}^s_{\mathcal{C}_{10}}(\partial_2\mathbb{K}) = \partial_{\mathcal{B}_2}\mathfrak{F}^s\big((\partial_2\mathbb{K})^0\big). \tag{4.13.16}$$

Since $\mathcal{B}_1$, $\widetilde{\mathcal{B}}_1$, and $\mathcal{B}_2$ are retractions from $\mathfrak{F}^s(\mathbb{K})$ onto $\partial_{\mathcal{B}_1}\mathfrak{F}^s(\partial_1\mathbb{K})$, $\partial_{\widetilde{\mathcal{B}}_1}\mathfrak{F}^s(\partial_1\mathbb{K})$, and $\partial_{\mathcal{B}_2}\mathfrak{F}^s(\partial_2\mathbb{K})$, respectively, it is not difficult to see that

$$\mathcal{C}_1\big(\mathfrak{F}^s_{\mathcal{B}_1}(\mathbb{K})\big) \cong \partial_{\widetilde{\mathcal{B}}_1}\mathfrak{F}^s(\partial_1\mathbb{K}), \quad \mathcal{C}_{12}\big(\mathfrak{F}^s_{\mathcal{B}_1}(\mathbb{K})\big) \cong \partial_{\widetilde{\mathcal{B}}}\mathfrak{F}^s(\partial_{12}\mathbb{K}).$$

From this, (4.13.15), and (4.13.16) it follows that there exists a retraction $\widetilde{\mathcal{R}}$ from $\mathfrak{F}^s_{\mathcal{B}_1}(\mathbb{K})$ onto

$$\partial_{\widetilde{\mathcal{B}}_1}\mathfrak{F}^s\big((\partial_1\mathbb{K})^0\big) \times \partial_{\mathcal{B}_2}\mathfrak{F}^s\big((\partial_2\mathbb{K})^0\big) \times \partial_{\widetilde{\mathcal{B}}}\mathfrak{F}^s(\partial_{12}\mathbb{K}).$$

Due to Theorem 4.12.1

$$\mathfrak{F}^s(\mathring{\mathbb{K}}) = \mathfrak{F}^s_{\mathcal{C}}(\mathbb{K}) = \mathfrak{F}^s_{(\mathcal{B}_1,\widetilde{\mathcal{B}}_1,\mathcal{B}_2)}(\mathbb{K}) = (\mathfrak{F}^s_{\mathcal{B}_1})_{\widetilde{\mathcal{B}}}(\mathbb{K}).$$

Observing (4.13.9) the assertion follows now from Lemma 4.1.5, since the analytic dependence on the coefficients of $\widetilde{\mathcal{B}}_1$ is clear by our previous considerations. □

Taking (4.13.10) and (4.13.11) into account it is not difficult to give a rather explicit description of this isomorphism. We leave the details to the interested reader.

The importance of Theorem 4.13.3 lies, similarly as that of Theorem 4.8.4, in the fact that it allows to represent the elements of $\mathfrak{F}^s_{\mathcal{B}_1}(\mathbb{K})'$ by distributions on $\mathbb{K}$, $\partial_1\mathbb{K}$, $\partial_2\mathbb{K}$, and $\partial_{12}\mathbb{K}$. This will become clear in Part 2.

Of course, it is also of interest to classify the structure of $\mathfrak{F}^s_{\mathcal{B}_1}(\mathbb{K})$ if the restrictions for s given in assumptions (4.13.12) are not satisfied. We refrain from doing it in the present generality. However, the following examples contain a complete analysis in the important and simple case of Dirichlet and Neumann boundary conditions for second order problems.

4.13.4 Examples We fix $n \in \dot{\mathbb{N}}$ and consider the closed wedge $\mathbb{W} := \mathbb{H}^n \times \mathbb{R}^+$ in $\mathbb{R}^{n+1}$ with generic point (x,t). Note $\partial_1\mathbb{W} = \mathbb{R}^{n-1} \times \mathbb{R}^+$ and $\partial_{n+1}\mathbb{W} = \mathbb{H}^n$, using standard identifications. For abbreviation, we also set

$$\gamma := \gamma_{\partial_1\mathbb{W}} = |_{x_1=0}, \quad \gamma_0 := \gamma_{\partial_{n+1}\mathbb{W}} = |_{t=0}, \quad \boldsymbol{n} := \boldsymbol{n}_1.$$

Furthermore, we assume $\boldsymbol{\omega} = (1,\ldots,1,2)$, the 2-parabolic weight vector.

(a) (Dirichlet boundary conditions) First we consider the Dirichlet operator on $\partial_1\mathbb{W}$; thus $\mathcal{B} = (\gamma, \gamma_0)$.

($\boldsymbol{\alpha}$) (Retractions) (i) Suppose

$$2/p < s < 2 + 2/p, \qquad s \neq 3/p.$$

Then (γ,γ_0) is a retraction from $H_p^{(s,s/2)}(\mathbb{W})$ onto

$$B_p^{(s-1/p)(1,1/2)}(\partial_1\mathbb{W}) \times B_p^{s-2/p}(\mathbb{H}^n) \quad \text{if } s < 3/p,$$

and, if $s > 3/p$, onto the closed linear subspace thereof consisting of all (g,w) satisfying the compatibility condition

$$g|_{t=0} = w|_{x^1=0}. \tag{4.13.17}$$

It possesses a universal coretraction. In particular, (γ,γ_0) is a retraction from

$$H_p^{(2,1)}(\mathbb{W}) \text{ onto } B_p^{(2-1/p)(1,1/2)}(\partial_1\mathbb{W}) \times B_p^{2-2/p}(\mathbb{H}^n)$$

if $p < 3/2$, and onto the closed linear subspace thereof determined by the compatibility condition (4.13.17) if $p > 3/2$.

(ii) Let $1/p \leq s < 2/p$. Then

$$H_p^{(s,s/2)}(\mathbb{W}) = H_p^{(s,s/2)}(\mathbb{H}^n \times \dot{\mathbb{R}}^+). \tag{4.13.18}$$

If $s > 1/p$, then γ is a retraction onto $B_p^{(s-1/p)(1,1/2)}(\partial_1\mathbb{W})$.

(iii) If $-1 + 1/p < s < 1/p$, then $H_p^{(s,s/2)}(\mathbb{W}) = H_p^{(s,s/2)}(\mathring{\mathbb{W}})$.

Proof. (i) follows from Theorem 4.11.6.

(ii) Theorem 4.11.1, applied to the last coordinate, implies (4.13.18). The second assertion is a consequence of Theorem 4.11.3.

(iii) follows from Corollary 4.11.2. $\square$

(β) (Isomorphisms) (i) Suppose

$$4 - 3/p < s < 3 - 1/p. \tag{4.13.19}$$

Then $H^{(s,s/2)}_{p',\gamma}(\mathbb{W})'$ is toplinearly isomorphic to

$$H_p^{-(s,s/2)}(\mathbb{W}) \times B_p^{(2-s-1/p)(1,1/2)}(\partial_1\mathbb{W}) \times B_p^{2-s-2/p}(\mathbb{H}^n) \times B_p^{4-s-3/p}(\mathbb{R}^{n-1}).$$

In particular, if $p < 3/2$, then

$$H^{(2,1)}_{p',\gamma}(\mathbb{W})' \cong H_p^{-(2,1)}(\mathbb{W}) \times B_p^{-(1/p,1/2p)}(\partial_1\mathbb{W}) \times B_p^{-2/p}(\mathbb{H}^n) \times B_p^{2-3/p}(\mathbb{R}^{n-1}).$$

(ii) Assume

$$2 - 1/p < s < 4 - 3/p. \tag{4.13.20}$$

Then

$$H^{(s,s/2)}_{p',\gamma}(\mathbb{W})' \cong H_p^{-(s,s/2)}(\mathbb{W}) \times B_p^{(2-s-1/p)(1,1/2)}(\partial_1\mathbb{W}) \times B_p^{2-s-2/p}(\mathbb{H}^n).$$

Thus, if $p > 3/2$,

$$H^{(2,1)}_{p',\gamma}(\mathbb{W})' \cong H_p^{-(2,1)}(\mathbb{W}) \times B_p^{-(1/p,1/2p)}(\partial_1\mathbb{W}) \times B_p^{-2/p}(\mathbb{H}^n).$$

(iii) If $-1/p < s < 2 - 1/p$, then $H^{(s,s/2)}_{p',\gamma}(\mathbb{W})' \cong H_p^{-(s,s/2)}(\mathbb{W})$.

Proof. (i) Set $\widetilde{\mathcal{B}}_1 := \partial_{\boldsymbol{n}}$. Then $\widetilde{\mathcal{B}}_1$ is complementary to $\mathcal{B}_1 = \gamma$ to order $m = 1$. Condition (4.13.19) is equivalent to $1 + 3/p' < s < 2 + 1/p'$. Hence Theorem 4.13.3 implies that $H^{(s,s/2)}_{p',\gamma}(\mathbb{W})$ is toplinearly isomorphic to

$$H^{(s,s/2)}_{p'}(\mathring{\mathbb{W}}) \times B_{p'}^{(s-1-1/p')(1,1/2)}\big((\partial_1\mathbb{W})^0\big) \times B_{p'}^{s-2/p'}(\mathring{\mathbb{H}}^n) \times B_{p'}^{s-1-3/p'}(\mathbb{R}^{n-1}).$$

Due to Theorems 3.3.3, 3.7.1(i), and 4.4.4 the assertion now follows by duality.

(ii) Assumption (4.13.20) means $1 + 1/p' < s < 1 + 3/p'$. Hence it follows from Theorem 4.12.1 that

$$\big(H^{(s,s/2)}_{p',\mathcal{B}_1}\big)_{\widetilde{\mathcal{B}}}(\mathbb{W}) = H^{(s,s/2)}_{p',(\gamma,\partial_{\boldsymbol{n}},\gamma_0)}(\mathbb{W}) = H^{(s,s/2)}_{p'}(\mathring{\mathbb{W}}). \tag{4.13.21}$$

Theorem 4.11.6 implies that $\widetilde{\mathcal{B}} = (\partial_{\boldsymbol{n}}, \gamma_0)$ is a retraction from $H^{(s,s/2)}_{p'}(\mathbb{W})$ onto

$$B_{p'}^{(s-1-1/p')(1,1/2)}(\partial_1\mathbb{W}) \times B_{p'}^{s-2/p'}(\mathbb{H}^n).$$

Since $s - 1 - 1/p' < 2/p'$ we deduce from Theorem 4.7.1 that

$$B_{p'}^{(s-1-1/p')(1,1/2)}(\partial_1\mathbb{W}) = B_{p'}^{(s-2+1/p)(1,1/2)}\big((\partial_1\mathbb{W})^0\big).$$

Similarly, since $s - 2/p' < 1 + 1/p'$, it follows

$$B_{p'}^{s-2/p'}(\mathbb{H}^n) = B_{p'}^{s-2+2/p}(\mathring{\mathbb{H}}^n).$$

Thus, see Lemma 4.1.5,

$$H^{(s,s/2)}_{p'}(\mathbb{W}) = H^{(s,s/2)}_{p',\widetilde{\mathcal{B}}}(\mathbb{W}) \oplus (\partial_{\boldsymbol{n}})^c B_{p'}^{(s-2+1/p)(1,1/2)}\big(\partial_1(\mathbb{W})^0\big) \times \gamma_0^c B_{p'}^{s-2+2/p}(\mathring{\mathbb{H}}^n).$$

From this and (4.13.21) we infer

$$H^{(s,s/2)}_{p',\gamma}(\mathbb{W}) \cong H^{(s,s/2)}_{p'}(\mathring{\mathbb{W}}) \times B_{p'}^{(s-2+1/p)(1,1/2)}\big(\partial_1(\mathbb{W})^0\big) \times B_{p'}^{s-2+2/p}(\mathring{\mathbb{H}}^n).$$

Finally, we obtain the assertion once more by duality.

(iii) This follows from (α)(iii) and duality. □

(b) (Neumann boundary conditions) Now we consider the Neumann boundary operator $\partial_{\boldsymbol{n}}$ on $\partial_1\mathbb{W}$, that is, $\mathcal{B} = (\partial_{\boldsymbol{n}}, \gamma_0)$.

($\boldsymbol{\alpha}$) (Retractions) Suppose

$$1 + 1/p < s < 3 - 1/p, \qquad s \neq 1 + 3/p.$$

Then $\mathcal{B}$ is a retraction from $H_p^{(s,s/2)}(\mathbb{W})$ onto

$$B_p^{(s-1-1/p)(1,1/2)}(\partial_1\mathbb{W}) \times B_p^{s-2/p}(\mathbb{H}^n) \quad \text{if } s < 1 + 3/p,$$

and, if $s > 1 + 3/p$, onto the closed linear subspace thereof consisting of all (g, w) satisfying the compatibility condition

$$g|_{t=0} = \partial_{\boldsymbol{n}} w|_{x^1=0}. \tag{4.13.22}$$

It possesses a universal coretraction.

In particular, if $p < 3$, then $(\partial_{\boldsymbol{n}}, \gamma_0)$ is a retraction from $H_p^{(2,1)}(\mathbb{W})$ onto

$$B_p^{(1-1/p)(1,1/2)}(\partial_1\mathbb{W}) \times B_p^{2-2/p}(\mathbb{H}^n)$$

and, if $p > 3$, onto the closed linear subspace thereof determined by (4.13.22).

Proof. Theorem 4.11.6. □

($\boldsymbol{\beta}$) (Isomorphisms) (i) Suppose

$$3 - 3/p < s < 3 - 1/p, \qquad s \neq 2 - 1/p. \tag{4.13.23}$$

Then $H_{p',\partial_{\boldsymbol{n}}}^{(s,s/2)}(\mathbb{W})'$ is toplinearly isomorphic to

$$H_p^{-(s,s/2)}(\mathbb{W}) \times B_p^{(1-s-1/p)(1,1/2)}(\partial_1\mathbb{W}) \times B_p^{2-s-2/p}(\mathbb{H}^n) \times B_p^{3-s-3/p}(\mathbb{R}^{n-1}),$$

where $H_{p',\partial_{\boldsymbol{n}}}^{(s,s/2)}(\mathbb{W}) := H_{p'}^{(s,s/2)}(\mathbb{W}^n)$ if $s < 1 + 1/p' = 2 - 1/p$. Thus, if $p < 3$, then

$$\begin{aligned} H_{p',\partial_{\boldsymbol{n}}}^{(2,1)}(\mathbb{W})' \cong H_p^{-(2,1)}(\mathbb{W}) &\times B_p^{-(1+1/p)(1,1/2)}(\partial_1\mathbb{W}) \\ &\times B_p^{-2/p}(\mathbb{H}^n) \times B_p^{1-3/p}(\mathbb{R}^{n-1}). \end{aligned}$$

(ii) If $2 - 2/p < s < 3 - 3/p$, then

$$H_{p',\partial_{\boldsymbol{n}}}^{(s,s/2)}(\mathbb{W})' \cong H_p^{-(s,s/2)}(\mathbb{W}) \times B_p^{(1-s-1/p)(1,1/2)}(\partial_1\mathbb{W}) \times B_p^{2-s-2/p}(\mathbb{H}^n).$$

In particular, if $p > 3$, then

$$H_{p',\partial_{\boldsymbol{n}}}^{(2,1)}(\mathbb{W})' \cong H_p^{-(2,1)}(\mathbb{W}) \times B_p^{-(1+1/p)(1,1/2)}(\partial_1\mathbb{W}) \times B_p^{-2/p}(\mathbb{H}^n).$$

(iii) Let $1 - 1/p < s < 2 - 2/p$. Then

$$H_{p',\partial_{\boldsymbol{n}}}^{(s,s/2)}(\mathbb{W})' \cong H_p^{-(s,s/2)}(\mathbb{W}) \times B_p^{-(1+1/p)(1,1/2)}(\partial_1\mathbb{W}).$$

Proof. (i) Condition (4.13.23) says $3/p' < s < 2 + 1/p'$ and $s \neq 1 + 1/p'$.

First suppose $s > 1 + 1/p'$. Then $\widetilde{\mathcal{B}}_1 := \gamma$ is complementary to $\mathcal{B}_1 = \partial_{\boldsymbol{n}}$ to order 1. Hence Theorem 4.13.3 implies that $H_{p',\partial_{\boldsymbol{n}}}^{(s,s/2)}(\mathbb{W})$ is toplinearly isomorphic to

$$\begin{aligned} H_{p'}^{(s,s/2)}(\mathring{\mathbb{W}}) &\times B_{p'}^{(s-1/p')(1,1/2)}\big((\partial_1\mathbb{W})^0\big) \\ &\times B_{p'}^{s-2/p'}(\mathring{\mathbb{H}}^n) \times B_{p'}^{s-3/p'}(\mathbb{R}^{n-1}). \end{aligned} \tag{4.13.24}$$

Now suppose $3/p' < s < 1 + 1/p'$. Put $\gamma_{10} := \gamma\,|\,\mathbb{H}^n$ and $\gamma_{02} := \gamma_0\,|\,\partial_1\mathbb{W}$. Then we deduce from Corollary 4.13.2 that $H_{p'}^{(s,s/2)}(\mathbb{W})$ is toplinearly isomorphic to

$$H_{p',\mathcal{B}}^{(s,s/2)}(\mathbb{W}) \times B_{p',\gamma_{02}}^{(s-1/p')(1,1/2)}(\partial_1\mathbb{W}) \times B_{p',\gamma_{10}}^{s-2/p'}(\mathbb{H}^n) \times B_{p'}^{s-3/p'}(\mathbb{R}^{n-1}).$$

Theorems 4.7.1 and 4.12.1 guarantee that this space also equals (4.13.24). Thus the assertion is obtained by duality.

(ii) Since $2/p' < s < 3/p'$ it follows from Theorem 4.11.6 that γ is a retraction from $H_{p'}^{(s,s/2)}(\mathbb{W})$ onto

$$B_{p'}^{(s-1/p')(1,1/2)}(\partial_1\mathbb{W}) \times B_p^{s-2/p'}(\mathbb{H}^n).$$

Due to $s - 1/p' < 2/p'$ and $s - 2/p' < 1/p'$ we infer from Theorem 4.7.1 that this space equals

$$B_{p'}^{(s-1+1/p)(1,1/2)}\big((\partial_1\mathbb{W})^0\big) \times B_{p'}^{s-2+2/p}(\mathring{\mathbb{H}}^n).$$

Arguments which are familiar by now conclude the proof.

(iii) This follows, similarly as above, by invoking Theorem 4.11.3. □

(c) It is clear that analogous assertions hold if we replace H everywhere by B. □

The reader is invited to draw the connection between the above isomorphism theorems for $H_{p',\gamma}^{(2,1)}(\mathbb{W})'$ and $H_{p',\partial_{\boldsymbol{n}}}^{(2,1)}(\mathbb{W})'$ and Theorem 0.5 in the introductory section on parabolic equations. For this Remark 4.4.5 has to be kept in mind. Of course, this will explained in detail in Part 2.

4.14 Parameter-dependent function spaces

For technical reasons, which will become clear in Part 2, it is useful to have parameter-dependent versions of Bessel potential and Besov spaces at our disposal. Thus we return to the general setting of Section 2.3 where parameter-dependent fractional power scales $[\,\mathfrak{F}_{\gamma,\eta}^s\ ;\ s \in \mathbb{R}\,]$ have been introduced. By specifying $\mathfrak{F}$ and γ we arrive at the desired concrete scales. Recall $1 < p < \infty$.

Parameter-dependent anisotropic Bessel potential spaces are naturally defined by

$$H_{p;\eta}^{s/\boldsymbol{\nu}} = H_{p;\eta}^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E) := J_\eta^{-s} L_p, \qquad s \in \mathbb{R}, \quad \eta \in \dot{\mathsf{H}}.$$

Thus the parameter-dependent anisotropic Bessel potential scale

$$[\,H_{p;\eta}^{s/\boldsymbol{\nu}}\ ;\ s \in \mathbb{R}\,]$$

is for $\eta \in \dot{\mathsf{H}}$ the fractional power scale generated by (L_p, J_η). It is a particularization of (2.3.7) with $\gamma = 0$.

To define **parameter-dependent anisotropic Besov spaces**, $\overset{\vartriangle}{B}{}_{q,r;\eta}^{s/\boldsymbol{\nu}}(\mathbb{R}^d, F)$, for $q, r \in [1, \infty]$, $s \in \mathbb{R}$, and an arbitrary Banach space F, we recall $\rho_t^\gamma = t^\gamma \sigma_t$ for $t > 0$ and $\gamma \in \mathbb{R}$. Hence $(\rho_t^\gamma)^{-1} = \rho_{1/t}^{-\gamma}$. Then, given $\eta \in \dot{\mathsf{H}}$,

$$\overset{\vartriangle}{B}{}_{q,r;\eta}^0 = \overset{\vartriangle}{B}{}_{q,r;\eta}^0(\mathbb{R}^d, F) := \rho_{|\eta|}^{|\boldsymbol{\omega}|/q}\, \overset{\vartriangle}{B}{}_{q,r}^0$$

and

$$\overset{\vartriangle}{B}{}_{q,r;\eta}^{s/\boldsymbol{\nu}} = \overset{\vartriangle}{B}{}_{q,r;\eta}^{s/\boldsymbol{\nu}}(\mathbb{R}^d, F) := J_\eta^{-s}\, \overset{\vartriangle}{B}{}_{q,r;\eta}^0$$

for $s \in \mathbb{R}$. These spaces are equipped with the image space norm

$$u \mapsto \|u\|_{B^{s/\boldsymbol{\nu}}_{q,r;\eta}} := \|\rho^{-|\boldsymbol{\omega}|/q}_{1/|\eta|} J^s_\eta u\|_{B^0_{q,r}}$$

(cf. Remark 2.2.1(a)). Thus, setting $\gamma := |\boldsymbol{\omega}|/q$ and using the notation of Section 2.3,

$$\overset{\scriptscriptstyle\triangle}{B}{}^{s/\boldsymbol{\nu}}_{q,r;\eta} = \mathfrak{F}^s_{\gamma,\eta} = J^{-s}_\eta \mathfrak{F}_{\gamma,\eta}, \quad \mathfrak{F}_{\gamma,\eta} = \rho^\gamma_{|\eta|} \overset{\scriptscriptstyle\triangle}{B}{}^0_{q,r}.$$

In other words: the parameter-dependent anisotropic Besov space scale

$$[\,\overset{\scriptscriptstyle\triangle}{B}{}^{s/\boldsymbol{\nu}}_{q,r;\eta}\ ;\ s \in \mathbb{R}\,]$$

is the fractional power scale generated by $(\overset{\scriptscriptstyle\triangle}{B}{}^0_{q,r;\eta}, J_\eta)$ for $\eta \in \dot{\mathsf{H}}$.

These definitions are justified by the following theorem which shows that the basic properties for the standard, that is parameter-free, Bessel potential and Besov spaces hold for the parameter-dependent versions also, uniformly with respect to $\eta \in \dot{\mathsf{H}}$.

4.14.1 Theorem

(i) *If* $u \in H^{s/\boldsymbol{\nu}}_p$, *then*

$$\|u\|_{H^{s/\boldsymbol{\nu}}_{p;\eta}} = |\eta|^{s-|\boldsymbol{\omega}|/p}\,\|\sigma_{1/|\eta|}u\|_{H^{s/\boldsymbol{\nu}}_p}, \qquad \eta \in \dot{\mathsf{H}}.$$

(ii) *For* $u \in \overset{\scriptscriptstyle\triangle}{B}{}^{s/\boldsymbol{\nu}}_{q,r}$,

$$\|u\|_{B^{s/\boldsymbol{\nu}}_{q,r;\eta}} = |\eta|^{s-|\boldsymbol{\omega}|/q}\,\|\sigma_{1/|\eta|}u\|_{B^{s/\boldsymbol{\nu}}_{q,r}}, \qquad \eta \in \dot{\mathsf{H}}.$$

(iii) $H^{s/\boldsymbol{\nu}}_{p;\eta}(\mathbb{R}^d, E)' = H^{-s/\boldsymbol{\nu}}_{p';\eta}(\mathbb{R}^d, E')$, $\eta \in \dot{\mathsf{H}}$, *and, if F is reflexive,*

$$B^{s/\boldsymbol{\nu}}_{q,r;\eta}(\mathbb{R}^d, F)' \underset{\eta}{\doteq} B^{-s/\boldsymbol{\nu}}_{q',r';\eta}(\mathbb{R}^d, F'), \qquad r \neq \infty,$$

with respect to the $L_p(\mathbb{R}^d, E)$, *resp.* $L_q(\mathbb{R}^d, F)$, *duality pairing.*

(iv) *For* $s_0, s_1 \in \mathbb{R}$ *with* $s_0 \neq s_1$ *and* $0 < \theta < 1$,

$$\overset{\scriptscriptstyle\triangle}{B}{}^{s_\theta/\boldsymbol{\nu}}_{p,q;\eta} \underset{\eta}{\doteq} (H^{s_0}_{p;\eta}, H^{s_1}_{p;\eta})^0_{\theta,q}.$$

Proof. (1) Assertions (i) and (ii) follow easily from Proposition 1.1.1(ii) and (2.3.2).

(2) For $u \in \mathcal{S}(\mathbb{R}^d, F)$ and $v \in \mathcal{S}(\mathbb{R}^d, F')$ one verifies

$$\int_{\mathbb{R}^d} \langle v, u\rangle_F\,dx = \int_{\mathbb{R}^d} \bigl\langle \rho^{-|\boldsymbol{\omega}|/q'}_{1/|\eta|} v, \rho^{-|\boldsymbol{\omega}|/q}_{1/|\eta|} u\bigr\rangle_F\,dx, \qquad \eta \in \dot{\mathsf{H}}.$$

Now (iii) is implied by (i), (ii), and the duality results for the parameter-free spaces.

(3) Put $F^s_\eta := |\eta|^{-s}\,H^{s/\boldsymbol{\nu}}_p$, that is, F^s_η is the image space of the multiplication operator $u \mapsto |\eta|^{-s}\,u$. Then, given $s_0 < s_1$ and $0 < \theta < 1$,

$$\begin{aligned} K_\eta(t,u) &:= \inf\bigl\{\,\|u_0\|_{F^{s_0}_\eta} + t\,\|u_1\|_{F^{s_1}_\eta}\ ;\ u = u_0 + u_1,\ u_j \in F^{s_j}_\eta\,\bigr\} \\ &= |\eta|^{s_0}\,K_1(|\eta|^{s_1 - s_0}\,t, u) \end{aligned}$$

for $t > 0$ and $u \in F_\eta^{s_1}$. Thus, setting $F^s := F_1^s$, the K-method of interpolation theory implies

$$\begin{aligned}\|u\|_{(F_\eta^{s_0}, F_\eta^{s_1})_{\theta,q}} &= \|t^{-\theta} K_\eta(t,u)\|_{L_q(\dot{\mathbb{R}}^+, dt/t)} = |\eta|^{s_\theta} \, \|\tau^{-\theta} K_1(\tau,u)\|_{L_q(\dot{\mathbb{R}}^+, d\tau/\tau)} \\ &= |\eta|^{s_\theta} \, \|u\|_{(F^{s_0}, F^{s_1})_{\theta,q}}.\end{aligned}$$

Consequently, by Theorem 3.7.1(iv) we infer

$$\big(|\eta|^{-s_0} \, H_p^{s_0/\boldsymbol{\nu}}, |\eta|^{-s_1} \, H_p^{s_1/\boldsymbol{\nu}}\big)_{\theta,q}^0 = |\eta|^{-s_\theta} \, \overset{\vartriangle}{B}{}_{p,q}^{s/\boldsymbol{\nu}}. \tag{4.14.1}$$

From (i) we see that $\rho_{1/|\eta|}^{-|\boldsymbol{\omega}|/p}$ is an isometric isomorphism from $H_{p;\eta}^{s/\boldsymbol{\nu}}$ onto $|\eta|^{-s} \, H_p^{s/\boldsymbol{\nu}}$. Thus, by interpolating and using (4.14.1), it follows that

$$\rho_{1/|\eta|}^{-|\boldsymbol{\omega}|/p} : \, (H_{p;\eta}^{s_0/\boldsymbol{\nu}}, H_{p;\eta}^{s_1/\boldsymbol{\nu}})_{\theta,q}^0 \to |\eta|^{-s_\theta} \, \overset{\vartriangle}{B}{}_{p,q}^{s_\theta/\boldsymbol{\nu}}$$

is an isomorphism as well, η-uniformly. Now (iv) follows from (ii). □

Theorem 2.3.8 shows explicitly the η-dependence of the norm of the parameter-dependent anisotropic Sobolev spaces

$$W_{p;\eta}^{m\nu/\boldsymbol{\nu}}(\mathbb{R}^d, E) = H_{p;\eta}^{m\nu/\boldsymbol{\nu}}(\mathbb{R}^d, E), \qquad m \in \mathbb{N}.$$

The following proposition shows that an analogous result holds for parameter-dependent anisotropic Slobodeckii spaces of positive order. For this we remind the reader of the definition of the seminorm $[\cdot]_{s/\boldsymbol{\nu},q,r}$ in (3.6.1) and (3.6.2).

4.14.2 Proposition *If $s > 0$, then*

$$\|\cdot\|_{B_{q,r;\eta}^{s/\boldsymbol{\nu}}} \underset{\eta}{\sim} |\eta|^s \, \|\cdot\|_{L_q} + [\cdot]_{s/\boldsymbol{\nu},q,r}.$$

Proof. For $u \in \mathcal{S}(\mathbb{R}^d, F)$, $y \in \mathbb{R}^d$, and $t > 0$ one verifies $\triangle_y^k \sigma_t = \sigma_t \triangle_{t\bullet y}^k$ for $k \in \mathbb{N}$. Hence, by Proposition 1.1.1(ii),

$$\|\triangle_{\check{h}_i}^{k\nu/\nu_i} \sigma_t u\|_q = t^{-|\boldsymbol{\omega}|/q} \, \|\triangle_{t\bullet\check{h}_i}^{k\nu/\nu_i} u\|_q.$$

Suppose $r \neq \infty$. Then

$$\begin{aligned}&\big\| \, |h_i|^{-s/\nu_i} \, \|\triangle_{\check{h}_i}^{k\nu/\nu_i} \sigma_t u\|_q \, \big\|_{L_r((\mathbb{R}^{d_i})^\bullet, dh_i/|h_i|^{d_i})} \\ &\qquad = t^{s-|\boldsymbol{\omega}|/q} \Big(\int_{\mathbb{R}^{d_i}} |t \bullet h_i|^{-sr/\nu_i} \, \|\triangle_{t\bullet\check{h}_i}^{k\nu/\nu_i} u\|_q^r \, \frac{d(t \bullet h_i)}{|t \bullet h_i|^{d_i}} \Big)^{1/r}\end{aligned}$$

due to $t \bullet \check{h}_i = t^{\nu_i} \check{h}_i$. Hence, by changing variables,

$$[\sigma_t u]_{s/\boldsymbol{\nu},q,r} = t^{s-|\boldsymbol{\omega}|/q} [u]_{s/\boldsymbol{\nu},q,r} \tag{4.14.2}$$

if $r \neq \infty$. It is easily verified that (4.14.2) holds also if $r = \infty$. Setting $t = 1/|\eta|$ it follows

$$|\eta|^{s-|\boldsymbol{\omega}|/q} [\sigma_{1/|\eta|} u]_{s/\boldsymbol{\nu},q,r} = [u]_{s/\boldsymbol{\nu},q,r}, \qquad \eta \in \dot{\mathsf{H}}.$$

Since by Proposition 1.1.1(ii)

$$|\eta|^{s-|\boldsymbol{\omega}|/q} \, \|\sigma_{1/|\eta|} u\|_q = |\eta|^s \, \|u\|_q, \qquad \eta \in \dot{\mathsf{H}},$$

we obtain the assertion from Theorems 3.6.1 and 4.14.1(ii). □

Now we turn to the half-space $\mathbb{H} = \mathbb{H}^d$ and standard corners.

4.14.3 Theorem *All extension and restriction results from and to corners derived in the preceding sections hold η-uniformly for parameter-dependent anisotropic Bessel potential and Besov spaces.*

PROOF. This is an easy consequence of Theorem 4.14.1 and the proofs for the parameter-free case. □

4.14.4 Theorem *The parameter-dependent anisotropic Bessel potential and Besov spaces possess the same interpolation properties, uniformly with respect to $\eta \in \dot{\mathsf{H}}$, as their parameter-free counterparts.*

PROOF. In the full space case, that is for spaces on $\mathbb{R}^d$, this follows from Theorems 2.3.2(v) and 4.14.1(iv), and by reiteration. For spaces on corners it is then a consequence of Theorems 4.4.1 and 4.14.3. □

If $u \in \mathcal{S}(\mathbb{H}, F)$, then

$$\gamma_{\partial\mathbb{H}}\sigma_t u(x') = \gamma_{\partial\mathbb{H}}\big(u(t \bullet x)\big) = u(0, t \bullet x') = \sigma_t \gamma_{\partial\mathbb{H}} u(x'), \qquad x' \in \partial\mathbb{H},$$

that is, $\gamma_{\partial\mathbb{H}}\sigma_t = \sigma_t\gamma_{\partial\mathbb{H}}$. Thus, since $|\boldsymbol{\omega}| = \omega_1 + |\boldsymbol{\omega}'|$, Theorem 4.14.1(ii) and the trace Theorem 4.5.4 imply

$$\begin{aligned}
\|\gamma_{\partial\mathbb{H}}u\|_{B_{q,r;\eta}^{(s-\omega_1/q)/\boldsymbol{\omega}'}(\partial\mathbb{H},F)} &= |\eta|^{s-\omega_1/q-|\boldsymbol{\omega}'|/q} \,\|\sigma_{1/|\eta|}\gamma_{\partial\mathbb{H}}u\|_{B_{q,r}^{(s-\omega_1/q)/\boldsymbol{\omega}'}(\partial\mathbb{H},F)} \\
&= |\eta|^{s-|\boldsymbol{\omega}|/q} \,\|\gamma_{\partial\mathbb{H}}\sigma_{1/|\eta|}u\|_{B_{q,r}^{(s-\omega_1/q)/\boldsymbol{\omega}'}(\partial\mathbb{H},F)} \\
&\leq c\,|\eta|^{s-|\boldsymbol{\omega}|/q} \,\|\sigma_{1/|\eta|}u\|_{B_{q,r}^{s/\boldsymbol{\nu}}(\mathbb{H},F)} = c\,\|u\|_{B_{q,r;\eta}^{s/\boldsymbol{\nu}}(\mathbb{H},F)}.
\end{aligned}$$

Thus, if $s > \omega_1/q$,

$$\gamma_{\partial\mathbb{H}} \in \mathcal{L}\big(\overset{\vartriangle}{B}{}_{q,r;\eta}^{s/\boldsymbol{\nu}}(\mathbb{H},F), \overset{\vartriangle}{B}{}_{q,r;\eta}^{(s-\omega_1/q)/\boldsymbol{\omega}'}(\partial\mathbb{H},F)\big), \quad \eta\text{-uniformly.}$$

Similarly,

$$\gamma_{\partial\mathbb{H}} \in \mathcal{L}\big(H_{p;\eta}^{s/\boldsymbol{\nu}}(\mathbb{H},E), B_{p;\eta}^{(s-\omega_1/p)/\boldsymbol{\omega}'}(\partial\mathbb{H},E)\big), \quad \eta\text{-uniformly,}$$

for $s > \omega_1/p$. This extends immediately to the higher order trace maps $\partial_{\boldsymbol{n}}^j$. In fact, the following important theorem is true.

4.14.5 Theorem *All retraction and coretraction results of the preceding sections hold η-uniformly for parameter-dependent anisotropic Bessel potential and Besov spaces.*

PROOF. (1) The η-uniform continuity of the various trace operators follows easily from the preceding considerations and Theorem 4.14.3.

(2) All coretraction results for the half-space of the foregoing sections are based on the coretractions γ_j^c for $\partial_{\boldsymbol{n}}^j$ constructed in the proof of Theorem 4.6.2. Thus we have to show that γ_j^c is continuous from the parameter-dependent spaces on $\partial\mathbb{H}$ to the corresponding parameter-dependent domains of $\partial_{\boldsymbol{n}}^j$, uniformly with respect to $\eta \in \dot{\mathsf{H}}$. As in the above considerations for $\gamma_{\partial\mathbb{H}}$, this will follow from (4.6.6)–(4.6.13), provided we show

$$\sigma_{|\eta|}A^k e^{-tA} = A^k e^{-tA}\sigma_{|\eta|}, \qquad \eta \in \dot{\mathsf{H}}, \quad k \in \mathbb{N}, \quad t \geq 0. \tag{4.14.3}$$

However, this follows from (2.2.2) and the representation formula for

$$e^{-tA} = (e^{-ta})(D)$$

of Theorem 2.2.6, where $a := \mathsf{K}_1^{\omega_1}$.

(3) Similar remarks apply to the case of retraction and coretraction theorems on wedges. This implies the assertion. □

Parameter-dependent function spaces occur naturally in resolvent constructions for elliptic and parabolic boundary value problems and in singular perturbation problems. It seems that parameter-dependent norms have first been used in connection with resolvent estimates by M.S. Agranovich and M.I. Vishik [1] in an L_2-setting.

Parameter-dependent isotropic and anisotropic L_2-Sobolev spaces of fractional order, H^s and $H^{(s,t)}$, have been extensively used by G. Grubb in numerous articles on a parameter-dependent Boutet de Monvel theory for pseudo-differential boundary value problems. This work and a functional calculus for such problems is well documented in her book [31]. In G. Grubb and N.J. Kokholm [32] the parameter-dependent calculus is extended to isotropic Bessel potential and Besov spaces in the L_p-setting for $1 < p < \infty$.

In all those papers the authors use, instead of $|\eta|$, the parameter $\langle\eta\rangle$ so that $1/\langle\eta\rangle$ does not blow up as $\eta \to 0$. This is, however, irrelevant for the above results.

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Herbert Amann

Anisotropic function spaces and maximal regularity for parabolic problems

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Jindřich Nečas

Jindřich Nečas was born in Prague on December 14th, 1929. He studied mathematics at the Faculty of Natural Sciences at the Charles University from 1948 to 1952. After a brief stint as a member of the Faculty of Civil Engineering at the Czech Technical University, he joined the Czechoslovak Academy of Sciences where he served as the Head of the Department of Partial Differential Equations. He held joint appointments at the Czechoslovak Academy of Sciences and the Charles University from 1967 and became a full time member of the Faculty of Mathematics and Physics at the Charles University in 1977. He spent the rest of his life there, a significant portion of it as the Head of the Department of Mathematical Analysis and the Department of Mathematical Modeling.

His initial interest in continuum mechanics led naturally to his abiding passion to various aspects of the applications of mathematics. He can be rightfully considered as the father of modern methods in partial differential equations in the Czech Republic, both through his contributions and through those of his numerous students. He has made significant contributions to both linear and non-linear theories of partial differential equations. That which immediately strikes a person conversant with his contributions is their breadth without the depth being compromised in the least bit. He made seminal contributions to the study of Rellich identities and inequalities, proved an infinite dimensional version of Sard's Theorem for analytic functionals, established important results of the type of Fredholm alternative, and most importantly established a significant body of work concerning the regularity of partial differential equations that had a bearing on both elliptic and parabolic equations. At the same time, Nečas also made important contributions to rigorous studies in mechanics. Notice must be made of his work, with his collaborators, on the linearized elastic and inelastic response of solids, the challenging field of contact mechanics, a variety of aspects of the Navier-Stokes theory that includes regularity issues as well as important results concerning transonic flows, and finally non-linear fluid theories that include fluids with shear-rate dependent viscosities, multi-polar fluids, and finally incompressible fluids with pressure dependent viscosities.

Nečas was a prolific writer. He authored or co-authored eight books. Special mention must be made of his book "Les méthodes directes en théorie des équations elliptiques" which has already had tremendous impact on the progress of the subject and will have a lasting influence in the field. He has written a hundred and forty seven papers in archival journals as well as numerous papers in the proceedings of conferences all of which have had a significant impact in various areas of applications of mathematics and mechanics.

Jindřich Nečas passed away on December 5th, 2002. However, the legacy that Nečas has left behind will be cherished by generations of mathematicians in the Czech Republic in particular, and the world of mathematical analysts in general.

Jindřich Nečas Center for Mathematical Modeling

The Nečas Center for Mathematical Modeling is a collaborative effort between the Faculty of Mathematics and Physics of the Charles University, the Institute of Mathematics of the Academy of Sciences of the Czech Republic and the Faculty of Nuclear Sciences and Physical Engineering of the Czech Technical University.

The goal of the Center is to provide a place for interaction between mathematicians, physicists, and engineers with a view towards achieving a better understanding of, and to develop a better mathematical representation of the world that we live in. The Center provides a forum for experts from different parts of the world to interact and exchange ideas with Czech scientists.

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