Shock profiles in the numerical analysis of hyperbolic systems of conservation laws

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1st order systems of conservation laws

Space-time domain:

$$t \geq 0, \qquad x = (x_1, \ldots, x_d).$$

Vector-valued unknown

$$(x,t) \quad \mapsto \quad u(x,t) \in \mathcal{U} \qquad \left(\subset \mathbb{R}^N\right),$$

having the meaning of physically conserved densities: mass density, energy-momentum, charge, electro-magnetic field, ...

Conservation laws:

$$\frac{\partial u}{\partial t} + \sum_{\alpha=1}^{d} \frac{\partial f^\alpha(u)}{\partial x_\alpha} = 0.$$

Examples

Gas Dynamics: $1 \leq d \leq 3$ and $N = 2 + d$. Unknowns $u = (\rho, \rho v, \rho \varepsilon)$. Euler equations:

- Conservation of mass $\partial_t \rho + \mathrm{div}(\rho v) = 0$,
- C. of momentum $\partial_t(\rho v) + \mathrm{div}(\rho v \otimes v) + \nabla p(\rho, e) = 0$,
- C. of energy

$$\partial_t \left(\rho e + \frac{1}{2}\rho|v|^2 \right) + \mathrm{div} \left(\left(\rho e + \frac{1}{2}\rho|v|^2 + p \right) v \right) = 0.$$

Traffic flow (Lighthill, Whitham): scalar unknown $u = \rho$, the density of cars along a road ($d = 1$).

Conservation of “mass”

$$\partial_t \rho + \partial_x q = 0, \quad q = f(\rho) := \kappa\rho(\rho_{\mathsf{max}} - \rho).$$

Maxwell’s equations: $d = 3$ and $N = 6$. Unknown $u = (B, D)$. Faraday and Ampère conservation laws

$$\partial_t B + \mathrm{curl} E = 0, \quad \partial_t D - \mathrm{curl} H = 0, \quad \mathrm{div} B = \mathrm{div} D = 0,$$

with equations of state

$$E = E(B, D), \qquad H = H(B, D).$$

The Cauchy Problem

For the sake of simplicity: $d = 1$ (planar waves).

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \qquad (1)$$

where

$$\begin{array}{ccc} u & \mapsto & f(u) \\ \mathcal{U} & \rightarrow & \mathbb{R}^N \end{array}$$

is called the *flux*.

Given an initial data $u^0 : \mathbb{R} \rightarrow \mathcal{U}$, find the solution such that

$$u(x, 0) \equiv u^0(x).$$

An example: the **Riemann problem**

- Hypothesis: u^0 is invariant under $x \mapsto \sigma x$:

$$u^0(x) \equiv a \quad \text{if } x < 0, \qquad u^0(x) \equiv b \quad \text{if } x > 0,$$

- The PDEs are invariant under $(x,t) \mapsto (\sigma x, \sigma t)$,

- Uniqueness is expected: The solution must be self-similar,

$$\boxed{u(x,t) = R\left(\frac{x}{t}\right).}$$

Solve the *implicit* Differential Equation

$$\frac{d}{d\xi} f(R(\xi)) = \xi \, \frac{dR}{d\xi} \,. \tag{2}$$

If $\xi \mapsto R(\xi)$ is Lipschitz:

$$(\mathrm{d} f(R(\xi)) - \xi) \frac{dR}{d\xi} = 0.$$

Whence either $\xi \mapsto R(\xi)$ is constant (locally), or

- $\frac{dR}{d\xi} = r_k(R(\xi))$ is an eigenvector,
- $\xi = \lambda_k(R(\xi))$ the corresponding eigenvalue.

$\longrightarrow$ Suggests to assume **hyperbolicity**: $\mathrm{d} f(u)$ *is diagonalisable with real eigenvalues.*

Differentiation yields

$$\mathrm{d}\lambda_{\mathrm{k}}(R(\xi)) \cdot \frac{dR}{d\xi} = 1, \tag{3}$$

which raises two obstacles:

1. (3) is impossible for linear systems ($\lambda_k \equiv \mathrm{cst}$), or more generally if

$$\mathrm{d}\lambda_{\mathrm{k}}(R(\xi)) \cdot r_k = 0.$$

2. (3) does not allow us to solve the Riemann problem for certain data.

   Example: the Burgers equation ($d = 1$),

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0.$$

   Then $\lambda(u) = f'(u) \equiv u$,

$$u(\xi) = \xi$$

yields the NC

$$a \le b.$$

What to do if $b < a$ instead ?

Answer: Accept discontinuous solutions.

Then (2) to be understood in the distributional sense,

$$\frac{d}{d\xi}\{f(R(\xi)) - \xi R(\xi)\} = -R(\xi).$$

Whence $\xi \mapsto f(R(\xi)) - \xi R(\xi)$ is Lipschitz continuous. Yields a jump relation,

...

the *Rankine–Hugoniot* condition:

$$[f(R)] = \xi[R], \qquad (4)$$

with

$$[R] := R(\xi + 0) - R(\xi - 0).$$

Warning. Not all discontinuities are admissible.

Example: in Burgers' equation, discontinuities are restricted by

$$R(\xi + 0) < R(\xi - 0).$$

Construction: to solve the Riemann problem, glue

- constant states,
- $\mathcal{C}^1$-solutions (*rarefaction waves*),
- discontinuities (*shock waves*).

Definition. $R = R\left(a, b; \frac{x}{t}\right)$ is the *Riemann solver*.

Conservative difference schemes

Choose $\Delta x > 0$, $\Delta t > 0$.

Rectangular grid: $t_n = n\Delta t$ and $x_j = j\Delta x$.

Aspect ratio:

$$\sigma := \frac{\Delta t}{\Delta x}$$

Dimensional analysis:

$$\frac{1}{\sigma} \quad \text{is a velocity.}$$

Discretized unknown:

$$u_j^n \sim u(x_j, t_n).$$

Driven by a *difference scheme*

$$\boxed{\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{g_{j+\frac{1}{2}}^n - g_{j-\frac{1}{2}}^n}{\Delta x} = 0,}$$

with g the *numerical flux*.

Initial sampling:

$$u_j^0 := u^0(j\Delta x) \qquad \text{or} \qquad u_j^0 := \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u^0(x)\,dx.$$

Designing a scheme

Choose a flux map

$$(\ldots, u_j, u_{j+1}, \ldots) \quad \mapsto \quad g_{j+\frac{1}{2}}$$

with shift invariance.

Example: Three-point schemes

$$g_{j+\frac{1}{2}} := F(u_j, u_{j+1}).$$

Yields

$$\boxed{u_j^{n+1} \quad = \quad u_j^n \quad + \quad \sigma \left(F(u_{j-1}^n, u_j^n) - F(u_j^n, u_{j+1}^n) \right).}$$

Reconstruction

$\longrightarrow$ Approximated solution $u^{\text{app}}(x,t)$.

Extra- / inter-polated from the points u_j^n.

- piecewise constant,
- piecewise linear,
- exact solution in strips $n\Delta t \le t < (n+1)\Delta t$ (uses the Riemann solver),
- ...

There remains to choose F.

Consistency

- Assume that σ is constant.

- Let $\triangle t \to 0^+$. Assume that u^{app} converges boundedly almost everywhere.

- **Theorem** (Lax–Wendroff). Then the limit $u(x,t)$ satisfies

$$\partial_t u + \partial_x F(u,u) = 0.$$

♡

- Want to fit $\partial_t u + \partial_x f(u) = 0$? Require that

$$\boxed{F(a,a) \equiv f(a), \qquad \forall a \in \mathbb{R}^N.}$$

Examples of schemes

Centered scheme (Von Neumann):

$$F_c(a,b) := \frac{1}{2}(f(a) + f(b)).$$

Highly unstable!!!

Lax–Friedrichs scheme:

$$F_{LF}(a,b) := \frac{1}{2}(f(a) + f(b)) + \frac{1}{2\sigma}(a-b).$$

Can be defined through the Riemann problem !

Lax–Wendroff scheme. A second-order variant of Lax–Friedrichs.

$$F_{LW}(a,b) := \frac{1}{2}(f(a)+f(b)) + \frac{\sigma}{2}\mathrm{d}f_m(f(a)-f(b)),$$

with $\mathrm{d}f_m$ a "middle point" between $\mathrm{d}f(a)$ and $\mathrm{d}f(b)$, *e.g.*

$$\mathrm{d}f\left(\frac{a+b}{2}\right), \qquad \frac{1}{2}(\mathrm{d}f(a)+\mathrm{d}f(b)), \qquad \int_0^1 \mathrm{d}f(\theta a+(1-\theta)b)\,d\theta.$$

Godunov scheme:

$$F_G(a,b) = f(c), \qquad c := R(a,b;0),$$

where $(x,t) \mapsto R(a,b;x/t)$ is the Riemann solver.

Nota. Godunov's flux $f(R(0))$ is well-defined even in the case of a stationary shock, since

$$f(R(0+)) = f(R(0-)).$$

Other schemes: Roe, Osher, Leveque, ...

Linearized stability of schemes

- Still assume σ constant.

- Let $\Delta t \to 0^+$.

- Write the scheme

$$u_j^{n+1} = G(u_{j-1}^n, u_j^n, u_{j+1}^n),$$

with

$$G(a_{-1}, a_0, a_1) := a_0 + \sigma \left(F(a_{-1}, a_0) - F(a_0, a_1)\right).$$

- Constants are solutions.

- Linearize about a constant state u^*:

$$w_j^{n+1} = \sum_{-1\le k\le 1} A_k w_{j+k}^n, \qquad A_k := \frac{\partial G}{\partial a_j}(u^*, u^*, u^*).$$

One has

$$\sum_{-1\le k\le 1} A_k = I_N.$$

- Linearized stability: Given $w(\cdot, 0) \in L^2(\mathbb{R})^N$, the approximate solution $w^{\text{app}}(\cdot, t)$, has to remain bounded in L^2 as $\Delta t \to 0^+$:

  There should exist $C(t)$ (independent of Δt), such that

$$\sum_{j=-\infty}^{+\infty} \|w_j^n\|^2 \le C(t) \sum_{j=-\infty}^{+\infty} \|w_j^0\|^2,$$

  with

$$n = E[t/\Delta t] \to +\infty.$$

- Apply discrete Fourier transform:

$$\widehat{w}(\xi) := \sum_{k=-\infty}^{\infty} e^{\imath k\xi} w_k.$$

The scheme becomes

$$\widehat{w}^{n+1}(\xi) = M(\xi)\widehat{w}^n(\xi),$$

with

$$M(\xi) := e^{\imath\xi} A_{-1} + A_0 + e^{-\imath\xi} A_1.$$

- Induction yields

$$\widehat{w}^n(\xi) = M(\xi)^n \widehat{w}^0(\xi).$$

- By Uniform Boundedness Principle, stability requires (Lax–Wendroff)

$$\sup\{\|M(\xi)^n\|\,;\,\xi\in\mathbb{R},\,n\geq 0\}<\infty.$$

  Depends on both u^* and σ.

- NC, the *Courant–Friedrichs–Lewy condition*:

$$\rho(M(\xi))\leq 1,\qquad \forall\xi\in\mathbb{R}.$$

Centered scheme: $\xi\neq 0$ implies $\rho(M(\xi))>1$.

Hadamard instability

Warning

Linearized analysis
is not appropriate
in presence of
shock waves

!!!

The **Courant–Friedrichs–Lewy condition** (again)

Propagation in the discrete world: u_j^n depends only on $u_{j-n}^0, \ldots, u_{j+n}^0$.

That is, $u^{\mathsf{app}}(x,t)$ depends only on the restriction of the initial data to

$$\left[x - \frac{t}{\sigma}, x + \frac{t}{\sigma}\right].$$

Propagation in the PDE world: at a linearized level,

$$\partial_t w + A\partial_x w = 0, \qquad A := \mathsf{d}f(u^*).$$

Decomposing the data and the solution onto the eigenbasis of $\mathsf{d}f(u^*)$ yields pure transport:

$$w(x,t) = \sum_{m=1}^{N} a_m(x - \lambda_m t) r_m, \qquad \mathsf{d}f(u^*) r_m = \lambda_m r_m.$$

Whence $w(x,t)$ depends on the restriction of the initial data to

$$[x + \lambda_1 t, x + \lambda_N t] \,.$$

Necessary condition for consistency:

The aspect ratio may not be so large that the waves travel slowlier in the discretized world than in the real world.

In other words, one needs

$$[x + \lambda_1 t, x + \lambda_N t] \subset \left[x - \frac{t}{\sigma}\,,\, x + \frac{t}{\sigma}\right].$$

Whence the C.-F.-L. condition:

For every likely u^*,

$$\sigma\,\rho(\mathrm{d}f(u^*)) \leq 1. \tag{5}$$

Lax–Friedrichs or Godunov schemes: CFL amounts precisely to (5).

The Cauchy problem: the state of the art

Only partial results for the Cauchy problem:

- Smooth initial data: Existence and uniqueness of classical solutions, on a strip

$$\mathbb{R}^d \times \Big[0, T(u^0)\Big) \,.$$

  Of little interest for applications.

- No other result for systems ($N \geq 2$) in several space dimensions ($d \geq 2$).

...

- Nice theory for the scalar case ($N = 1$, $d \geq 1$), Volpert, Kruzhkov (1970).
  - Existence and uniqueness for L^∞-data.
  - Contraction in the L^1-distance.
  - Error estimates for approximate solutions (Kutznetsov).
  - Kinetic formulation (Perthame, P.-L. Lions & Tadmor)
- Systems ($N \geq 2$) when $d = 1$, Glimm (1965), Bressan (1994–ff):
  - Existence for quite general systems and small initial data in $BV(\mathbb{R})$.
  - Uniqueness, L^1-continuous semi-group.

...

- Systems with many entropies (mainly $N = 2$ and $d = 1$) and some convexity, DiPerna (1983):
  - Existence of solutions for arbitrarily large initial data in L^∞,
  - Uniqueness is not known.

One cause of troubles: shock waves.

A related one is: irreversibility.

Entropies

In physics and mechanics, $\mathcal{C}^1$-solutions of

$$\partial_t u + \mathrm{Div}_x f(u) = 0$$

do satisfy an additional conservation law

$$\partial_t \phi(u) + \mathrm{div}_x \vec{q}(u) = 0,$$

where $\mathrm{D}^2\phi > 0_n$.

Terminology (math[al]):

- ϕ is an *entropy* (!?!), q its *entropy flux*.

Proposition (Godunov, Lax & Friedrichs). A strongly convex entropy ensures the hyperbolicity: $\mathrm{d}f(u)$ diagonalizable with real eigenvalues.

◊

Example (gas dynamics):

Define the physical entropy $s = s(\rho, e)$ by

$$\theta \mathrm{d}s = \mathrm{d}e + p(\rho, e)\mathrm{d}\frac{1}{\rho}.$$

Then smooth flows satisfy

$$\frac{\partial}{\partial t}(\rho s) + \mathrm{Div}(\rho s \mathbf{v}) = 0.$$

Whence

$$\phi = -\rho s, \qquad \vec{q} = -\rho s \mathbf{v} = \phi \mathbf{v}.$$

Shock waves

Typical solutions of $\partial_t u + \partial_x f(u) = 0$ display discontinuities along curves $x = X(t)$.

Limits $u(X(t) \pm 0, t) =: u^{\pm}(t)$ are expected, together with a *shock speed*

$$s := \frac{dX}{dt}.$$

The PDEs translate into jump relations: the Rankine–Hugoniot condition,

$$\boxed{f(u^+) - f(u^-) = s(u^+ - u^-).}$$

Nota: the shock velocity is a $\lambda_k(u^*)$ (Taylor formula).

Irreversibility: the Lax entropy inequality

Relevant to thermodynamics and its 2nd principle.

Translates through a differential inequality.

For *genuinely nonlinear systems*, the R.-H. condition is not compatible with the jump relation

$$q(u^+) - q(u^-) = s\left(\phi(u^+) - \phi(u^-)\right) \tag{6}$$

associated to the additional conservation law.

So what ?

Example: *Burgers equation*, $N = 1$ and $f(u) = \frac{1}{2}u^2$.

- Rankine–Hugoniot:

$$s = \frac{f(u^+) - f(u^-)}{u^+ - u^-} = \frac{u^+ + u^-}{2}.$$

- With $\phi(u) := u^2/2$ (thus $q(u) = u^3/3$), (6) reads

$$s = \frac{q(u^+) - q(u^-)}{\phi(u^+) - \phi(u^-)} = \frac{2}{3} \times \frac{(u^+)^2 + u^+u^- + (u^-)^2}{u^+ + u^-}.$$

- Together, these identities give $u^- = u^+$.

Means that for discontinuous solutions,

$$\partial_t \phi(u) + \partial_x q(u) \neq 0.$$

So what ?

Require only the *Lax entropy inequality* (say $d \geq 1$)

$$\partial_t \phi(u) + \mathsf{div}_x \vec{q}(u) \leq 0,$$

in the sense of distributions.

Translated as

$$q(u^+) - q(u^-) \leq s\left(\phi(u^+) - \phi(u^-)\right)$$

across discontinuities.

$\longrightarrow$ irreversibility.

Entropy consistent schemes

Definition $(d = 1)$. Have a discrete entropy flux $Q(a,b)$ with $Q(a,a) \equiv q(a)$, such that

$$\phi(u_j^{n+1}) \leq \phi(u_j^n) + \sigma \left(Q(u_{j-1}^n, u_j^n) - Q(u_j^n, u_{j+1}^n) \right)$$

for every sequency $(u_j^m)_{j,m}$ generated by the scheme.

♣

Lax & Wendroff: one recovers again

$$\partial_t \phi(u) + \mathrm{div}_x \vec{q}(u) \leq 0$$

in the limit.

Shock profile

Principle: Every admissible solution of $\partial_t u + \partial_x f(u) = 0$, depending only on $d' = 0$ or 1 variable should have a counterpart at the discrete level.

- Constants $\longrightarrow$ constants !

OK for conservative finite differences:

$$\left(u_{j-1}^n = u_j^n = u_{j+1}^n = a\right) \Longrightarrow \left(u_j^{n+1} = a\right).$$

- Discontinuous travelling waves (shocks)

$$u(x,t) = \begin{cases} u^-, & x < st, \\ u^+, & x > st. \end{cases}$$

$\longrightarrow$ “discrete” shock profile (DSP).

What is a DSP ?

- Look for a travelling wave in

$$x - ct = j\Delta x - cn\Delta t.$$

Normalized variable

$$y := \frac{x - ct}{\Delta x} = j - \sigma cn.$$

- Look for a travelling discrete wave

$$u_j^n = U(y) = U\left(\frac{x - ct}{\Delta x}\right).$$

...

- Plug into the difference scheme:

$$\boxed{U(y-\sigma c) = U(y) + \sigma\{F(U(y-1),U(y)) - F(U(y),U(y+1))\}.}$$

Terminology: the *Profile Equation*.

- Ask for limits

$$U(y) \to u^{\pm}, \qquad x \to \pm\infty.$$

Then

$$u^{\mathsf{app}}(x,t) \xrightarrow{\Delta x \to 0} \begin{cases} u^{-}, & x < ct, \\ u^{+}, & x > ct. \end{cases}$$

The velocity of a discrete shock

Integrate the profile equation over $y \in (-\infty, +\infty)$:

$$\int_{-\infty}^{+\infty} (U(y) - U(y - \sigma c))\, dy = \sigma \int_{-\infty}^{+\infty} \begin{array}{l} \{F(U(y), U(y+1)) \\ -F(U(y-1), U(y))\}\, dy. \end{array}$$

Apply twice the formula

$$\int_{-\infty}^{+\infty} (Z(y) - Z(y-h))\, dy = h(Z(+\infty) - Z(-\infty)).$$

$\longrightarrow$

$$F(u^+, u^+) - F(u^-, u^-) = c(u^+ - u^-).$$

Remember the consistency

$$F(a,a) = f(a).$$

$\longrightarrow$ The Rankine–Hugoniot condition for $(u^-, u^+; c)$!

Proposition: If a DSP exists from a state u^- to a state u^+, then

1. (u^-, u^+) satisfy the Rankine–Hugoniot condition,

2. the velocity c of the DSP and the shock speed s coincide.

$\diamondsuit$

The latter is specific to conservation laws. When discretizing reaction-diffusion equations, say

$$\partial_t v - \Delta v = g(v),$$

then

- the velocity of a discrete front differs from the front speed in the PDE,
- the velocity may not be unique,
- there is a “pinning” phenomenon: as parameters in the PDE vary smoothly, the velocity of the discrete front may vary as in a “devil staircase”.

Example: KPP–Fisher equation.

Integration also gives:

Proposition. Assume that the scheme be entropy-consistent. Let U be a DSP with limits $u^{\pm}$ and velocity s.

Then the shock $(u^-, u^+; s)$ satisfies the Lax entropy inequality

$$q(u^+) - q(u^-) \leq s\left(\phi(u^+) - \phi(u^-)\right).$$

♣

Thus DSPs are a valuable tool. They represent faithfully shock waves.

Existence of DSPs

Question.

? Given a shock wave $(u^-, u^+; s)$, does there exist a profile $y \mapsto U(y)$, satisfying

- the limits $U(\pm\infty) = u^\pm$,
- the profile equation

$$U(y-\sigma s) = U(y) + \sigma\{F(U(y-1), U(y)) - F(U(y), U(y+1))\}.$$

Notation: the equation involves a dimensionless parameter, the ‘grid velocity’

$$\boxed{\eta := \sigma s}$$

The domain $\mathcal{D}$ of a DSP

$$y \in \mathcal{D} \quad \mapsto \quad U(y).$$

For the PE to make sense, $\mathcal{D}$ must be invariant under both

$$y \quad \mapsto \quad y \pm 1 \qquad \text{and} \qquad y \quad \mapsto \quad y - \eta.$$

Simplest choice:

$$\mathcal{D} = \mathbb{Z} + \eta\mathbb{Z}.$$

Rational case: If $\eta = \frac{p}{q}$, then

$$\mathcal{D} = \frac{1}{q}\mathbb{Z}$$

is OK.

Irrational case: If $\eta \notin \mathbb{Q}$, then $\mathcal{D}$ is dense in $\mathbb{R}$. Take

$$\mathcal{D} = \mathbb{R}$$

instead.

Ask that $y \mapsto U(y)$ be continuous.

Existence: the rational case

$$\eta = \frac{p}{q}, \qquad p \wedge q = 1.$$

General method:

- “Integrate” once the profile equation (Benzoni).

  Example: if $\eta = \frac{1}{2}$, then

  $U(y) - \sigma\{F(U(y-1/2), U(y+1/2)) + F(U(y), U(y+1))\} \equiv$ cst .

  Calculation of the constant:

  - Take the limit as $y \to -\infty$,
  - use $\eta = \sigma s$ and apply consistency.

In the example:

$$U(y) \quad - \quad \sigma\left(F(U(y-\frac{1}{2}),U(y+\frac{1}{2}))+F(U(y),U(y+1))\right)$$
$$= u_- - \frac{1}{s}f(u_-).$$

- This integrated form encodes the conditions at infinity $U(\pm\infty) = u^\pm$.

- More generally, rewrite the profile equation as

$$G\left(V_k, V_{k+1}; u^-, \sigma\right) = 0$$

for the extended state

$$V_k = \left(U\left(\frac{k}{q}-1\right), U\left(\frac{k+1}{q}-1\right), \ldots, U\left(\frac{k-1}{q}+1\right)\right).$$

- If possible, apply the IFT, to convert the integrated profile equation into a *discrete dynamical system*

$$V_{k+1} = H\left(V_k; u^-, \sigma\right).$$

- $V^- = (u^-, \ldots, u^-)$ is a rest point (obvious).

- $V^+ = (u^+, \ldots, u^+)$ is a rest point (Rankine–Hugoniot).

- Look for a heteroclinic orbit between V^- and V^+.

- **Tools**: bifurcation theory, center manifold theorem applied to the map

$$(V, u, \sigma) \quad \mapsto \quad \widehat{H}(V, u, \sigma) := (H(V; u), u, \sigma).$$

Results in the rational case

Theorem (Majda & Ralston, 1979). Under the assumptions that

- the scheme is “non-resonant” and “linearly stable”,
- the system is “genuinely non-linear”,
- $(u^-, u^+; s)$ is an admissible shock,
- $\|u^+ - u^-\| \ll \frac{1}{q}$,

there exists a one-parameter family of DSPs with limits $u^\pm$.

Sketch of the proof ($\eta = 0$)

For steady shocks ($s = 0$), one has $\eta = 0$.

1- Geometry of the R.–H. condition. Select an index $1 \le k \le N$. Select a state u^* such that

$$\lambda_k(u^*) = 0, \qquad \mathrm{d}\lambda_k(u^*) r_k(u^*) \neq 0.$$

- Define locally

  $$\Sigma := \{u \in \mathcal{U}\,;\ \lambda_k(u) = 0\}.$$

  Σ is a hypersurface, transversal to $r_k(u)$.

- $f(\Sigma)$ is a hypersurface too.

- Locally, $f(\Sigma)$ splits $\mathbb{R}^N$ into two open sets $\mathcal{O}_0$ and $\mathcal{O}_2$.

- The graph of $u \mapsto f(u)$ folds over Σ. The equation

$$f(v) = \bar{f}$$

has zero, one or two solutions, depending on whether

$$\bar{f} \in \mathcal{O}_0, \quad \in f(\Sigma), \quad \in \mathcal{O}_2.$$

- In a neighbourhood $\mathcal{U}^*$ of u^*,

$$f(v) = f(v')$$

defines a smooth involution

$$v \mapsto v',$$

such that

$$(v' = v) \Longleftrightarrow (v \in \Sigma).$$

- One has

$$\lambda_k(v)\lambda_k(v') < 0, \qquad \forall v \notin \Sigma.$$

2- The dynamical system.

- Define $M(a,v)$ by I.F.T.:

$$F(a,M(a,v)) = f(v).$$

  Works for Lax–Friedrichs, but not for Godunov.

- Write the Profile Equation $F(u_j,u_{j+1}) = f(u^-)$ in the form

$$(u_{j+1},v_{j+1}) = H(u_j,v_j), \qquad H(a,v) := (M(a,v),v). \tag{7}$$

  Meaning that $v_j \equiv$ cst.

- Fixed points correspond to $f(a) = f(v)$. Two families:
  - (v,v) for $v \in \mathcal{U}^*$,
  - (v',v) for $v \in \mathcal{U}^*$.

- These N-dimensional manifolds intersect transversally along $\mathtt{diag}(\Sigma \times \Sigma)$.

3- Center manifold theory.

- Compute

$$\mathsf{D}H(u^*, u^*) = \begin{pmatrix} \mathsf{d}_a M & \mathsf{d}_b M \\ 0_N & I_N \end{pmatrix}.$$

- Differentiating, one has

$$\mathsf{d}_a F + \mathsf{d}_b F \, \mathsf{d}_a M = 0, \qquad \mathsf{d}_b F \, \mathsf{d}_v M = \mathsf{d} f.$$

- Recall that

$$\mathsf{d}_a F + \mathsf{d}_b F = df,$$

along the diagonal.

- Whence

$$1 \in \mathsf{Sp}\left(\mathsf{d}_a M(u^*, u^*)\right),$$

- and $\mu = 1$ is an eigenvalue of $\mathsf{D}H(u^*, u^*)$,

$$\#\{\mu = 1\} \geq N + 1.$$

- Non-resonnance:
  - the multiplicity is exactly $N + 1$,
  - no other eigenvalue on the unit circle.

- **Center Manifold Theorem.** There exists locally a smooth manifold $\mathcal{M}$ of dimension $N+1$, invariant under the dynamics, containing every trajectory which remains globally in $\mathcal{U}^*$.

  The center manifold is tangent at (u^*, u^*) to

$$\ker \mathsf{D}H(u^*, u^*).$$

$\diamondsuit$

- Here, $\ker \mathsf{D}H(u^*, u^*)$ is made of vectors

$$\begin{pmatrix} X \\ X + \alpha r_k(u^*) \end{pmatrix}, \qquad \forall X \in \mathbb{R}^N,\ \alpha \in \mathbb{R}.$$

- The center manifold contains
  - fixed points in $\mathcal{U}^*$ (two hypersurfaces),
  - heteroclinic orbits within $\mathcal{U}^*$.

- Since $v_{j+1} = v_j$, $\mathcal{M}$ is foliated by curves

$$\delta(\bar{v}) := \{(a, v) \in \mathcal{M}\,;\, v = \bar{v}\},$$

invariant under the dynamics.

- These curves are transversal to the fixed point locuses. Each $\delta(\bar{v})$ contains exactly two fixed points:

$$P := (\bar{v}, \bar{v}) \qquad \text{and} \qquad Q := (\bar{v}', \bar{v}).$$

- The restriction of H to $\delta(\bar{v})$ is orientation-preserving: H maps the arc PQ onto itself PQ, monotonically.

- Every point R in PQ yields a heteroclinic orbit such that

$$(u_0, v_0) = R.$$

Other values of η (sketchy)

1. Still use the integrated profile equation

2. Pretend that u^- and σ are not constant, and write the dynamics as

$$V_{k+1} = H(V_k, z_k, \sigma_k), \qquad z_{k+1} = z_k, \qquad \sigma_{k+1} = \sigma_k, \qquad (!!)$$

   that is

$$(V_{k+1}, z_{k+1}, \sigma_{k+1}) = \widehat{H}(V_k, z_k, \sigma_k),$$

   but with z_k and σ_k constant ...

3. Given $u^- \in \mathcal{U}$ and $1 \leq j \leq N$, the state (V^-, u^-, σ^-) is a fixed point, where

$$V^- := (u^-, \ldots, u^-), \qquad \sigma^- \in \mathbb{R}$$

   is arbitrary

4. Nearby fixed points are of the form (V^+, u^+, σ^+) with $V^+ := (u^+, \dots, u^+)$ and $(u^-, u^+; \eta/\sigma^+)$ satisfying R–H.

5. The dynamics stands in a space of dimension $(2q + 1)N + 1$... but The Center Manifold Theorem reduces the dynamics to an $(N + 2)$-dimensional manifold $\mathcal{M}$.

6. There are $N + 1$ constants of the dynamics: (u, σ). Thus $\mathcal{M}$ is foliated by curves invariant under the dynamics.

7. ...

QED

- In other words, there exists a *continuous* “D”SP

$$U : \mathbb{R} \to \mathbb{R}^N \qquad !$$

- For every $h \in \mathbb{R}$, the following defines a travelling wave

$$u_j^n = U\left(h + j - \frac{pn}{q}\right).$$

- **Re-parametrization**: If U is a continuous DSP, then so is $U \circ \psi$ for every one-to-one mapping $\psi : \mathbb{R} \to \mathbb{R}$ with (circle homeomorphism)

$$\psi\left(y + \frac{1}{q}\right) = \psi(y) + \frac{1}{q}.$$

- The theorem applies mainly to Lax “compressive” shocks.

Non-resonance *vs* Lax–Friedrichs

Lax–Friedrichs scheme:

$$u_j^{n+1} = \frac{1}{2}\left(u_{j-1}^n + u_{j+1}^n\right) + \frac{1}{2\sigma}\left(f(u_{j-1}^n) - f(u_{j+1}^n)\right).$$

The odd / even subgrids ignore each other:

$$j + n \in 2\mathbb{Z}, \qquad / \qquad j + n + 1 \in 2\mathbb{Z}.$$

$\longrightarrow$ L.–F. is resonant.

To apply Majda–Ralston Theorem: iterate the scheme

$$u_j^{n+2} = \frac{1}{4}\left(u_{j-2}^n + 2u_j^n + u_{j+2}^n\right) + \cdots$$

Doubling the scales Δt and Δx yields

$$v_k^m := u_{2k}^{2m},$$

which obeys a conservative difference scheme with numerical flux

$$\begin{aligned} F_{LF2}(a,b) \;&:=\; \frac{1}{4\sigma}(a-b)+\frac{1}{4}(f(a)+f(b)) \\ &\quad +\frac{1}{2}f\left(\frac{a+b}{2}+\frac{\sigma}{2}(f(a)-f(b))\right). \end{aligned}$$

This scheme is non-resonant.

The irrational case

Warning: $\mathbb{Z} + \eta\mathbb{Z}$ is dense in $\mathbb{R}$.

$\longrightarrow$ Search for a *continuous* DSP

$$U : \mathbb{R} \to \mathbb{R}^N.$$

First attempt: Pass to the limit as rationals tend to irrationals.

Failure, because of the restriction

$$\|u^+ - u^-\| \ll \frac{1}{q}$$

in Majda–Ralston Theorem.

In the limit, $q \to +\infty$. There remains the useless situation

$$u^+ = u^-.$$

A complete theory: the scalar case ($N = 1$)

Scalar conservation laws satisfy a comparison principle (Kruzkhov): If u and v solve the Cauchy problem, then

$$(u^0 \leq v^0, \quad a.e.) \Longrightarrow (u \leq v, \quad \forall t > 0).$$

Suggests to employ monotone schemes

$$u_j^{n+1} = G\left(u_{j-1}^n, u_j^n, u_{j+1}^n\right),$$

with

$$(a, b, c) \quad \mapsto \quad G(a, b, c)$$

(componentwise) monotonous non-decreasing.

Often related to the CFL condition.

Examples:

- Lax–Friedrichs and Godunov schemes are monotone under $\sigma|f'| \leq 1$,

$$G_{LF}(a,b,c) = \frac{1}{2}(a + \sigma f(a)) + \frac{1}{2}(c - \sigma f(c)).$$

$$G_G(a,b,c) = b + \sigma\left(f_G(a,b) - f_G(b,c)\right)$$

  with

$$f_G(a,b) := \begin{cases} \inf\{f(u)\,;\; u \in [a,b]\}, \\ \sup\{f(u)\,;\; u \in [b,a]\}. \end{cases}$$

- Lax–Wendroff is never monotone (2nd order).

- Monotone schemes are only first-order.

Theorem (G. Jennings).

For scalar equations and monotone schemes, continuous DSPs

1. *exist for every admissible shock with* $\eta \in \mathbb{Q}$*,*
2. *are strictly monotone,*
3. *are essentially unique,*
4. *are Lipschitz:*

$$|U(x+h) - U(x)| \leq |h(u^{+} - u^{-})|, \qquad \forall\, x,\, h \in \mathbb{R}.$$

♡

“Admissible shocks”: those satisfying the Oleinik condition.

The latter justifies the passage to the limit:

Theorem (H. Fan , D. S.).

The same existence / uniqueness / monotonicity result holds true regardless the (ir)rationality of η, for every (weakly) monotone scheme.

♢

Sketch of proof:

- Apply Ascoli–Arzela
- Pass to the limit in the “integrated form” of the profile equation.
- From 1– monotonicity of the profile U, 2– the integrated profile equation, 3– the Oleinik inequality, prove that $U(\pm\infty) = u^{\pm}$.

The shift function

Back to systems. Let $U : \mathbb{R} \to \mathcal{U}$ be a DSP, with bounded variations.

Given $h \in \mathbb{R}$, define

$$Y(h) := \sum_{j \in \mathbb{Z}} (U(j+h) - U(j)) \qquad \left(Y(h) \in \mathbb{R}^N\right).$$

Properties:

- Because $U(\pm\infty) = u^{\pm}$,

$$Y(h+1) - Y(h) = u^{+} - u^{-}.$$

- Because of the profile equation (+ Rankine–Hugoniot and $\sigma s = \eta$):

$$Y(h+\eta) - Y(h) = \eta(u^+ - u^-).$$

$\Longrightarrow$

$$Y(h) = h(u^+ - u^-), \qquad \forall h \in \mathbb{Z} + \eta\mathbb{Z}. \tag{8}$$

Application:

The scalar case with a monotone scheme. The monotonicity of U together with (8) imply

$$|U(y+h) - U(y)| \leq |h(u^+ - u^-)|$$

(see above).

Irrational case. By continuity and density of $\mathbb{Z} + \eta\mathbb{Z}$, (8) yields

$$Y(h) = h(u^{+} - u^{-}), \qquad \forall h \in \mathbb{R}. \tag{9}$$

But $\mathbb{R} \setminus \mathbb{Q}$ is dense ...

Thus (9) is expected to hold even when $\eta \in \mathbb{Q}$.

In particular for

$$h \notin \frac{1}{q}\mathbb{Z},$$

... well, if the life is smooth.

Something must go wrong !

In the rational case, the shift function compares two profiles

$$\mathbf{u} = (u_y)_{y \in \frac{1}{q}\mathbb{Z}} \quad \text{and} \quad \mathbf{v} = (v_y)_{y \in \frac{1}{q}\mathbb{Z}} \,,$$

$$U(j) = u_j, \qquad U(j+h) = v_j.$$

- If $h \in \frac{1}{q}\mathbb{Z}$, $\mathbf{u}$ and $\mathbf{v}$ are identical, up to a shift ; (9) is OK because it is (8).

- But if $h \notin \frac{1}{q}\mathbb{Z}$, $\mathbf{u}$ and $\mathbf{v}$ are distinct.

  If $N \geq 2$, there is no reason why $Y(h)$ should be parallel to $u^+ - u^-$.

Counter-example

Here is a construction with

$$\boxed{Y(h) \not\parallel u^+ - u^-.}$$

- $\eta = 0$: the shock (u^-, u^+) is stationnary,
- The scheme is Godunov's (Lax–Wendroff scheme works too).
- The "integrated" profile equation for steady shocks:

$$f\left(R(u_j, u_{j+1}; 0)\right) = f(u^-) = f(u^+).$$

- $\longrightarrow$ Typically:

$$R(u_j, u_{j+1}; 0) \in \{u^-, u^+\}, \qquad \forall j \in \mathbb{Z}.$$

Lemma. If $(u^-, u^+; 0)$ is an admissible shock, it is not possible that

$$R(u_{j-1}, u_j; 0) = u^+ \quad \text{and} \quad R(u_j, u_{j+1}; 0) = u^-.$$

♠

Proof: 1- Since $R(u_j, u_{j+1}; 0) = u^-$, the Riemann problem from u_j to u^- consists only in backward waves.

2- One passes from u^- to u^+ by a steady admissible shock.

3- Since $R(u_{j-1}, u_j; 0) = u^+$, the Riemann problem from u^+ to u_j consists only in forward waves.

Gluing these pieces, the Riemann problem from u_j to u_j admits a non-constant solution. This contradicts the Lax entropy inequality.

QED

Consequence: up to a shift,

$$R(u_j, u_{j+1}; 0) = \begin{cases} u^-, & j < 0, \\ u^+, & j \geq 0. \end{cases}$$

Same idea as in the proof above: if $j < 0$, the solution of the Riemann problem from u^- to itself passes through u_j. Likewise, if $j > 0$, ... Whence

$$u_j = \begin{cases} u^-, & j < 0, \\ u^+, & j > 0. \end{cases}$$

There remains

$$\boxed{R(u^-, u_0; 0) = u^-, \qquad R(u_0, u^+; 0) = u^+.}$$

These conditions define an arc $\gamma \subset \mathcal{U}$ with ends u^- and u^+.

[*For specialists only: if* $(u^-, u^+; 0)$ *is an* N*-shock, then* γ *is the portion of the shock curve* $\mathcal{S}_N(u^-)$ *between* u^- *and* u^+*.*]

The continuous DSP:

Arbitrary parametrization of γ

$$y \in [0, 1] \quad \mapsto \quad U(y), \qquad U(0) = u^-, \quad U(1) = u^+.$$

Extend it by

$$U(y) \equiv \begin{cases} u^-, & y < 0, \\ \\ u^+, & y > 1. \end{cases}$$

To every point $a = U(h) \in \gamma$, there corresponds a DSP

$$u_j = U(h+j) = \begin{cases} u^-, & j < 0, \\ a, & j = 0, \\ u^+, & j > 0. \end{cases}$$

The shift function Y measures the difference between two DSPs. If a is as above, then

$$Y(h) = \sum_{j \in \mathbb{Z}} (u_j - v_j) = a - u^-.$$

Not parallel to $u^+ - u^-$, unless $\gamma = [u^-, u^+]$.

QED

Thus (9) does not pass to the limit from irrationals to rationals.

The alternative

1. Either DSPs do not exist for irrationals too close to rationals (non-Diophantine numbers),

2. or their have an infinite total variation,

3. or they do not depend smoothly on the data $(u^-, u^+; s, \sigma)$.

Causes:

- Small divisors problem,
- Resonance between the shock front and the grid.

Why the scalar case is not that bad

For a monotone scheme:

- DSPs do exist,
- they have a finite total variation $|u^+ - u^-|$,
- they depend smoothly on the data.

So what ?

Two vectors in $\mathbb{R}$ are always parallel !

$$\longrightarrow \qquad Y(h) \parallel u^+ - u^-.$$

Monotonicity forbids infinite total variation.

(back to systems) **The Diophantine case**

Definition. A real number η is *Diophantine* if there exists $C = C(\eta) < \infty$ and $\nu = \nu(\eta) > 0$ such that

$$\left|\eta - \frac{r}{\ell}\right| \geq \frac{C}{\ell^\nu}, \qquad \forall \frac{r}{\ell} \in \mathbb{Q}, \quad r \wedge \ell = 1.$$

♣

- Lebesgue-almost every number is Diophantine of degree $\nu = 2$.

- $\pi = 3.14159...$ is Diophantine of degree $\nu \leq 8.0161...$.

- $\zeta(3)$ is Diophantine of degree $\nu \leq 5.513891...$.

- But

$$\sum_{m=1}^{\infty} 10^{-m!} \qquad \text{is not (Liouville).}$$

The small divisor problem

- Look at the integrated profile equation

$$\int_{x-\eta}^{x} U(y)\,dy - \sigma \int_{x}^{x+1} F(U(y-1), U(y))\,dy = \eta u^{-} - \sigma f(u^{-}).$$

- Linearize the r.-h.-s.:

$$Lv(x) = \int_{x-\eta}^{x} v(y)\,dy - \sigma \left(A \int_{x-1}^{x} v(y)\,dy + B \int_{x}^{x+1} v(y)\,dy \right).$$

- The operator L diagonalizes *via* Fourier transform:

$$e^{-i\xi x} L \left[e^{i\xi x} X \right] = M(\xi) X,$$

with

$$M(\xi) := \frac{1}{i\xi} \Big((1 - e^{-i\xi\eta}) I_N - \sigma((1 - e^{-i\xi})A - \sigma(e^{i\xi} - 1)B \Big).$$

- The operator L is not Fredholm:

$$M(2\pi\ell) = \frac{1}{2i\pi\ell}\left(1 - e^{-2i\pi\ell\eta}\right) I_N.$$

The right-hand side is

$$O\left(\frac{1}{\ell^2}\right) \qquad \text{for infinity many } \ell\text{'s.}$$

- If η is not Diophantine: $\forall\nu > 2$, $\exists\,\frac{r}{\ell} \in \mathbb{Q}$ with

$$\left|\eta - \frac{r}{\ell}\right| \leq \frac{1}{\ell^\nu}.$$

Then

$$\|M(2\pi\ell)\| \leq \frac{1}{\ell^\nu}.$$

- Very fast decay !!

  Even Nash–Moser technique does not apply in this case.

- Diophantine case:

  $\exists\, \nu \geq 2$ such that

$$\|M(2\pi\ell)\| = \mathcal{O}\left(\frac{1}{\ell^\nu}\right).$$

$\longrightarrow$ Tame estimates for the Green function of the linearized scheme.

Theorem (T.-P. Liu & S.-H. Yu).

Assume that the scheme is dissipative and non-resonant.

Assume that η is Diophantine and that $(u^-, u^+; s)$ is a small enough ($|u^+ - u^-| \ll 1$) admissible shock.

Then there exists a continuous DSP.

♠

Smallness is measured in terms of $C(\eta)$ and $\nu(\eta)$.

These DSPs are orbitally stable for the numerical scheme.

Large total variation problem

(Baiti, Bressan & Jenssen) consider semi-decoupled systems

$$\begin{aligned} \partial_t v + \partial_x f(v) &= 0, & (10) \\ \partial_t w + \partial_x(\lambda w + g(v)) &= 0. & (11) \end{aligned}$$

- Either apply Jennings Theorem to (10), a scalar equation.

  Or compute explicit DSPs (Lax) for certain fluxes f.

- Evaluate Green function for the linear part (11)

  $$(\partial_t + \lambda\partial_x)w = \text{r.h.s.}$$

  Resonance may occur, depending on $\lambda\sigma$.

Lax–Friedrichs scheme. Here $\sigma_m \to \sigma \in \mathbb{Q}$.

The DSP U_m converges uniformly but its total variation increases unboundedly.

The variations concentrate on an interval

$$\left[-a(\sigma_m - \sigma)^{-2}, -b(\sigma_m - \sigma)^{-2}\right],$$

far away the shock front.

Godunov scheme. More or less the same result.

By-products

- The schemes (L.-F. or G.) produce sequences $(a_\nu, u_\nu^{\mathsf{app}})$ with
  - initial data a_ν whose total variation remains bounded as $\nu \to \infty$.
  - approximate solution u_ν^{app} whose total variation over $\mathbb{R} \times \{T\}$ does not remain bounded as $\nu \to \infty$.

- Considering a_ν and $a_\nu(\cdot - h)$, the approximations are unstable in the L^1-norm, with respect to the initial data:

$$\sup_{\nu,h} \frac{1}{h}\|a_\nu(\cdot - h) - a_\nu\|_{L^1(\mathbb{R})} < \infty,$$

$$\lim_{\nu\to\infty}\left(\sup_{0<h<1} \frac{1}{h}\|u_\nu^{\mathsf{app}}(\cdot - h, T) - u_\nu^{\mathsf{app}}(\cdot, T)\|_{L^1(\mathbb{R})}\right) = \infty.$$

- However, compensated-compactness method yields convergence $u^{\mathsf{app}} \to u$ towards an admissible solution of the Cauchy problem.

  This convergence cannot be very strong; at least, it is not uniform.

- The convergence of finite difference schemes cannot be proven by *a priori* BV bounds.

- For small initial data, BV-bounds do hold (Glimm, Bressan & coll.). Thus the counter-example build by Baiti & coll. are not that small.

  The mathematics of the stability / convergence of conservative difference schemes must be very hard !

Comparison with Viscous Shock Profiles

Shortcoming: VSP

Approximate (1) by some amount of viscosity:

$$\partial_t u + \partial_x f(u) = \epsilon \partial_x (B(u) \partial_x u).$$

Examples:

- Euler *vs* Navier–Stokes in gas dynamics,
- Viscoelasticity,
- second-order model of traffic flow,

Normalized travelling wave

$$u^\epsilon(x,t) = U\left(\frac{x-st}{\epsilon}\right).$$

with

$$(B(U)U')' = f(U)' - sU', \qquad U(\pm\infty) = u^\pm. \tag{12}$$

Integrate once:

$$\boxed{B(U)' = f(U) - sU - f(u^-) + su^-. \qquad (13)}$$

(13) includes:

- Conditions at infinity,
- Rankine–Hugoniot.

Existence theory for VSPs

- A VSP is a heteroclinic orbit of a continuous dynamical system.

- VSPs form the intersection of $\mathcal{W}^u(u^-)$ and $\mathcal{W}^s(u^+)$, unstable / stable invariant manifolds of $u^\pm$ for (13).

- If
$$\dim \mathcal{W}^u(u^-) + \dim \mathcal{W}^s(u^+) \geq N + 1,$$
then generically,
$$\dim \left(\mathcal{W}^u(u^-) \cap \mathcal{W}^s(u^+)\right) = \dim \mathcal{W}^u(u^-) + \dim \mathcal{W}^s(u^+) - N.$$

Tools: again, bifurcation analysis, Center Manifold Theorem.

The case of a Lax shock

Notation: The k-th characteristic field

$$\mathrm{d}f(u)r_k(u) = \lambda_k(u)r_k(u).$$

Definition: A discontinuity $(u^-, u^+; s)$ is a Lax shock if $\exists\, k$ such that

$$\boxed{\lambda_{k-1}(u^-) < s < \lambda_k(u^-), \qquad \lambda_k(u^+) < s < \lambda_{k+1}(u^+).}$$

♠

Interpretation: *Among the $2N$ characteristic curves*

$$\dot{x} = \lambda_j(u(x,t))$$

(N curves at right of the shock and N at left), $N+1$ enter the shock.

Lemma (Lax).

1. Small discontinuities are approximately parallel to one of the eigenvectors r_k:

   $$u^+ - u^- \sim \rho r_k(u^-)$$

   for some $1 \leq k \leq N$.

2. Assume that the k-th characteristic field is *genuinely nonlinear*:

   $$\mathrm{d}\lambda_k(u) \cdot r_k(u) \neq 0.$$

   Then small k-discontinuities are Lax shocks, up to a switch $u^- \longleftrightarrow u^+$.

♡

For a Lax shock,

$$\dim \mathcal{W}^u(u^-) = N - k + 1, \qquad \dim \mathcal{W}^s(u^+) = k.$$

$\longrightarrow$ Generically (always true for small shocks)

$$\dim\left(\mathcal{W}^u(u^-)\cap\mathcal{W}^s(u^+)\right) = 1.$$

Whence the existence and uniqueness of a VSP, up to a shift.

This is a one-parameter family of VSPs.

Parameter = shift.

Qualitatively similar to DSPs.

Question. Does this similarity occur for non-Lax shocks ?

Non-Lax shocks: VSPs

- Undercompressive shocks

$$\lambda_k(u^-) < s < \lambda_{k+1}(u^-), \qquad \lambda_k(u^+) < s < \lambda_{k+1}(u^+).$$

Only N characteristics enter the shock:

$$\dim \mathcal{W}^u(u^-) + \dim \mathcal{W}^s(u^+) = N.$$

- Overcompressive shocks

$$\lambda_{k-2}(u^-) < s < \lambda_{k-1}(u^-), \qquad \lambda_k(u^+) < s < \lambda_{k+1}(u^+).$$

$N + 2$ characteristics enter the shock.

$$\dim \mathcal{W}^u(u^-) + \dim \mathcal{W}^s(u^+) = N + 2.$$

Undercompressive shocks: VSPs

Generically,

$$\dim \left(\mathcal{W}^u(u^-) \cap \mathcal{W}^s(u^+)\right) \leq N - N = 0.$$

But $\mathcal{W}^u(u^-) \cap \mathcal{W}^s(u^+)$ is made of integral curves of the field

$$u \quad \mapsto \quad B(u)^{-1}\left(f(u) - su - f(u^-) + su^-\right).$$

Therefore

$$\mathcal{W}^u(u^-) \cap \mathcal{W}^s(u^+) = \emptyset$$

Principle. Most undercompressive shocks do not admit a VSP.

The existence of a shock profile is a codimension-1 property.

♣

Undercompressive shocks: DSPs

Assume $\eta \in \mathbb{Q}$. Example: $\eta = 0$.

Recall:

Integrated profile equation:

$$F(u_j, u_{j+1}) = f(u^-) \overset{\text{(R.–H.)}}{=} f(u^+).$$

When IFT applies, rewrite

$$u_{j+1} = H(u_j). \tag{14}$$

Then

$$\text{DSP} \quad \longleftrightarrow \quad \text{heteroclinic orbit from } u^- \text{ to } u^+$$

Again, DSPs correspond to an intersection

$$\mathcal{W}^u(u^-) \cap \mathcal{W}^s(u^+),$$

unstable / stable manifolds for H, a *diffeormorphism*.

Undercompressive shock:

$$\text{dim}\, \mathcal{W}^u(u^-) + \text{dim}\, \mathcal{W}^s(u^+) = N,$$

whence (generically)

$$\text{dim}\left(\mathcal{W}^u(u^-) \cap \mathcal{W}^s(u^+)\right) \leq N + N - 2N = 0.$$

Special: in discrete dynamics, an invariant subset under H may be discrete !

Thus the intersection may have $\dim = N - N = 0$.

Principle. Undercompressive shocks may admit a DSP.

The existence of a shock profile is a generic property (stable under small disturbances of the data).

A DSP is now isolated, instead of a one-parameter family.

♠

Undercompressive shocks: DSPs *vs* VSPs

Discrete SP. Generic property.

Discrete set, with a $\mathbb{Z}$-action.

An even number of orbits. Often 2 orbits.

Viscous SP. Codimension-one property.

One-parameter set if any, with an $\mathbb{R}$-action.

Moral: in the theory of profiles for undercompressive shocks

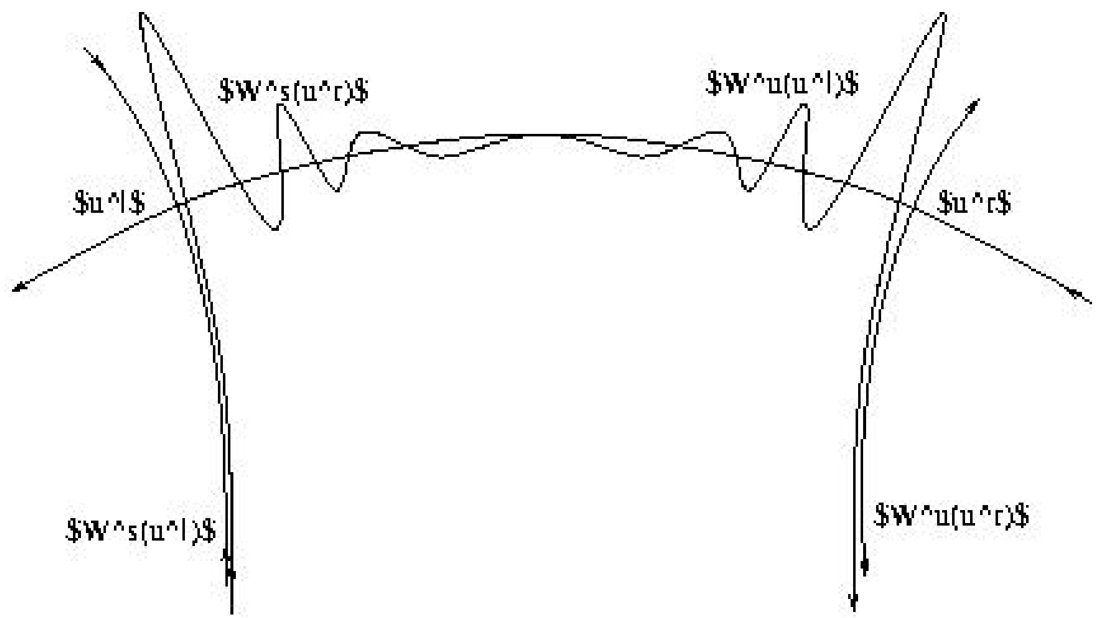
$$0 \cdot \infty = 2 \quad \text{or} \quad \mathbb{R}^{-1} \times \mathbb{R} = \mathbb{Z}/2\mathbb{Z}.$$

Why two DSPs ?

Say $N = 2$, $\eta = 0$. Then

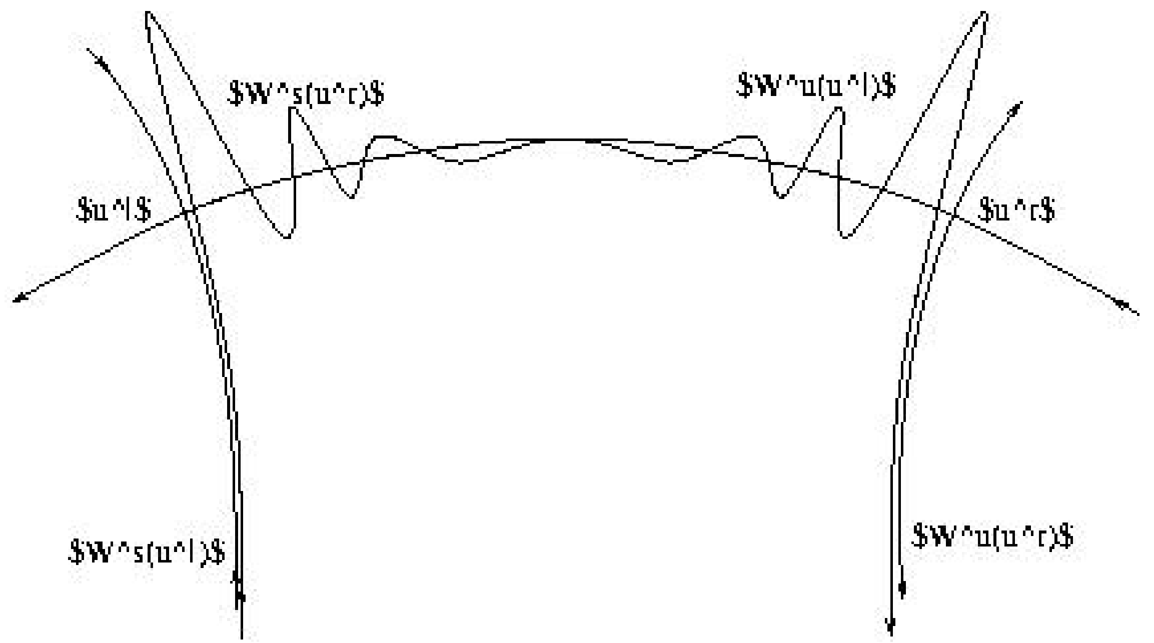
$$\dim \mathcal{W}^s(u^+) = \dim \mathcal{W}^u(u^-) = 1.$$

$u^\pm$ are saddle points of (14)



$$U_{j+1} = H(U_j).$$

$$H(u^{\pm}) = u^{\pm}.$$



Principle.

- H is orientation preserving.
- Let $\tau^s(u)$ be the tangent to $W^s(u^+)$ at u, oriented towards u^+. Likewise, let $\tau^u(u)$ ...

- (Generic) At an intersection point,

$$\mathcal{B}(u) = \{\tau^s(u), \tau^u(u)\}$$

  is a basis.

- Define the "sign" of the intersection:

$$\sigma(u) := \begin{cases} +1, & \text{direct basis,} \\ \\ -1, & \text{reverse basis.} \end{cases}$$

- The sign of the intersection is constant along an orbit.

- Two consecutive intersection points u and $\bar{u}$ have opposite intersection signs

Thus u and $\bar{u}$ correspond to distinct orbits,

$\longrightarrow$ distinct DSPs.

An example taken from reaction-diffusion

Consider the KPP equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \phi'(u), \qquad \phi(u) := \frac{1}{4}\left(u^2 - 1\right)^2.$$

Steady states are

Constants:

$$u \equiv \pm 1.$$

Fronts:

$$\frac{d^2 u}{dx^2} = \phi'(u),$$

whence

$$\frac{1}{2}\left(\frac{du}{dx}\right)^2 = \phi(u), \qquad u(\pm\infty) = \pm 1.$$

Lemma. Fronts minimize the functional

$$J[v] := \int_{\mathbb{R}} \left(\frac{1}{2} \left(\frac{du}{dx} \right)^2 + \phi(u) \right) dx,$$

under the constraint

$$u(\pm\infty) = \pm 1.$$

♠

The front is unique up to a shift.

It is odd:

$$u(-x) = -u(x).$$

Actually, $-u$ is another front, from $+1$ to -1.

Discretization of KPP

$$\frac{u_j^{m+1} - u_j^m}{\Delta t} = \frac{u_{j+1}^m - 2u_j^m + u_{j-1}^m}{(\Delta x)^2} - \phi'(u_j^m).$$

Standing discrete waves:

$$\frac{u_{j+1}^m - 2u_j^m + u_{j-1}^m}{(\Delta x)^2} = \phi'(u_j^m). \tag{15}$$

Interpretation in the phase space

Define

$$v_j := \frac{u_{j+1} - u_j}{\Delta x}.$$

Then

$$\begin{pmatrix} u_j \\ v_j \end{pmatrix} = \begin{pmatrix} u_{j-1} + \Delta x\, v_{j-1} \\ v_{j-1} + \Delta x\, \phi'\big(u_{j-1} + \Delta x\, v_{j-1}\big) \end{pmatrix} =: H \begin{pmatrix} u_{j-1} \\ v_{j-1} \end{pmatrix}.$$

Two fixed points:

$$H \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}.$$

These are saddle points.

Discrete fronts from -1 to $+1$ correspond to heteroclinic orbits of H.
They are parametrized by

$$\mathcal{W}^u \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cap \mathcal{W}^s \begin{pmatrix} +1 \\ 0 \end{pmatrix}.$$

Existence of discrete fronts

Lemma. Discrete fronts minimize the functional

$$J_\triangle[v] := \frac{1}{2\Delta x} \sum_{j\in\mathbb{Z}} (u_j - u_{j-1})^2 + \Delta x \sum_{j\in\mathbb{Z}} \phi(u_j),$$

under the constraint

$$u_{\pm\infty} = \pm 1.$$

♣

Whence the *idea*:

Minimize $J_\triangle$ over the set of odd sequences.

Still, minimizers are fronts.

There are two fronts

Two ways to express the oddness:

- Either u is odd with respect to 0,

$$u_{-j} = -u_j.$$

- Or u is odd with respect to $\frac{1}{2}$,

$$u_{1-j} = -u_j.$$

This yields two disjoint sets of odd sequences, whence two distinct minimizers.

Theorem. The KPP equation admits at least two distinct discrete fronts from -1 to $+1$.

They are monotonous.

♡

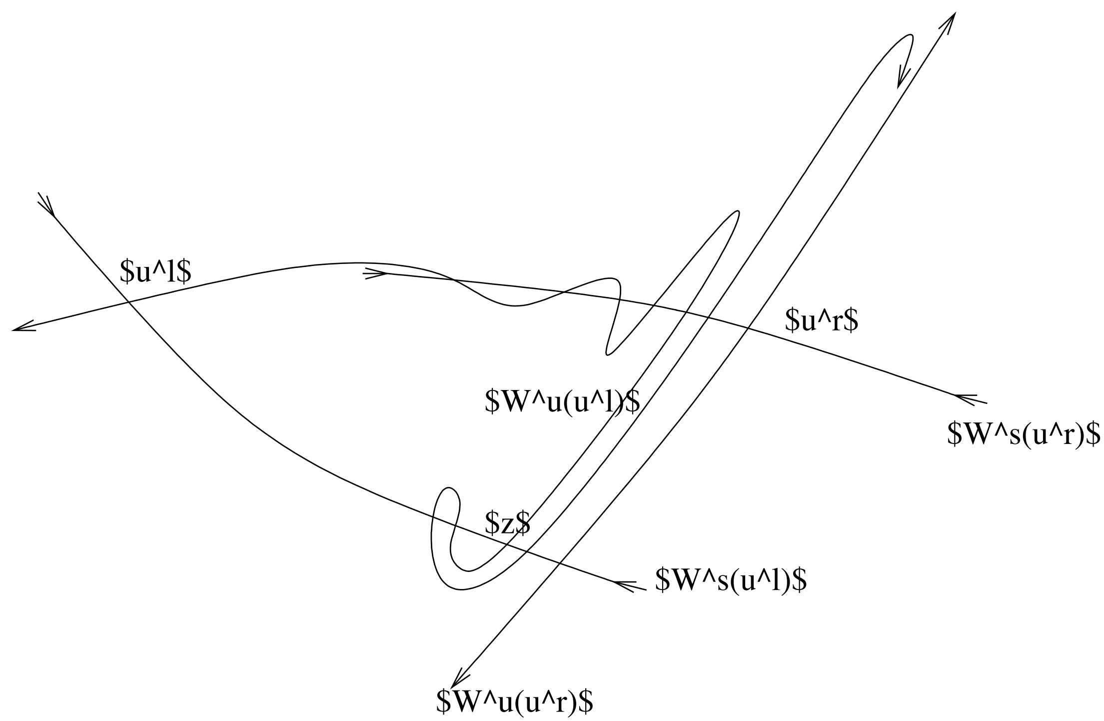
There are many fronts !

- We proved that $\mathcal{W}^u \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ intersect transversally $\mathcal{W}^s \begin{pmatrix} +1 \\ 0 \end{pmatrix}$.

- By symmetry, $\mathcal{W}^u \begin{pmatrix} +1 \\ 0 \end{pmatrix}$ intersect transversally $\mathcal{W}^s \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

- $\mathcal{W}^u \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ folds infinitely many times and approaches $\mathcal{W}^u \begin{pmatrix} +1 \\ 0 \end{pmatrix}$, being squeezed.

- Ultimately, $\mathcal{W}^u \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ intersect transversally $\mathcal{W}^s \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.



u^l
u^r
$W^u(u^l)$
$W^s(u^r)$
z
$W^s(u^l)$
$W^u(u^r)$

- Whence a *Smale horse-shoe* configuration.

Theorem. There are countably many discrete fronts from -1 to $+1$.

Most of them are non-monotone.

$\diamondsuit$

Theorem. There are as well countably many discrete fronts homoclinic to -1 (or to $+1$).

$\spadesuit$

There are also chaotic trajectories, approaching ± 1 infinitely many times on intervals of arbitrary lengths.

The case of a non-even potential ϕ

On the one hand, the Smale horse-shoe configuration is *structurally stable*: it persists under small disturbance of the dynamical systems.

On the other hand, the saddle-saddle connection of

$$\frac{d^2u}{dx^2} = \phi'(u)$$

does not persist, if the wells of ϕ are not equal.

Application: choose ϕ, close enough to an even, double-well potential ϕ_0.

- Then countably many heteroclinic discrete fronts.

- However, there is no viscous front, when unequal wells.