Convergence results : from the Boltzmann equation to incompressible hydrodynamic models

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The mathematical framework

▶ Renormalized solutions to the Boltzmann equation

Theorem (DiPerna & Lions) : **Assume that b satisfies Grad's cutoff assumption. Let $f_{in} \in L^1_{loc}(\Omega \times \mathbb{R}^3)$ be such that**

$$H(f_{in}|M) \stackrel{\text{def}}{=} \int_\Omega \int \left(f_{in} \log \frac{f_{in}}{M} - f_{in} + M \right) (x,v) \; dv \, dx < +\infty,$$

Then there exists (at least) one global renormalized solution $f \in C(\mathbb{R}^+, L^1_{loc}(\Omega \times \mathbb{R}^3))$ to the Boltzmann equation : for any $\Gamma \in C^\infty_c(\mathbb{R}^+)$,

$$\mathrm{Ma}\partial_t \Gamma(f) + v \cdot \nabla_x \Gamma(f) = \frac{1}{\mathrm{Kn}} \Gamma'(f) Q(f,f) \text{ on } \mathbb{R}^+ \times \Omega \times \mathbb{R}^3,$$
$$f(0,x,v) = f_{in}(x,v) \text{ on } \Omega \times \mathbb{R}^3.$$

Moreover, f satisfies

- the continuity equation

$$\mathrm{Ma}\partial_t \int f dv + \nabla_x \cdot \int f v dv = 0;$$

- **the momentum equation with defect measure** m

$$\mathrm{Ma}\partial_t \int fvdv + \nabla_x \cdot \int fv \otimes vdv + \nabla_x \cdot m = 0$$

- **the entropy inequality with defect measure**

$$H(f|M)(t) + \int \mathrm{Tr}\, m(t) + \frac{1}{\mathrm{MaKn}} \int_0^t \int_\Omega D(f)(s,x)dsdx \leq H(f_{in}|M)$$

▶ Proceeding by analogy

The main idea is then to recognize in the scaled Boltzmann equation the **same mathematical structure** as in the asymptotic hydrodynamic equations

- weak stability (controlled by some dissipation)
- strong-weak stability (controlled by some energy functional)

▶ Leray solutions to the Navier-Stokes equations

Theorem : Let $u_{in} \in L^2(\Omega)$ be a divergence free vector field. Then there exists (at least) one global weak solution $u \in L^2_{loc}(\mathbb{R}^+, H^1(\Omega)) \cap C(\mathbb{R}^+, w - L^2(\Omega))$ to the incompressible Navier-Stokes equations

$$\begin{aligned} &\nabla \cdot u = 0, \\ &\partial_t u + (u \cdot \nabla) u + \nabla p = \mu \Delta u, \end{aligned} \tag{1}$$

It further satisfies the energy inequality

$$\|u(t)\|^2_{L^2(\Omega)} + 2\mu \int_0^t \|\nabla u(s)\|^2_{L^2(\Omega)} ds \leq \|u_{in}\|^2_{L^2(\Omega)}$$

The Leray energy inequality and the DiPerna-Lions entropy inequality are very similar objects : in both cases,

- the dissipation controls the spatial regularity of the moments
- the global inequality controls the weak stability of solutions

▶ Dissipative solutions to the Euler equations

Theorem : **Let $u_{in} \in L^2(\Omega)$ be a divergence free vector field. Then there exists at least one global dissipative solution $u \in L^\infty([0,T), L^2(\Omega)) \cap C([0,T), w - L^2(\Omega))$ to the incompressible Euler equations**

$$\nabla \cdot u = 0, \quad \partial_t u + (u \cdot \nabla) u + \nabla p = 0. \tag{2}$$

meaning that, for all t and all $\tilde{u} \in C_c^\infty(\mathbb{R}^+ \times \Omega)$,

$$\begin{aligned} \|u(t) - \tilde{u}(t)\|^2_{L^2(\Omega)} \quad &\leq \|u_{in} - \tilde{u}_{in}\|^2_{L^2(\Omega)} \exp\left(\int_0^t \|D\tilde{u}(s)\|_{L^\infty(\Omega)} ds\right) \\ &+ \int_0^t \int \quad A(\tilde{u}) \cdot (\tilde{u} - u)(s,x) dx \exp\left(\int_s^t \|D\tilde{u}(\tau)\|_{L^\infty(\Omega)} d\tau\right) \end{aligned}$$

A similar stability inequality will be established for the solutions to the Boltzmann equation. In particular, we will have

- uniqueness and convergence as long as the smooth solution exists

From Boltzmann to Navier-Stokes

▶ Statement of the result

Theorem : **Let $f_{\varepsilon,in} \in L^1_{loc}(\Omega \times \mathbb{R}^3)$ be a family of initial fluctuations around a global equilibrium M, i.e. such that**

$$\frac{1}{\varepsilon^2} H(f_{\varepsilon,in}|M) \leq C_{in},$$

Let (f_ε) be a family of renormalized solutions to

$$\varepsilon\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon = \frac{1}{\varepsilon} Q(f_\varepsilon, f_\varepsilon) \text{ on } \mathbb{R}^+ \times \Omega \times \mathbb{R}^3,$$
$$f_\varepsilon(0,x,v) = f_{\varepsilon,in}(x,v) \text{ on } \Omega \times \mathbb{R}^3.$$

Then the family (g_ε) defined by $f_\varepsilon = M(1+\varepsilon g_\varepsilon)$ is relatively weakly compact in $L^1_{loc}(dtdx, L^1(Mdv))$; and for any limit point g of (g_ε),

$$g(t,x,v) = u(t,x) \cdot v + \theta(t,x)\frac{|v|^2 - 5}{2}$$

where u is a weak solution to the Navier-Stokes equations (1) and θ satisfies some convection-diffusion equation.

Strategy of the proof : the moment method

- From the relative entropy bound, we deduce that

$$\hat{g}_\varepsilon \rightharpoonup g \text{ in } w - L^2_{loc}(dt, L^2(dxMdv)),$$

up to extraction of a subsequence.

- By the entropy dissipation bound, we have

$$\mathcal{L}\hat{g}_\varepsilon = \frac{\varepsilon}{2}\mathcal{Q}(\hat{g}_\varepsilon, \hat{g}_\varepsilon) - 2\varepsilon\hat{q}_\varepsilon \to 0 \text{ in } L^1_{loc}(dtdx, L^2(Mdv))$$

from which we deduce that

$$g(t,x,v) = \rho(t,x) + u(t,x)\cdot v + \frac{1}{2}\theta(t,x)(|v|^2 - 3).$$

- Passing to the limit in the local conservations of mass and momentum, we get

$$\nabla_x \cdot u = 0, \quad \nabla_x(\rho + \theta) = 0.$$

- The core of the proof is to derive the equations for u and θ.

Start from the **formal conservation laws**

$$\partial_t \int M g_\varepsilon v dv + \nabla_x \cdot \frac{1}{\varepsilon} \int M g_\varepsilon \Phi(v) dv + \nabla_x \left(\frac{1}{3\varepsilon} \int M g_\varepsilon |v|^2 dv \right) = 0,$$

$$\partial_t \int M g_\varepsilon \frac{1}{2}(|v|^2 - 5) dv + \nabla_x \cdot \frac{1}{\varepsilon} \int M g_\varepsilon \Psi(v) dv = 0$$

As Φ, Ψ belong to $\mathrm{Ker}(\mathcal{L}_M)$,

$$\Phi = \mathcal{L}_M \tilde{\Phi}, \quad \Psi = \mathcal{L}_M \tilde{\Psi} \text{ for some } \tilde{\Phi}, \tilde{\Psi} \in \mathrm{Ker}(\mathcal{L}_M)$$

We then use the skew-symmetry of $\mathcal{L}_M$ together with the identity

$$\frac{1}{\varepsilon} M \mathcal{L}_M g_\varepsilon = -v \cdot \nabla_x M g_\varepsilon + Q(M g_\varepsilon, M g_\varepsilon) + O(\varepsilon)$$

$$\partial_t \int M g_\varepsilon v dv + \nabla_x \cdot \int \Big(\underbrace{Q(Mg_\varepsilon, Mg_\varepsilon)}_{\text{convection}} - \underbrace{v \cdot \nabla_x M g_\varepsilon}_{\text{viscous diffusion}} \Big) \tilde{\Phi}(v) dv + \nabla_x p_\varepsilon = O(\varepsilon),$$

$$\partial_t \int M g_\varepsilon \frac{|v|^2 - 5}{2} dv + \nabla_x \cdot \int \Big(\underbrace{Q(Mg_\varepsilon, Mg_\varepsilon)}_{\text{convection}} - \underbrace{v \cdot \nabla_x M g_\varepsilon}_{\text{thermal diffusion}} \Big) \tilde{\Psi}(v) dv = O(\varepsilon).$$

Using the **relaxation estimate** $g_\varepsilon - \Pi g_\varepsilon = O(\varepsilon)$ together with the identity

$$2Q(M\Pi g, M\Pi g) = M\mathcal{L}_M(\Pi g)^2,$$

we get explicit formulas for the convection terms

$$\int Q(Mg_\varepsilon, Mg_\varepsilon)\tilde{\Phi} dv \sim u_\varepsilon^{\otimes 2} - \frac{1}{3}|u_\varepsilon|^2 Id, \quad \int Q(Mg_\varepsilon, Mg_\varepsilon)\tilde{\Psi} dv \sim \frac{5}{2} u_\varepsilon \theta_\varepsilon$$

Taking limits as $\varepsilon \to 0$ and assuming some strong convergence on the moments, we get the **motion and heat equations**.

Convergence of the conservation defects

Because **renormalized solutions** are not known to satisfy the Boltzmann equation in distributional sense, we use

- a truncation of large tails (renormalization)
- a truncation of large velocities

and start from

$$\partial_t \int M g_\varepsilon \gamma_\varepsilon \mathbf{1}_{|v|^2 \le K_\varepsilon} \xi(v) dv + \nabla_x \cdot \int M g_\varepsilon \gamma_\varepsilon \mathbf{1}_{|v|^2 \le K_\varepsilon} v \xi(v) dv = D_\varepsilon(\xi)$$

The first step is therefore to prove that the **conservation defect** $D_\varepsilon(\xi)$ converges to 0 for any collision invariant ξ. We use

- the decomposition of the collision integrand
$$f'_\varepsilon f'_{\varepsilon *} - f_\varepsilon f_{\varepsilon *} = (\sqrt{f'_\varepsilon f'_{\varepsilon *}} - \sqrt{f_\varepsilon f_{\varepsilon *}})^2 + 2(\sqrt{f'_\varepsilon f'_{\varepsilon *}} - \sqrt{f_\varepsilon f_{\varepsilon *}})\sqrt{f_\varepsilon f_{\varepsilon *}}$$
- the bound coming from the entropy dissipation
- some symmetrization based on the invariance $\xi + \xi_* = \xi' + \xi'_*$
- the equiintegrability of $M\hat{g}_\varepsilon^2$ coming from the relaxation estimate and the (x, v)-mixing property (see lecture 2)

Decomposition of the flux terms

The asymptotic behaviour of the **flux terms**

$$\int M g_\varepsilon \gamma_\varepsilon \mathbf{1}_{|v|^2\leq K_\varepsilon} \zeta(v) dv - \frac{1}{2}\int M(\Pi \hat{g}_\varepsilon)^2 \zeta dv + 2\int M\hat{q}_\varepsilon \tilde{\zeta} dv \to 0,$$

comes from

- a suitable decomposition based on the identities

$$\begin{aligned} g_\varepsilon &= \hat{g}_\varepsilon + \varepsilon \hat{g}_\varepsilon^2/4, \\ \frac{1}{\varepsilon} M \mathcal{L}_M \hat{g}_\varepsilon &= \frac{1}{2} Q(M\hat{g}_\varepsilon, M\hat{g}_\varepsilon) - 2M\hat{q}_\varepsilon \end{aligned}$$

  together with the skew-symmetry of $\mathcal{L}_M$
- the equiintegrability of $M\hat{g}_\varepsilon^2(1+|v|^p)$ $(p<2)$
- the relaxation estimate : $\hat{g}_\varepsilon - \Pi\hat{g}_\varepsilon \to 0$ in some weighted L^2 space

The convergence of the **diffusion term** is obtained using the weak compactness on $(\hat{q}_\varepsilon)$ as well as

$$v\cdot\nabla_x g = 2q$$

Filtering of acoustic waves

The convergence of the **convection term** (depending nonlinearly on the moments of $\hat{g}_\varepsilon$) requires some strong convergence

- the spatial regularity comes from averaging lemma
- the regularity with respect to time is valid only for some projections $\mathbb{P}u_\varepsilon$ and $(3\theta_\varepsilon - 2\rho_\varepsilon)/5$

To deal with **acoustic waves**, i.e. with the fast oscillating components $\nabla\psi_\varepsilon = (Id - \mathbb{P})u_\varepsilon$ and $\pi_\varepsilon = 3(\rho_\varepsilon + \theta_\varepsilon)/5$

$$\partial_t \pi_\varepsilon + \frac{1}{\varepsilon}\Delta_x \psi_\varepsilon = o\left(\frac{1}{\varepsilon}\right),$$
$$\partial_t \nabla\psi_\varepsilon + \frac{5}{3\varepsilon}\nabla_x \pi_\varepsilon = \mathrm{o}\left(\frac{1}{\varepsilon}\right),$$

we use some **compensated compactness** argument

$$P\nabla_x \cdot ((\nabla\psi_\varepsilon)^{\otimes 2}) \to 0, \qquad \text{and } \nabla_x \cdot (\pi_\varepsilon \nabla\psi_\varepsilon) \to 0$$

in the sense of distributions on $\mathbb{R}^+ \times \Omega$.

From Boltzmann to Euler

▶ The convergence result for well-prepared initial data

Theorem : **Let $f_{\varepsilon,in} \in L^1_{loc}(\Omega \times \mathbb{R}^3)$ be a family of initial fluctuations around a global equilibrium M, satisfying**

$$\frac{1}{\varepsilon^2} H(f_{\varepsilon,in} | \mathcal{M}_{1,\varepsilon u_{in},1}) \to 0 \text{ as } \varepsilon \to 0,$$

for some given divergence-free vector field $u_{in} \in L^2(\Omega)$.
Let (f_ε) be a family of renormalized solutions to ($q > 1$)

$$\varepsilon \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon = \frac{1}{\varepsilon^q} Q(f_\varepsilon, f_\varepsilon) \text{ on } \mathbb{R}^+ \times \Omega \times \mathbb{R}^3,$$
$$f_\varepsilon(0,x,v) = f_{\varepsilon,in}(x,v) \text{ on } \Omega \times \mathbb{R}^3.$$

Then the family (u_ε) defined by $u_\varepsilon = \varepsilon^{-1} \int f_\varepsilon v dv$ is relatively weakly compact in $L^1_{loc}(dtdx)$, and any limit point u of (u_ε) is a dissipative solution to the incompressible Euler equations (2).

▶ Strategy of the proof : the modulated entropy method

- From the relative entropy bound, we deduce that

$$\hat{g}_\varepsilon \rightharpoonup g \text{ in } w - L^2_{loc}(dt, L^2(dxMdv)),$$

up to extraction of a subsequence.

- By the entropy dissipation bound, we have

$$\mathcal{L}\hat{g}_\varepsilon = \frac{\varepsilon}{2}\mathcal{Q}(\hat{g}_\varepsilon, \hat{g}_\varepsilon) - 2\varepsilon\hat{q}_\varepsilon \to 0 \text{ in } L^1_{loc}(dtdx, L^2(Mdv))$$

from which we deduce that

$$g(t,x,v) = \rho(t,x) + u(t,x)\cdot v + \frac{1}{2}\theta(t,x)(|v|^2 - 3).$$

- Passing to the limit in the local conservation of mass, we get

$$\nabla_x \cdot u = 0$$

- The core of the proof is to establish the **stability inequality**

$$\begin{aligned}
&\frac{1}{\varepsilon^2} H(f_\varepsilon | \mathcal{M}_{1,\varepsilon\tilde{u},1})(t) + \frac{1}{2\varepsilon^{q+3}} \int_0^t \iint D(f_\varepsilon) ds dx \\
&\leq \frac{1}{\varepsilon^2} H(f_{\varepsilon,in} | \mathcal{M}_{1,\varepsilon\tilde{u}_{in}1}) \exp\left(C \int_0^t \|D\tilde{u}(s)\|_{L^\infty(\Omega)} ds \right) \\
&-\frac{1}{\varepsilon} \int_0^t \iint f_\varepsilon (v - \varepsilon\tilde{u}) \cdot A(\tilde{u}) \exp\left(C \int_s^t \|D\tilde{u}(s)\|_{L^\infty(\Omega)} ds \right) dv dx ds
\end{aligned}$$

- The conclusion follows then from some **convexity argument** giving

$$\begin{aligned}
\frac{1}{2} \int (u - \tilde{u})^2 dx \quad &\leq \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon^2} H(\mathcal{M}_{f_\varepsilon} | \mathcal{M}_{1,\varepsilon\tilde{u},1}) \\
&\leq \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon^2} H(f_\varepsilon | \mathcal{M}_{1,\varepsilon\tilde{u},1})
\end{aligned}$$

▶ The modulated entropy inequality

Start from the **entropy inequality** with defect measure :

$$H(f_\varepsilon(t)|M) + \int_{\mathbb{R}^3} \mathrm{Tr}(m_\varepsilon)(t) + \frac{1}{\varepsilon^{q+1}} \int_0^t \iint D(f_\varepsilon)(s,x)dsdx \leq H(f_{\varepsilon,in}|M)$$

By definition of the modulated entropy,

$$H(f_\varepsilon|\mathcal{M}_{1,\varepsilon\tilde{u},1})(t) + \int_{\mathbb{R}^3} \mathrm{Tr}(m_\varepsilon)(t) + \frac{1}{\varepsilon^{q+1}} \int_0^t \int D(f_\varepsilon)dsdx$$
$$\leq H(f_{\varepsilon,in}|\mathcal{M}_{1,\varepsilon\tilde{u}_{in},1}) + \int_0^t \frac{d}{dt} \iint \frac{1}{2}(\varepsilon^2\tilde{u}^2 - 2\varepsilon v\cdot\tilde{u})f_\varepsilon(s,x,v)dvdxds$$

Then use

- the **continuity equation**
- the **local conservation of momentum** with defect measure.

Integrating by parts leads finally to

$$
\begin{aligned}
&H(f_\varepsilon|\mathcal{M}_{1,\varepsilon\tilde{u},1})(t) + \int_{\mathbb{R}^3} \mathrm{Tr}(m_\varepsilon)(t) + \frac{1}{\varepsilon^{q+1}} \int_0^t \int D(f_\varepsilon) ds dx \\
&\leq H(f_{\varepsilon,in}|\mathcal{M}_{1,\varepsilon\tilde{u}_{in},1}) + \int_0^t \iint \varepsilon\partial_t \tilde{u}\cdot(\varepsilon\tilde{u}-v) f_\varepsilon(s,x,v) dv dx ds \\
&- \int_0^t \int \nabla_x \tilde{u} : \left(\int (v-\varepsilon\tilde{u})^{\otimes 2} f_\varepsilon(s,x,v) dv dx + m_\varepsilon(s,x) \right) dx ds \\
&- \int_0^t \int \varepsilon \nabla_x \tilde{u} : \left(\int (v-\varepsilon\tilde{u}) \otimes \tilde{u} f_\varepsilon(s,x,v) dv \right) ds
\end{aligned}
$$

In order to obtain the expected stability inequality, it remains then to control the **flux term**

$$
\begin{aligned}
&\nabla_x \tilde{u} : \frac{1}{\varepsilon^2} \int (v-\varepsilon u)^{\otimes 2} f_\varepsilon dv \\
&= \nabla_x \tilde{u} : \frac{1}{\varepsilon^2} \int \left((v-\varepsilon u)^{\otimes 2} - \frac{1}{3}|v-\varepsilon u|^2 Id \right) f_\varepsilon dv
\end{aligned}
$$

and to apply **Gronwall's lemma**.

▶ Decomposition of the flux term

Because

$$\Phi_\varepsilon = (v - \varepsilon\tilde{u})^{\otimes 2} - \frac{1}{3}|v - \varepsilon\tilde{u}|^2 Id \quad \text{belongs to } (\mathrm{Ker}\,\mathcal{L}_{\mathcal{M}_\varepsilon})^\perp$$

we have

$$\Phi_\varepsilon = \mathcal{L}_{\mathcal{M}_\varepsilon}\tilde{\Phi}_\varepsilon \text{ for some } \tilde{\Phi}_\varepsilon \in (\mathrm{Ker}\,\mathcal{L}_{\mathcal{M}_\varepsilon})^\perp$$

Then, using the identity

$$\frac{1}{\varepsilon}\mathcal{M}_\varepsilon\mathcal{L}_\varepsilon\tilde{g}_\varepsilon = -\frac{2}{\varepsilon^2}Q(\sqrt{\mathcal{M}_\varepsilon f_\varepsilon}, \sqrt{\mathcal{M}_\varepsilon f_\varepsilon}) + \frac{1}{2}Q(\mathcal{M}_\varepsilon\tilde{g}_\varepsilon, \mathcal{M}_\varepsilon\tilde{g}_\varepsilon)$$

together with the skew-symmetry of $\mathcal{L}_{\mathcal{M}_\varepsilon}$, we can prove

$$\begin{aligned} -\frac{1}{2\varepsilon^2}\int_0^t \iint \nabla_x\tilde{u} : \Phi_\varepsilon(f_\varepsilon - \mathcal{M}_\varepsilon)(s,x,v)dvdxds \\ \leq \frac{C}{\varepsilon^2}\int_0^t \|\nabla_x\tilde{u}\|_{L^2\cap L^\infty(\Omega)} H(f_\varepsilon|\mathcal{M}_\varepsilon)(s)ds + o(1) \end{aligned}$$

▶ Some improvements of the modulated entropy method

Under an additional non-uniform estimate on f_ε (that guarantees the local conservation of momentum and energy), we can

Take into account the acoustic waves

- replacing A by some penalized acceleration operator $A_\varepsilon(\tilde\rho, \tilde u, \tilde\theta)$ defined by

$$\begin{pmatrix} \partial_t \tilde\rho + (\tilde u \cdot \nabla_x)\tilde\rho + \frac{1}{\varepsilon}\nabla_x \cdot \tilde u \\ \partial_t \tilde u + (\tilde u \cdot \nabla_x)\tilde u + \left(\frac{e^{\varepsilon\tilde\theta} - 1}{\varepsilon}\right)\nabla_x\left(\tilde\rho - \frac{3}{2}\tilde\theta\right) + \frac{1}{\varepsilon}\nabla_x(\tilde\rho + \tilde\theta) \\ \partial_t \tilde\theta + (\tilde u \cdot \nabla_x)\tilde\theta + \frac{2}{3\varepsilon}\nabla_x \cdot \tilde u \end{pmatrix}$$

- building approximate solutions to the acoustic equations $A_\varepsilon(\tilde\rho, \tilde u, \tilde\theta) = 0$

Take into account the initial kinetic layer

- modulating also the entropy dissipation

$$\begin{aligned} D(f_\varepsilon|f_{app}) &= \frac{1}{4}\iint (f'_\varepsilon f'_{\varepsilon 1} - f_\varepsilon f_{\varepsilon 1}) \log\left(\frac{f'_\varepsilon f'_{\varepsilon 1} f_{app} f_{app1}}{f_\varepsilon f_{\varepsilon 1} f'_{app} f'_{app1}}\right) \\ &\quad - (f'_{app} f'_{app1} - f_{app} f_{app1})\left(\frac{f'_\varepsilon f'_{\varepsilon 1}}{f'_{app} f'_{app1}} - \frac{f_\varepsilon f_{\varepsilon 1}}{f_{app} f_{app1}}\right) b dv dv_1 d\sigma \end{aligned}$$

- building approximate solutions to the relaxation equation in the initial layer

$$\partial_t f = \frac{1}{\varepsilon^{q+1}} Q(f,f)$$

- using the previous argument outside from the initial layer