Analysis of a model of elastic plastic mixtures ("Prandtl-Reuss-mixtures")

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Outline

We refer to a paper

J. Kratochvíl, J. Málek, K.R. Rajagopal and A.R. Srinivasa: *Modelling of the response of elastic plastic materials treated as mixture of hard and soft regions*, ZAMP 55 (2004), 500-518

and to

Phd-Thesis of Luba Khasina

adviced by Kratochvíl-Málek-Frehse, where first steps are done to formulate the model of KMRS in the framework of Sobolev spaces nad variational inequalities.

KMRS consider a body Ω consisting of soft and hard material and loading.
During the loading process, soft material is assumed to be perfectly elastic plastic, the hard material is assumed to satisfy a certain hardening rule in the non elastic region.
The beautiness of the theory consists in the fact, that no artificial internal variables enter, the history of the plastic deformation of the soft material replaces the internal variable.

Purpose of the present talk

Presenting a mathematical formulation in the framework of quasi-variational inequalities and Sobolev spaces.
The theory turns out to be very similar to the Prandtl-Reuss-law for single materials; all known regularity results hold also for the mixture model.

KMRS-model after simplification and choice of example for illustration

Ω	basic domain $\subset \mathbb{R}^3$
$[0, T]$	loading interval
$\boldsymbol{\sigma}_s = \boldsymbol{\sigma}_s(x,t), x \in \Omega, t \in [0, T]$	stress of the soft material
$\boldsymbol{\sigma}_h$	stress of the hard material
$\mathbf{e} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}})$	strain
$\mathbf{e}_{ps}$	plastic strain of the soft material
$\mathbf{e}_{hs}$	plastic strain of the hard material
$\alpha = \alpha(x,t)$	volume fraction of the soft material
$1 - \alpha$	volume fraction of the hard material

$0 \leq \alpha \leq 1$, (later $0 < \varepsilon_0 < \alpha < 1 - \varepsilon_0$)
$\alpha(x, 0) = 0$, (resp. ε_0)
$\alpha(x,t) = \alpha(x,t;\mathbf{e}_{ps})$, α monotone increasing in t

Governing equations

Balance of forces

$$-\mathrm{div}(\alpha\boldsymbol{\sigma}_s + (1-\alpha)\boldsymbol{\sigma}_h) = \mathbf{f} \text{ in } \Omega$$

Weak formulation with boundary values

$$(\alpha\boldsymbol{\sigma}_s + (1-\alpha)\boldsymbol{\sigma}_h, \nabla\varphi)_{L^2(\Omega)} = (\mathbf{f}, \varphi)_{L^2(\Omega)} + \int_{\partial\Omega} p_0 \varphi d\mathbf{o}, \ \forall \varphi \in H^{1,2}_{\Gamma}(\Omega; \mathbb{R}^3)$$

Symmetry

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}$$

Hooke's law in the elastic region

$$\boldsymbol{\sigma}_s = \mathbf{C}(\mathbf{e} - \mathbf{e}_{ps}), \quad \boldsymbol{\sigma}_h = \mathbf{C}(\mathbf{e} - \mathbf{e}_{ph}),$$

$\mathbf{C}$ elasticity tensor, of course different $\mathbf{C}_s$ and $\mathbf{C}_h$ can be used.

Plastic incompressibility

$$\text{trace } \mathbf{e}_{ps} = \text{trace } \mathbf{e}_{ph} = 0.$$

Yield function and yield condition

Let

$$\mathbf{B}_D = \mathbf{B} - \frac{1}{3}(\text{trace } \mathbf{B})\mathbf{I} \quad \text{“Deviator”}$$

$$F_s(\boldsymbol{\sigma}_s) = |\boldsymbol{\sigma}_s| - \kappa_s$$

$$F_h(\boldsymbol{\sigma}_h) = |\boldsymbol{\sigma}_h| - \kappa_h(t), \quad \kappa_h(t) = \kappa_h(t, \mathbf{e}_{ps})$$

These are v Mises type yield functions.
Yield conditions

$$F_s(\boldsymbol{\sigma}_s) \leq 0, \quad F_h(\boldsymbol{\sigma}_h) \leq 0$$

Kuhn-Tucker-conditions

$$\dot{\mathbf{e}}_s = \lambda_s \frac{\boldsymbol{\sigma}_{sD}}{|\boldsymbol{\sigma}_{sD}|}, \quad \dot{\mathbf{e}}_h = \lambda_h \frac{\boldsymbol{\sigma}_{hD}}{|\boldsymbol{\sigma}_{hD}|},$$

$$\lambda_s, \lambda_h \geq 0, \quad \lambda_s F_s(\boldsymbol{\sigma}_s) = 0, \quad \lambda_h F_h(\boldsymbol{\sigma}_h) = 0.$$

Model for the volume fraction α

$$\ell = \int_0^t |\dot{\mathbf{e}}_{ps}|$$
$$\alpha(\cdot, t) = \alpha_0 + (1 - \alpha_0) e^{-c_0 \ell}$$

Model for the hardening rule and yield parameters

$$\kappa_s = const > 0, \quad \kappa_h = \kappa_s + r_0 \ell.$$

Later we will work with Lipschitz α and κ_h.

Prandtl-Reuss law

Formulation as variational inequality
Find $\boldsymbol{\sigma} \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^{3\times 3}))$ such that $\dot{\boldsymbol{\sigma}} \in L^\infty(L^2)$ and

$$\mathbb{K}\begin{cases} \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \\ \boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}} \\ (\boldsymbol{\sigma}, \nabla\varphi)_{L^2(\Omega)} = (\mathbf{f}, \varphi)_{L^2(\Omega)} + \int_{\partial\Omega} p_0 \varphi d\mathbf{o}, \ \forall \varphi \in H^{1,2}_{\Gamma}(\Omega;\mathbb{R}^3) \\ F(\boldsymbol{\sigma}) = |\boldsymbol{\sigma}_D| - \kappa \\ F(\boldsymbol{\sigma}) \le 0 \end{cases}$$

$$(\mathbf{A}\dot{\boldsymbol{\sigma}}, \boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}})_{L^2(\Omega)} \le 0 \quad \forall \tilde{\boldsymbol{\sigma}} \text{ satisfying } \mathbb{K}$$

Theorem

$\exists$ unique solution
Regularity results: $\dot{\boldsymbol{\sigma}} \in L^\infty(L^{2+\delta})$, $\boldsymbol{\sigma} \in L^\infty(H^{1,2}_{loc})$ (1992),
$\dot{\boldsymbol{\sigma}} \in H^{1/2,2}(L^2)$ (2011)

Strain and plastic strain can be derived via penalty approximation.

Prandtl-Reuss law. Penalty approximation.

(Penalty)

$$(\mathbf{A}\dot{\boldsymbol{\sigma}}, \boldsymbol{\tau}) + (\mu^{-1} \max[|\boldsymbol{\sigma}_D| - \kappa]_+ \frac{\boldsymbol{\sigma}_D}{|\boldsymbol{\sigma}_D|}, \boldsymbol{\tau}) = 0 \quad \mu \to 0_+$$

$$\forall \boldsymbol{\tau} \text{ such that } \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathrm{T}}$$

$$(\boldsymbol{\tau}, \nabla \boldsymbol{\varphi}) = 0$$

Initial condition, symmetry, balance of force remain as before, yield condition is replaced by the penalty term.

How to prove existence of solutions to the Prandtl-Reuss law. Estimates

Step 0 Solve the "ODE" in an abstract Hilbert space setting or use Rothe approximation $\Rightarrow$ (Penalty) is solvable

Step 1 Assume safe load condition, i.e. existence of a "good" $\hat{\boldsymbol{\sigma}}$ satisfying $\mathbb{K}$ $\Rightarrow$ $L^\infty(L^2)$ estimate for $\boldsymbol{\sigma}$ and $L^1(L^1)$ estimate for the penalty term.

Step 2 Test by $\dot{\boldsymbol{\sigma}} - \dot{\hat{\boldsymbol{\sigma}}}$ $\Rightarrow$ $L^2(L^2)$ estimate for $\dot{\boldsymbol{\sigma}}$

Step 3 Test by $\ddot{\boldsymbol{\sigma}}$ $\Rightarrow$ $L^\infty(L^2)$ estimate for $\dot{\boldsymbol{\sigma}}$

Step 4 Go to Step 1 once more $\Rightarrow$ $L^\infty(L^1)$ estimate for penalty term (all estimates uniformly for $\mu \to 0$)

Converge and existence for Prandtl-Reuss

Step 5 Use symmetric Helmholtz decomposition $\Rightarrow \exists \mathbf{v} \in H^{1,2}_{\Gamma}(\Omega;\mathbb{R}^3)$ such that $\frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^{\mathrm{T}}) = \mathbf{A}\dot{\boldsymbol{\sigma}} +$ Penalty term

$$\text{with } \|\nabla \mathbf{v} + (\nabla \mathbf{v})^{\mathrm{T}}\|_{L^\infty(L^1)} \le K \text{ uniformly as } \mu \to 0$$

$\Rightarrow$ The strain velocities are only measures for $\mu = 0$

$\frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^{\mathrm{T}}) \in C^*$, similar, for the penalty term

$$\mu^{-1}\max[|\boldsymbol{\sigma}_D| - \kappa]_+ \frac{\boldsymbol{\sigma}_D}{|\boldsymbol{\sigma}_D|} \rightharpoonup \dot{\mathbf{\Lambda}} \in C^*(0,T;\mathbb{R}^{3\times 3})$$

$$\frac{\mu^{-1}\max[|\boldsymbol{\sigma}_D| - \kappa]_+}{|\boldsymbol{\sigma}_D|} \rightharpoonup \lambda$$

$\dot{\mathbf{\Lambda}}$ is the plastic strain velocity

If one had $\boldsymbol{\sigma}_\mu \to \boldsymbol{\sigma}$ uniformly $\Rightarrow$ representation as multiplier

$$\dot{\mathbf{\Lambda}} = \lambda\boldsymbol{\sigma}, \quad \lambda \ge 0, \quad \lambda F(\boldsymbol{\sigma}) = 0.$$

Theorem

$\boldsymbol{\sigma}_\mu \to \boldsymbol{\sigma}, \boldsymbol{\sigma}$ solution of the Prandtl-Reuss variational inequality and

$$\frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^{\mathrm{T}}) = \mathbf{A}\dot{\boldsymbol{\sigma}} + \dot{\boldsymbol{\Lambda}}.$$

A similar procedure can be done for the mixture MDE model of KMRS.

Constitutive elastic law

Let

$$\boldsymbol{\tau} = \begin{pmatrix} \boldsymbol{\tau}_s \\ \boldsymbol{\tau}_h \end{pmatrix}, \quad \hat{\boldsymbol{\tau}} = \begin{pmatrix} \hat{\boldsymbol{\tau}}_s \\ \hat{\boldsymbol{\tau}}_h \end{pmatrix}$$

and

$$\mathbf{Q}(\hat{\boldsymbol{\tau}}, \boldsymbol{\tau}) = \int_\Omega [\mathbf{A}_s \hat{\boldsymbol{\tau}}_s : \boldsymbol{\tau}_s + \mathbf{A}_h \hat{\boldsymbol{\tau}}_h : \boldsymbol{\tau}_h] \, dx$$

Here $\mathbf{A}_s$ and $\mathbf{A}_h$ are inverse elasticity tensors, say Lame-Navier structure. They model the elastic interaction between the soft and hard material. (It is possible to treat additional interaction terms $\mathbf{A}_{sh} \hat{\boldsymbol{\tau}}_h : \boldsymbol{\tau}_s$)

Penalty approximation for Prandtl-Reuss mixtures

Find $\boldsymbol{\sigma}_s, \boldsymbol{\sigma}_h \in L^\infty(0, T; L^2(\Omega; \mathbb{R}^{3\times 3}))$ such that $\dot{\boldsymbol{\sigma}}_s, \dot{\boldsymbol{\sigma}}_h \in L^\infty(L^2)$ and

$$\boldsymbol{\sigma}_s(0) = \boldsymbol{\sigma}_{s0}, \boldsymbol{\sigma}_h(0) = \boldsymbol{\sigma}_{h0}$$
$$\boldsymbol{\sigma}_s = \boldsymbol{\sigma}_s^{\mathrm{T}}, \boldsymbol{\sigma}_h = \boldsymbol{\sigma}_h^{\mathrm{T}}$$
$$(\alpha\boldsymbol{\sigma}_s + (1-\alpha)\boldsymbol{\sigma}_h, \nabla\varphi)_{L^2(\Omega)} = (\mathbf{f}, \varphi)_{L^2(\Omega)} + \int_{\partial\Omega} p_0 \varphi d\mathbf{o}, \ \forall \varphi \in H^{1,2}_\Gamma(\Omega; \mathbb{R}^3)$$

(Penmix)

$$\mathbf{Q}\begin{pmatrix} \alpha\dot{\boldsymbol{\sigma}}_s, \alpha\boldsymbol{\tau}_s \\ (1-\alpha)\dot{\boldsymbol{\sigma}}_h, (1-\alpha)\boldsymbol{\tau}_h \end{pmatrix} + \left(\alpha\mu^{-1}[|\boldsymbol{\sigma}_{sD}| - \kappa_s]_+ \frac{\boldsymbol{\sigma}_{sD}}{|\boldsymbol{\sigma}_{sD}|}, \boldsymbol{\tau}_s\right) + \left((1-\alpha)\mu^{-1}[|\boldsymbol{\sigma}_{hD}| - \kappa_h]_+ \frac{\boldsymbol{\sigma}_{hD}}{|\boldsymbol{\sigma}_{hD}|}, \boldsymbol{\tau}_h\right) = 0$$

for all $\boldsymbol{\tau}_s = \boldsymbol{\tau}_s^{\mathrm{T}}, \boldsymbol{\tau}_h = \boldsymbol{\tau}_h^{\mathrm{T}}$ such that $\operatorname{div}(\alpha\boldsymbol{\tau}_s + (1-\alpha)\boldsymbol{\tau}_h) = 0$.

Consequence of the penalty equation

Choose $\boldsymbol{\tau}_h = \mathbf{0}, \operatorname{div}(\alpha\boldsymbol{\tau}_s) = 0, \alpha\boldsymbol{\tau}_s = \boldsymbol{\tau}_0 \Rightarrow$

$$\int_\Omega \left(\alpha \mathbf{A}_s \dot{\boldsymbol{\sigma}}_s + \mu^{-1}[|\boldsymbol{\sigma}_{sD}| - \kappa_s]_+ \frac{\boldsymbol{\sigma}_{sD}}{|\boldsymbol{\sigma}_{sD}|} \right) : \boldsymbol{\tau}_0 dx = 0$$

By symmetric Helmholtz decomposition

$$\frac{1}{2}\left((\nabla \dot{\mathbf{u}}_s) + (\nabla \dot{\mathbf{u}}_s)^{\mathrm{T}}\right) = \alpha \mathbf{A}_s \dot{\boldsymbol{\sigma}}_s + \underbrace{\mu^{-1}[|\boldsymbol{\sigma}_{sD}| - \kappa_s]_+ \frac{\boldsymbol{\sigma}_{sD}}{|\boldsymbol{\sigma}_{sD}|}}_{\substack{\text{approximate plastic strain} \\ \text{velocity of the soft material}}}$$

Similarly

$$\frac{1}{2}\left((\nabla \dot{\mathbf{u}}_h) + (\nabla \dot{\mathbf{u}}_h)^{\mathrm{T}}\right) = (1-\alpha) \mathbf{A}_h \dot{\boldsymbol{\sigma}}_h + \mu^{-1}[|\boldsymbol{\sigma}_{hD}| - \kappa_h]_+ \frac{\boldsymbol{\sigma}_{hD}}{|\boldsymbol{\sigma}_{hD}|}$$

Theorem

Assume α, κ_h Lipschitz, $\mathbf{Q}$ positively definite and smooth data. Assume a safe load condition with smooth stresses $\hat{\boldsymbol{\sigma}}_s$ and $\hat{\boldsymbol{\sigma}}_h$. Let $0 \leq \varepsilon_0 \leq \alpha \leq 1 - \varepsilon_0$. Then

1. The solutions $\boldsymbol{\sigma}_s^n, \boldsymbol{\sigma}_h^n$ are uniformly bounded in the norms $L^\infty(L^2), L^\infty(L^{2+\delta}), H^{1,\infty}(L^2)$. Furthermore $\dot{\boldsymbol{\sigma}} \in H^{1/2}(L^2)$.
2. The partial strain velocities and the partial plastic strain velocities are bounded in $L^\infty(L^1)$.
3. The approximate stresses converge weakly $\boldsymbol{\sigma}_s^n \rightharpoonup \boldsymbol{\sigma}_s, \boldsymbol{\sigma}_h^n \rightharpoonup \boldsymbol{\sigma}_h$ $(\mu \to 0_+)$ in $H^{1,2}(L^2) \cap L^2(H^{1,2}_{loc})$.

Theorem (continuation)

The limit satisfies the variational inequality

$$\mathbf{Q}\left(\begin{pmatrix}\alpha\dot{\boldsymbol{\sigma}}_s\\(1-\alpha)\dot{\boldsymbol{\sigma}}_h\end{pmatrix},\begin{pmatrix}\alpha(\boldsymbol{\sigma}_s-\boldsymbol{\tau}_s)\\(1-\alpha)(\boldsymbol{\sigma}_h-\boldsymbol{\tau}_h)\end{pmatrix}\right)\leq 0 \qquad \text{(V)}$$

with respect to the yield conditions and the balance of forces defining the convex set.

Since in the application $\alpha=\alpha(t,x;\boldsymbol{\sigma}_s,\boldsymbol{\sigma}_h)$ and similar κ_h, inequality (V) is a quasi-variational inequality.

Safe load condition for mixtures

$\exists \hat{\boldsymbol{\sigma}}_s, \hat{\boldsymbol{\sigma}}_h$ such that

$$\hat{\boldsymbol{\sigma}}_s = \hat{\boldsymbol{\sigma}}_s^{\mathrm{T}}, \hat{\boldsymbol{\sigma}}_h = \hat{\boldsymbol{\sigma}}_h^{\mathrm{T}}$$

$$(\alpha \hat{\boldsymbol{\sigma}}_s + (1-\alpha)\hat{\boldsymbol{\sigma}}_h, \nabla \varphi)_{L^2(\Omega)} = (\mathbf{f}, \varphi)_{L^2(\Omega)} + \int_{\partial\Omega} p_0 \varphi \, d\mathbf{o}, \ \forall \varphi \in H_\Gamma^{1,2}(\Omega; \mathbb{R}^3)$$

$$F_s(\boldsymbol{\sigma}_s) \leq -\varepsilon_0 < 0, \quad F_h(\boldsymbol{\sigma}_h) \leq -\varepsilon_0 < 0$$

Smoothness: $\hat{\boldsymbol{\sigma}}, \ddot{\hat{\boldsymbol{\sigma}}}, \nabla \dot{\hat{\boldsymbol{\sigma}}} \in L^\infty$ is more than necessary for the regularity theory.

$L^1(L^1)$ estimate using safe load

Test the penalty equation with

$$\boldsymbol{\tau}_s = \boldsymbol{\sigma}_s - \hat{\boldsymbol{\sigma}}_s, \quad \boldsymbol{\tau}_h = \boldsymbol{\sigma}_h - \hat{\boldsymbol{\sigma}}_s$$

This admissible since

$$\operatorname{div}(\alpha(\boldsymbol{\sigma}_s - \hat{\boldsymbol{\sigma}}_s) + (1-\alpha)(\boldsymbol{\sigma}_h - \hat{\boldsymbol{\sigma}}_s)) = 0$$

Hence

$$\mathbf{Q}\left(\begin{pmatrix} \alpha\dot{\boldsymbol{\sigma}}_s \\ (1-\alpha)\dot{\boldsymbol{\sigma}}_h \end{pmatrix}, \begin{pmatrix} \alpha(\boldsymbol{\sigma}_s - \hat{\boldsymbol{\sigma}}_s) \\ (1-\alpha)(\boldsymbol{\sigma}_h - \hat{\boldsymbol{\sigma}}_h) \end{pmatrix}\right) + \left(\alpha\mu^{-1}[|\boldsymbol{\sigma}_{sD}| - \kappa_s]_+ \frac{\boldsymbol{\sigma}_{sD}}{|\boldsymbol{\sigma}_{sD}|}\right) + \text{penalty for } \boldsymbol{\sigma}_{hD} = 0$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{Q}(\cdots) + \text{pollution} +$$

$$\left(\alpha\mu^{-1}[|\boldsymbol{\sigma}_{sD}|-\kappa_s]_+\left(\underbrace{|\boldsymbol{\sigma}_{sD}|-\frac{\boldsymbol{\sigma}_{sD}}{|\boldsymbol{\sigma}_{sD}|}\hat{\boldsymbol{\sigma}}_{sD}}_{\geq -\kappa_s+\varepsilon_0}\right)\right)+\text{penalty for } \boldsymbol{\sigma}_{hD} = 0$$

Integration with respect to t

$$\frac{1}{2}\mathbf{Q}\begin{pmatrix} \alpha\boldsymbol{\sigma}_s, & \alpha\boldsymbol{\sigma}_s \\ (1-\alpha)\boldsymbol{\sigma}_h, & (1-\alpha)\boldsymbol{\sigma}_h \end{pmatrix}\Bigg|_T - \int_0^T K|\boldsymbol{\sigma}|^2 dxdt +$$

$$\varepsilon_0 \int\int \alpha\mu^{-1}[|\boldsymbol{\sigma}_{sD}|-\kappa_s]_+ dxdt \leq K(\hat{\boldsymbol{\sigma}})$$

Gronwall $\Rightarrow L^\infty(L^2)$ estimate for $\boldsymbol{\sigma}$, L^1 estimate for $\mu^{-1}[|\boldsymbol{\sigma}_{sD}|-\kappa_s]_+$
Inspection return $\Rightarrow \mu^{-1}[|\boldsymbol{\sigma}_{sD}|-\kappa_s]_+|\boldsymbol{\sigma}_{sD}| \in L^1(L^1)$
Similar for $\boldsymbol{\sigma}_h$.

Refined regularity results for $(\mathbf{u}, \boldsymbol{\sigma})$ (solution to Prandtl Reuss law) or $(\mathbf{u}_s, \mathbf{u}_h, \boldsymbol{\sigma}_s, \boldsymbol{\sigma}_h)$ (solution to the Prandtl-Reuss mixture).

Theorem

Assumption: reasonably smooth data, safe load condition $\Rightarrow$

$$\sup_t \int_{B_R} |\nabla \mathbf{u} + \nabla \mathbf{u}^T| \, dx \leq K R^{\alpha} \quad \text{(uniformly for } \mu \to 0\text{)}$$

as well as for $\mathbf{u}_s$ and $\mathbf{u}_h$.

- Technique of proof: Cf. L^1-estimates before & hole filling procedure for the penalty term.
- Consequence the measure $\lim_{\mu \to 0} \nabla \mathbf{u} + \nabla \mathbf{u}^T$ cannot concentrate on sets of "low" Hausdorff-dimension (e.g. no pointwise concentration).

Theorem

Assumption as before, $\Omega_0 \Subset \Omega$,

$$\sup_t \int_{\Omega_0} |\nabla \boldsymbol{\sigma}|^2 \, dx \leq K.$$

- Technique of proof: Difference quotient technique. The pollution terms due to $\mathbf{u}$ give trouble since $\nabla \mathbf{u} + \nabla \mathbf{u}^T$ is estimated only in L^1.

Boundary differentiability:

Let D_τ be the tangential derivative at the boundary.

- Malek, JF $\Rightarrow D_\tau \boldsymbol{\sigma} \in L^2$ up to the boundary for $n = 2$, Ω circle or cirlce$\setminus$smaller circles.
- Bulicek, Malek, JF $\Rightarrow D_\tau \boldsymbol{\sigma} \in L^2$ for $n \geq 3$, Ω circle or cirlce$\setminus$smaller concentric circles.
- Seregin: Normal derivatives to certain approximations are unbounded, $\Omega =$circle.

These results are published for Hencky's law but hold for Prandtl-Reuss-problem and Prandtl-Reuss-mixtures as well.

Fractional differentiability for $\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{\sigma}}_h, \dot{\boldsymbol{\sigma}}_s$.

Theorem

$\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{\sigma}}_h, \dot{\boldsymbol{\sigma}}_s$ have fractional derivatives in the sense of Besov spaces in time direction of order $\frac{1}{2} - \delta$, in space direction (interior regularity) of order $\frac{1}{6} - \delta$. This is claimed in the limit $\mu = 0$.

- Proof: Use the function $h^{-1}(\dot{\boldsymbol{\sigma}}(t+s) - \dot{\boldsymbol{\sigma}}(t))$ as a testfunction and perform $h^{-1} \int ds$. The spatial derivatives are obtained via cross interpolation.

Dziekuje za uwage.

Thank you for your attention.