June 18, 2008

Plan

In $\Omega = \mathbb{R}^2_+ = \{x > 0\}$ consider the linear IBVP

$$\begin{cases} u_t + u_x + v_y = 0 \\ v_t + u_y = 0 \\ u_{|x=0} = 0 \\ (u, v)_{|t=0} = (u_0, v_0), \end{cases} \tag{1}$$

In matrix form

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_y \begin{pmatrix} u \\ v \end{pmatrix} = 0.$$

- Symmetric hyperbolic system
- The boundary is characteristic
- The boundary condition is maximally nonnegative

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We look for a priori estimates of the solution.
Assume that

$$(u_0, v_0) \in H^1(\Omega) \quad \text{with} \quad ||(u_0, v_0)||_{H^1(\Omega)} \leq K.$$

(I) We multiply the first equation by u, the second one by v, integrate over $(0,t) \times \Omega$ and obtain ($|| \cdot ||$ stands for $|| \cdot ||_{L^2(\Omega)}$)

$$||u(t,)||^2 + ||v(t,)||^2 = ||u_0||^2 + ||v_0||^2 \qquad \forall t > 0.$$

It follows that

$$||u(t,)|| + ||v(t,)|| \leq C(K) \qquad \forall t > 0.$$

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(II) Consider the tangential derivatives (u_y, v_y). By taking the $y-$derivative of the problem we see that (u_y, v_y) solves the same problem as (u, v) with initial data (u_{0y}, v_{0y}). In particular it satisfies the same boundary condition as (u, v). It follows that

$$||u_y(t,)||^2 + ||v_y(t,)||^2 = ||u_{0y}||^2 + ||v_{0y}||^2 \qquad \forall t > 0.$$

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$$||u_y(t,)|| + ||v_y(t,)|| \leq C(K) \qquad \forall t > 0.$$

(III) By taking the $t-$derivative of the equations we see that (u_t, v_t) is also a solution. This yields

$$||u_t(t,)||^2 + ||v_t(t,)||^2 = ||u_t(0,)||^2 + ||v_t(0,)||^2 = ||u_{0x} + v_{0y}||^2 + ||u_{0y}||^2,$$

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(IV) Normal derivative u_x.
From

$$u_x = -u_t - v_y$$

$$||u_x(t,)|| \leq ||u_t(t,)|| + ||v_y(t,)|| \leq C(K) \qquad \forall t > 0.$$

Let P be the orthogonal projection onto $ker\, A_\nu(x,t)^\perp$. Then

$$P \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix}$$

(noncharacteristic component of $(u,v)^T$).

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(V) Normal derivative v_x.

$$(I-P)\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}0\\v\end{pmatrix}$$

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Take the $x-$derivative of the second equation in (1)

$$v_{tx}+u_{xy}=0. \tag{2}$$

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Multiply (2) by v_x and integrate over Ω. Then ($\int = \int_\Omega dxdy$)

$$\frac{1}{2}\frac{d}{dt}||v_x||^2 = -\int u_{xy}v_x = \int (u_{ty} + v_{yy})v_x$$

$$= \frac{d}{dt}\int u_y v_x - \int u_y v_{tx} + \int v_{yy} v_x$$

$$= \frac{d}{dt}\int u_y v_x + \int u_y u_{xy} - \int v_y v_{xy}$$

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$$= \frac{d}{dt}\int u_y v_x - \underbrace{\frac{1}{2}\int_{|x=0} u_y^2(t,0,y)dy}_{=0} + \frac{1}{2}\int_{|x=0} v_y^2(t,0,y)dy.$$

From $v_{ty} = -u_{yy}$, we have

$$\frac{1}{2}\frac{d}{dt}\int_{|x=0} v_y^2 dy = \int_{|x=0} v_y v_{ty} dy = -\int_{|x=0} v_y u_{yy} dy = 0$$

Then

$$\int_{|x=0} v_y^2(t,0,y)dy = \int_{|x=0} v_{0y}^2(y)dy = \text{ constant in time.}$$

We then obtain

$$\frac{d}{dt}||v_x||^2 = 2\frac{d}{dt}\int u_y v_x + \int_{|x=0} v_{0y}^2(y)dy$$

Integrate in time between 0 and $t > 0$

$$||v_x(t,)||^2 = ||v_{0x}||^2 + 2\int u_y v_x - 2\int u_{0y} v_{0x} + t\int_{|x=0} v_{0y}^2(y)dy.$$

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By the Young's inequality we finally obtain

$$\frac{1}{2}\, t \int_{|x=0} v_{0y}^2(y)dy - C_1(K) \quad \leq ||v_x(t,)||^2 \leq$$

$$\leq 2\, t \int_{|x=0} v_{0y}^2(y)dy + C_2(K), \quad t > 0.$$

This shows that

$$v_x(t,) \in L^2(\Omega) \text{ for } t > 0 \text{ if and only if } v_0 \in H^1(\partial\Omega).$$

By the trace theorem, $v_0 \in H^1(\Omega)$ only gives $v_{0|\partial\Omega} \in H^{1/2}(\partial\Omega)$. Therefore

$$\boxed{(u_0, v_0) \in H^1(\Omega) \not\Rightarrow (u(t,), v(t,)) \in H^1(\Omega) \text{ for } t > 0.}$$

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PROBLEM:
Which space X for the persistence of regularity

$$(u_0, v_0) \in X \Rightarrow (u(t,), v(t,)) \in X, \quad \forall t > 0\,?$$

We assume

$$(u_0, v_0) \in H^2(\Omega) \quad \text{with} \quad ||(u_0, v_0)||_{H^2(\Omega)} \leq K_2.$$

After the above analysis, we don't expect to obtain $(u(t,), v(t,)) \in H^2(\Omega)$.

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Calculations as above give

$$\partial_t^h \partial_y^k u(t,), \, \partial_t^h \partial_y^k v(t,) \in L^2(\Omega), \quad t > 0, \, h + k \leq 2,$$

with norms bounded by $C(K_2)$. By the t and y differentiation of the first equation in (1) we readily obtain

$$u_{tx} = -u_{tt} - v_{ty} \in L^2(\Omega), \quad ||u_{tx}(t,)|| \leq C(K_2), \quad t > 0,$$

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$v_0 \in H^2(\Omega)$ yields $v_{0|\partial\Omega} \in H^1(\partial\Omega)$, so that by the above analysis

$$v_x(t,) \in L^2(\Omega), \quad ||v_x(t,)|| \leq C(K_2), \quad 0 < t < T,$$

for any $T < +\infty$.

We look for an estimate of the mixed derivative v_{xy}. Here we start from

$$u_{tyy} + u_{xyy} + v_{yyy} = 0,$$

$$v_{txy} + u_{xyy} = 0.$$

Multiply the second equation by v_{xy} and integrate over Ω. Then

$$\frac{1}{2}\frac{d}{dt}||v_{xy}||^2 = -\int u_{xyy}v_{xy} = \int (u_{tyy} + v_{yyy})v_{xy}$$

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Since $v_{tyy} = -u_{yyy}$, we have

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again by the boundary condition on u. It follows that

$$\int_{|x=0} v_{yy}^2(t,0,y)dy = \int_{|x=0} v_{0yy}^2(y)dy = \text{ constant in time.}$$

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Integrating in time between 0 and $t > 0$ yields

$$\frac{1}{2}\, t \int_{|x=0} v_{0yy}^2(y)dy - C_1(K_2) \quad \leq ||v_{xy}(t,)||^2 \leq$$

$$\leq 2\, t \int_{|x=0} v_{0yy}^2(y)dy + C_2(K_2), \quad t > 0.$$

It follows that,
if $v_0 \in H^2(\Omega)$, but $v_{0|\partial\Omega} \notin H^2(\partial\Omega)$, then $v_{xy}(t,) \notin L^2(\Omega)$.
Since $u_{xx} = -u_{tx} - v_{xy}$ and $u_{tx}(t,) \in L^2(\Omega)$, then $u_{xx}(t,) \notin L^2(\Omega)$.
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