Regularity theory for elliptic and parabolic systems and problems in continuum mechanics

Workshop, 2[nd] May – 5[th] May 2018, Telč

Schedule

Thursday, May 3

Time	Speaker	Title
8:00		Breakfast
8:50		Opening
9:00	Jan Kristensen	Regularity and uniqueness results for minimizers
10:00	Filip Rindler	Existence and regularity of solutions to rate-independent systems
10:30		Coffee & Refreshment
11:00	Vesa Julin	The surface diffusion flow with elasticity
11:30	Václav Mácha	Global BMO estimates for non-Newtonian fluids with perfect slip boundary conditions
11:50	Erika Maringová	On the regularity of minimizers of convex variational problems
12:10	Yeonghun Youn	Partial regularity via potentials
12:30		Lunch
15:20		Coffee & Refreshment
15:50	Ángel Castro	Degraded mixing solutions for the incompressible porous media
16:50	Sukjung Hwang	Hölder regularity of parabolic quasi-linear degenerate and singular equations
17:10	Anna Abbatiello	Regularity of solutions to flow of non-Newtonian fluids with concentration dependent power-law index
17:30		Break
17:50	Jens Frehse	A regularity criterium for Navier-Stokes-equations based on signed pressures
18:50	Josef Málek	On several regularity results for PDE problems in fluid mechanics
19:15		Dinner

Friday, May 4

Time	Speaker	Title
8:00		Breakfast
8:50	Giuseppe Mingione	A soup of Lipschitz estimates with uniformly and non-uniformly elliptic ingredients
9:50	Paolo Baroni	On the Cauchy-Dirichlet problem for a general class of parabolic equations
10:20	Sebastian Schwarzacher	Parabolic Lipschitz truncation and applications
10:40		Coffee & Refreshment
11:10	Barbora Benešová	Non-interpenetration in thin-film models
11:40	Iwona Skrzypczak	Gradient estimates for problems with Orlicz growth
12:00	Michal Bathory	Identification of outflow boundary conditions on artificial boundary
12:20	Tomáš Los	On three-dimensional flows of internal pore pressure activated Bingham fluids
12:40		Lunch
15:00		Coffee & Refreshment
15:30	Ireneo Peral	Principal eigenvalue of mixed problem for the fractional Laplacian: Moving the boundary conditions. Applications
16:30	Stanislav Hencl	Weak regularity of the inverse under minimal assumptions
16:50	Grzegorz Serafin	Two-sided estimates for solutions of fractal Burgers equation
17:10	Franz Gmeineder	Regularity for semiconvex problems on BD
17:35		Break
17:50	Jan Burczak	Stokes flow with rough drifts
18:10	Alexander Ukhlov	Spectral estimates of nonelastic vibrating membranes
18:30	Tomasz Jakubowski	Critical and subcritical Schrödinger perturbations of fractional Laplacian
18:50	Ismail Ali	Uniqueness and Stability of Nonnegative Solutions for Semipositone Problems
19:15		Dinner

Saturday, May 5

Time	Speaker	Title
8:00		Breakfast
8:50	Sun-Sig Byun	Regularity of solutions to quasilinear parabolic equations
9:50	Michał Miśkiewicz	Higher differentiability of solutions to the inhomogeneous p-Laplace system
10:10	Petr Kaplický	Gradient L^q theory for a class of nondiagonal elliptic systems
10:30	Miłosz Krupski	A framework for non-local, non-linear diffusion
10:50		Coffee & Refreshment
11:10	Petr Pelech	Gradient polyconvexity in the framework of rate-independent processes
11:30	Victor Kovtunenko	On solution of initial boundary value problems in hypoplasticity
11:50	Lyoubomira Softova	Morrey regularity of the solutions to a kind of nonlinear systems with Morrey data
12:10		Closing
12:20		Lunch
13:30		Departure

Abstracts of main talks

Sun-Sig Byun, Regularity of solutions to quasilinear parabolic equations. We consider quasilinear parabolic equations of p-Laplacian type. We discuss sharp weighted norm inequalities for the gradient of solutions.

Ángel Castro, Degraded mixing solutions for the incompressible porous media. I will present the construction of degraded mixing solutions for the incompressible porous media (IPM) system. This system models the dynamics of an incompressible and viscous fluid in a porous media under the gravitational force. When the initial density of the fluid just take two values the existence of solutions for IPM is known as the Muskat problem. In a previous work, together with D. Córdoba and D. Faraco, we showed the existence of solutions in the unstable regime which consist of the mixing of the two densities. In this talk I will sketch a new construction in which we show that the solutions, in average, mix in a linear way. This is a work in collaboration with D. Faraco and F. Mengual Bretón.

Jens Frehse, A regularity criterium for Navier-Stokes-equations based on signed pressures. We present two alternative conditions which establish full regularity for the solutions to Navier-Stokes-equation (incompressible case), which are based on certain sign conditions on the pressure. The method does not rely on the well known theory of regular points or the blow-up-method. It works also for p-fluids in the sense that one gets an $L^q(L^q)$ improvement for the solution.

Jan Kristensen, Regularity and uniqueness results for minimizers. It is well-known that minimizers of variational integrals, even under favourable conditions, might not be fully regular nor unique in the multi-dimensional vectorial case. However under suitable smallness conditions it is possible to regain both. In this talk we discuss such results under natural conditions on the integrand. These conditions are flexible and cover in particular the case of nonconvex variational integrals that are merely coercive on the space of maps of bounded variation. Parts of the talk is based on joint work with Judith Campos Cordero (Mexico City) and Franz Gmeineder (Bonn).

Giuseppe Mingione, A soup of Lipschitz estimates with uniformly and non-uniformly elliptic ingredients. Lipschitz estimates are very often a turning point in regularity theory. For instance, when dealing with uniformly elliptic equations, they are a necessary step towards establishing gradient continuity of solutions and, eventually, to use boot-strap methods. In the case of non-uniformly elliptic problems, this is often the real focal point of regularity, as, after knowing that the gradient is bounded, uniformly elliptic theorems apply to several non-uniformly elliptic model cases. I will present a list of recent approaches to Lipschitz (and sometimes higher) continuity, for both uniformly and non-uniformly elliptic problems. The results are from joint work with Paolo Baroni (Parma), Lisa Beck (Augburg), Maria Colombo (Lausanne), Cristiana De Filippis (Oxford) and Tuomo Kuusi (Oulu).

Ireneo Peral, Principal eigenvalue of mixed problem for the fractional Laplacian: Moving the boundary conditions. Applications. Let Ω be a bounded domain in $\mathbb{R}^N$ and $D, N \in \mathbb{R}^N \setminus \Omega$ open sets such that $D \cap N = \emptyset$, $\overline{D} \cup \overline{N} = \mathbb{R}^N \setminus \Omega$. We analyze the behavior of the eigenvalues of the following non local mixed problem

$$\begin{cases} (-\Delta)^s u = \lambda_1(D) u & \text{in } \Omega, \\ u = 0 & \text{in } D, \\ \mathcal{N}_s u = 0 & \text{in } N, \end{cases}$$

where

$$\mathcal{N}_s u = a_{N,s} \int_\Omega \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy, \qquad x \in \mathbb{R}^N \setminus \overline{\Omega}$$

is a natural Neumann boundary condition.

Our first goal is to construct different sequences of problems by modifying the configuration of the sets D and N, and to provide sufficient and necessary conditions on the size and the location of these sets in order to obtain sequences of eigenvalues that in the limit recover the eigenvalues of the Dirichlet or Neumann problem. We will see that the non locality plays a crucial role here, since the sets D and N can have infinite measure, a phenomenon that does not appear in the local case. The second aim of the talk is to study the mixed Dirichlet-Neumann boundary problem

$$P_\lambda \equiv \begin{cases} (-\Delta)^s u = \lambda \dfrac{u}{|x|^{2s}} + u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \mathcal{B}_s u := u\chi_D + \mathcal{N}_s u \chi_N = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \end{cases}$$

where $N > 2s$, $l > 0$ and $0 < p \leq 2^* - 1$, $2^* = \frac{2N}{N-2s}$. We emphasize that the nonlinear term can be critical.

Abstracts of invited short talks

Paolo Baroni, On the Cauchy-Dirichlet problem for a general class of parabolic equations. We consider a class of parabolic equations extending the evolutionary p-Laplacian in a natural way and we present some regularity results for solutions to the related Cauchy-Dirichlet problem. We shall also compare the local behaviour of solutions with the typical one of solutions to parabolic p-Laplace-type equations, both of singular and degenerate type, stressing differences and describing some open questions. The talk is based on a joint work with C. Lindfors (Aalto University, Helsinki).

Barbora Benešová, Non-interpenetration in thin-film models. Within this talk, we consider the dimension-reduction procedure in the membrane regime in order to obtain thin-film models of elastic solids. One of the crucial requirements on such models is that penetration of matter is prohibited. While in the bulk this translates to injectivity of deformations, in the film this is no longer true because a thin film can be folded. Thus, we introduce suitable conditions on thin-film deformations that characterize non-interpenetration and show these conditions are enforced by thin-film Γ-limits of bulk energies assuring injectivity. This is a joint work with Martin Kružík (Prague).

Vesa Julin, The surface diffusion flow with elasticity. I will discuss about short-time existence of a smooth solution to the surface diffusion equation with an elastic term, without an additional curvature regularization. I will also discuss about the asymptotic stability of strictly stable stationary sets. This is a joint work with Nicola Fusco (Naples) and Massimiliano Morini (Parma).

Filip Rindler, Existence and regularity of solutions to rate-independent systems. Rate-independent systems are time-dependent PDEs, where the time derivative only occurs inside a positively zero-homogeneous dissipational force term. As such, these systems can be thought of as about 'half-way' between elliptic and parabolic systems. In particular, the evolution is quasi-static and jumps may occur in time. If one sees these processes as rescaling limits of parabolic equations, then it becomes clear that the jump evolution may be complicated and in fact needs to be carefully analyzed in order to prove well-posedness and the physically relevant energy dissipation balance. While there are several approaches to the analysis of rate-independent systems, so far questions of regularity, which are intimately tied to the development of a satisfactory solution theory, have not been considered in great detail. In this talk I will present some very recent results on existence, uniqueness, regularity, and approximation for rate-independent systems. This is joint work with S. Schwarzacher, E. Süli, and J. Velazquez.

Abstracts of short talks

Anna Abbatiello, Regularity of solution to flow of non-Newtonian fluids with concentration dependent power-law index. Steady flows of an incompressible homogeneous chemically reacting fluid are described by a coupled system, consisting of the generalized Navier–Stokes equations and convection-diffusion equation with diffusivity dependent on the concentration and the shear rate. The form for the Cauchy stress corresponds to power-law fluid with the exponent depending on the concentration. It makes the problem much more difficult than the standard model for power-law fluid in the analysis of the system of PDEs, since the variable exponent space $W^{1,p(x)}$ is a priori unknown. We prove the existence of a classical solution for the two dimensional space-periodic case and $p(\cdot) \in (\frac{4}{3}, 4)$. This is a joint work with M. Bulíček and P. Kaplický.

[1] A. Abbatiello, M. Bulíček and P. Kaplický, *On the existence of classical solution to the steady flows of generalized Newtonian fluid with concentration dependent power-law index*, preprint.

Ismail Ali, Uniqueness and Stability of Nonnegative Solutions for Semipositone Problems. We study the uniqueness and stability of nonnegative solutions for classes of nonlinear elliptic Dirichlet problems on a ball, when the nonlinearity is monotone, negative at the origin, and either concave or convex.

Michal Bathory, Identification of outflow boundary conditions on artificial boundary. A proper choice of boundary conditions for flows of fluid belongs to one of the most fundamental problem in continuum mechanic as well as in numerical mathematics. Such boundary conditions are often chosen *ad hoc* so that they simply fit to the laminar flow. However, such a setting cannot be justified in general geometries or for higher Reynolds numbers. Our idea is to chose such boundary conditions on artificial boundary so that they lead to the smallest dissipation. This setting naturally leads to the minimization of the energy functional among the class of weak solution with arbitrary boundary conditions on a part of the boundary. We show rigorously that such a minimum is attained and fulfills certain necessary constraints. Based on these constraints we show that for Stokes system, this condition implies certain modification of the "do-nothing" boundary condition. For the Navier-Stokes system, we foreshadow the "correct" explicit boundary condition.

Jan Burczak, Stokes flow with rough drifts. I will present a recent result on a slightly improved local regularity for Stokes system with drifts in Navier-Stokes-critical, endpoint space $L_\infty(BMO^{-1})$, obtained in collaboration with Gregory Seregin.

Franz Gmeineder, Regularity for semiconvex problems on BD. We survey recent regularity results for minima of linear growth functionals that are either strongly symmetric quasiconvex or symmetric-rank-one convex. Being posed on BD, the space of functions of bounded deformation, the issues addressed in this talk complement J. Kristensen's lecture from a symmetric gradient perspective.

Stanislav Hencl, Weak regularity of the inverse under minimal assumptions. Let $\Omega \subset R^3$ be a domain and let $f : \Omega \to R^3$ be a homeomorphism in BV. We show that the inverse is also in BV if the distributional adjoint of f is a Radon measure. This is a joint work with A. Kauranen and R. Luisto.

Sukjung Hwang, Hölder regularity of parabolic quasi-linear degenerate and singular equations. In this talk, we introduce a class of quasi-linear parabolic equations with the principal part in divergence form, that are either degenerate or singular due to the vanishing of a solution or its gradient. In particular, we mainly discuss the local Hölder continuity of weak solutions to equations of p-Laplace type and of porous medium type. The generalized structure in the setting from Orlicz spaces and quantitative methods of proofs are main subjects. Details of different geometric characters of degenerate and singular solutions are discussed. Furthermore, we may discuss Hölder continuity of porous medium equations on the drift vector field.

Tomasz Jakubowski, Critical and subcritical Schrödinger perturbations of fractional Laplacian. Let $p(t, x, y)$ be the fundamental solution of equation

$$\partial_t u(t, x) = (\Delta)^{\alpha/2} p(t, x).$$

I will consider the integral equation

$$\tilde{p}(t, x, y) = p(t, x, y) + \int_0^t \int_{\mathbb{R}^d} p(t-s, x, z) q(s, z) \tilde{p}(s, z, y) dz ds,$$

where $q(s, z)$ is some nonnegative function. The function $\tilde{p}$ solving this equation will be called the Schrödinger perturbations of the function p by q. I will focus on the critical perturbations given by $q(x) = \kappa|x|^{-\alpha}$, where κ is some positive constant. I will present the estimates of the function $\tilde{p}$ and discuss the blow-up phenomena for large values of κ. The talk is based on the joint work with Krzysztof Bogdan, Tomasz Grzywny and Dominika Pilarczyk.

Petr Kaplický, Gradient L^q theory for a class of nondiagonal elliptic systems. We show that a recent result on Hölder continuity of solutions to a class of nondiagonal elliptic systems with p-growth in [M. Bulíček and J. Frehse, C^α-regularity for a class of non-diagonal elliptic systems with p-growth, Calc. Var. Partial Differential Equations 43 (2012), no. 3-4, 441–462] can be used to improve the L^q theory for such systems.

Victor Kovtunenko, On solution of initial boundary value problems in hypoplasticity. A class of hypoplastic models describing granular materials is investigated with respect to proper mathematical formulations which ensure the existence of a solution. The problem is split into two particular formulations which are rigorously proved: a nonlinear boundary value problem written in rates in the case of prescribed stress field, and a semi-discrete nonlinear Cauchy problem in the case of prescribed velocity. Suitable hypoplastic laws are justified in a general form and supported by specific models known from the geotechnical literature. Analytical and numerical examples are provided.

Miłosz Krupski, A framework for non-local, non-linear diffusion. Diffusion is a ubiquitous notion in the theory of PDEs. The most obvious example is the heat equation and it has many derivations, including both non-local and non-linear examples (fractional Laplacian, fractional p-Laplacian, porous medium). We will discuss how to make your own diffusion operator from scratch and why it will have (some of) the properties you would like it to have.

Tomáš Los, On three-dimensional flows of internal pore pressure activated Bingham fluids. We are concerned with a system of partial differential equations describing internal flows of homogeneous incompressible fluids of Bingham type with activated boundary conditions. The Bingham activation threshold depends on internal pore pressure in the material, which is governed by an advection-diffusion equation. This model may be suitable for description of certain class of granular water-saturated materials. By suitably extending recent approaches by Chupin and Maté (European Journal of Mechanics. B. Fluids, 61(part 1):135–143, 2017) and Bulíček and Málek (in Recent Developments of Mathematical Fluid Mechanics. (Eds. H. Amann, Y. Giga, H. Okamoto, H. Kozono, M. Yamazaki) Birkhauser-Verlag, 2014) or also a closely related work by Maringová and Žabenský (Nonlinear Anal. Real World Appl., 41:152–178, 2018), we prove long time and large data existence of weak solutions. This is a joint work with A. Abbatiello, J. Málek, and O. Souček.

Erika Maringová, On the regularity of minimizers of convex variational problems. We study the existence and regularity of a minimizer to a wide class of convex, variational integrals having radial structure. For the linear growth setting, we sharply identify the class of problems which admit globally Lipschitz solution. Further, similar method is applied for superlinear growth. On the other hand, for a general BV minimizer, we study the regularity structure of its absolutely continuous and singular part. The result is not restricted to any geometrical assumption on the domain. The talk is based on works with L. Beck, M. Bulíček, B. Stroffolini and A. Verde.

Václav Mácha, Global BMO estimates for non-Newtonian fluids with perfect slip boundary conditions. We study the generalized stationary Stokes system in a bounded domain in the plane equipped with perfect slip boundary conditions. We show natural stability results in oscillatory spaces, i.e. Hölder spaces and Campanato spaces including the border line spaces of bounded mean oscillations (BMO) and vanishing mean oscillations (VMO). Especially we show that under appropriate assumptions gradients of solutions are globally continues. Since the stress tensor is assumed to be governed by a general Orlicz function, our theory includes various cases of (possibly degenerate) shear thickening and shear thinning fluids; including the model case of power law fluids.

Josef Málek, On several regularity results for PDE problems in fluid mechanics. We recall several important regularity results proven in last decades and present an overview of newly developed techniques that were later extended and generalized to be used in more complex problems.

Michał Miśkiewicz, Higher differentiability of solutions to the inhomogeneous p-Laplace system. We are interested in regularity of $W^{1,p}$-solutions to the problem

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f$$

with $p \geq 3$ and regular enough f. For any s in the range $(\frac{p-1}{2}, \frac{p}{2}]$, we show that the nonlinear expression $|\nabla u|^{s-1}\nabla u$ belongs to $W^{1,2}_{\mathrm{loc}}$. In consequence, the gradient ∇u lies in the fractional Nikol'skii space $\mathcal{N}^{1/s,2s}_{\mathrm{loc}}$. The argument is based on choosing suitable test functions, which can seen as an interpolation between the methods used by G. Mingione and A. Cellina to obtain $\mathcal{N}^{2/p,p}$ (for $p \geq 2$) and $W^{1,2}$ (for $2 \leq p < 3$) estimates for the gradient. To the author's knowledge, the results are new even in the case of p-harmonic functions (i.e. $f \equiv 0$), but the method extends to non-zero f and systems of equations.

Petr Pelech, Gradient polyconvexity in the framework of rate-independent processes. The talk treats mathematical aspects of evolutionary material models for shape-memory alloys at finite-strains. The difficulty of related mathematical analysis consists in the non-linear and non-convex dependence of the energy on the deformation gradient. One possible way, how maintain the analysis tractable, is to suppose that the energy depends also on the second deformation gradient and is convex in it. We relax this assumption by using the recently proposed concept of gradient-polyconvexity. Namely, we consider energies which are convex only in gradients of non-linear minors (i.e. cofactor and determinant in three dimension) of the deformation gradient. As a result, the whole second deformation gradient needs not to be integrable. Yet, at the same time, the obtained compactness is sufficient and, moreover, additional physically desirable properties (e.g. local invertibility) can be shown. We extend the previous result for hyperelastic materials by incorporating a rate-independent dissipation to our model and by proving existence of an energetic solution to it. It is a joint work with Martin Kružík (Institute of Information Theory and Automation of the Czech Academy of Sciences) and Anja Schloemerkemper (University of Wuerzburg).

Sebastian Schwarzacher, Parabolic Lipschitz truncation and applications. I will present some recent existence and regularity results that are built upon the so called Lipschitz truncation method. The Lipschitz truncation method (first introduced by Acerbi and Fusco in the 80s) allows to approximate a Sobolev function by Lipschitz continuous functions, by modifying the function only on a small set; i.e. on some level set of the maximal operator of the gradient. This method has been proven very useful for PDEs. In my talk I will focus on parabolic Lipschitz truncations in the case when the time derivative is only in a negative Sobolev space. The talk is related to results achieved in collaboration with M. Bulíček, J. Burczak, L. Diening, B. Stroffolini and A. Verde.

Grzegorz Serafin, Two-sided estimates for solutions of fractal Burgers equation. We provide two-sided pointwise estimates for solutions to critical fractal Burgers equation. Both source solution and a solution to the equation with a non-negative initial condition in L^1 are considered.

Iwona Skrzypczak, Gradient estimates for problems with Orlicz growth. We study a general nonlinear elliptic equation in the Orlicz setting with data not belonging to the dual of the energy space. We provide several Lorentz-type and Morrey-type estimates for the gradients of solutions under various conditions on the data.

Lyoubomira Softova, Morrey regularity of the solutions to a kind of nonlinear systems with Morrey data. We consider a Dirichlet problem for a kind of non-linear elliptic systems satisfying suitable structural conditions. We firstly obtain essential boundedness of the weak solution that permits to establish Morrey regularity of its gradient. As a consequence, applying an iteration procedure, we obtain precise Morrey regularity for the solutions of a kind of quasilinear elliptic systems with Morrey data.

Alexander Ukhlov, Spectral estimates of nonelastic vibrating membranes. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. The Neumann eigenvalue problem for the two-dimensional degenerate p-Laplace operator ($p > 2$)

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \mu_p |u|^{p-2}u \text{ in } \Omega,\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,$$

arises in study of free vibrations of nonelastic membranes. We suggest spectral estimates of the first non-trivial Neumann eigenvalue $\mu_p(\Omega)$ of the p-Laplace operator in the terms of the (quasi)conformal geometry of domains. The main result is:

Theorem A. *Let $\Omega \subset \mathbb{R}^2$ be a K-quasidisc. Then*

$$\mu_p(\Omega) \geq \frac{M_p(K)}{|\Omega|^{\frac{p}{2}}} = \frac{M_p^*(K)}{R_*^p},$$

where R_* is a radius of a disc Ω^* of the same area as Ω and $M_p^*(K) = M_p(K)\pi^{-p/2}$ depends on p and a quasiconformality coefficient K only.

(Joint works with Vladimir Gol'dshtein and Valerii Pchelintsev)

Yeonghun Youn, Partial regularity via potentials. I will present results about potential estimates for p-Laplace type systems without Uhlenbeck structure condition. In contrary to lack of Uhlenbeck structure, I consider operators those are asymptotically close to p-Laplace operator at the origin. Partial regularities for gradient of solutions can be understood via modified Riesz potential of the right hand side data. The crucial methods that I use are higher integrability and harmonic approximation lemmas. This is a joint work with S-S. Byun.