Mathematical tools for the study of hydrodynamic limits

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Notations

- Nondimensional form of the Boltzmann equation

$$\mathrm{Ma}\partial_t f + v \cdot \nabla_x f = \frac{1}{\mathrm{Kn}} Q(f,f)$$

- Fluctuations around a global equilibrium M

$$f = M(1 + \mathrm{Ma}\, g)$$

controlled by the relative entropy

$$H(f|M) = \iint \left(f \log \frac{f}{M} - f + M \right) dv dx \leq C \mathrm{Ma}^2$$

- Perturbative form of the Boltzmann equation

$$\mathrm{Ma}\partial_t g + v \cdot \nabla_x g = -\frac{1}{\mathrm{Kn}} \mathcal{L} g + \frac{\mathrm{Ma}}{\mathrm{Kn}} \mathcal{Q}(g,g)$$

Physical a priori estimates

▶ The entropy inequality

Starting from

- the local conservation of mass, momentum and energy
- the local entropy inequality

and integrating by parts using

- Maxwell's boundary condition with accomodation coefficient α

we get formally the **entropy inequality**

$$H(f|M)(t) + \frac{1}{\mathrm{KnMa}} \int_0^t \int_\Omega D(f)(s,x)dsdx + \frac{\alpha}{\mathrm{Ma}} \int_0^t \int_{\partial\Omega} E(f|M)(s,x)d\sigma_x ds$$

$$\leq H(f_{in}|M) \leq C\mathrm{Ma}^2$$

(which will be actually satisfied even for very weak solutions of the Boltzmann equation)

The three controlled quantities are

- the relative entropy

$$H(f|M) = \iint M h(\mathrm{Mag}) dv dx \quad \text{with } h(z) = (1+z)\log(1+z) - z$$

- the entropy dissipation

$$\begin{aligned} D(f) &= -\int Q(f,f) \log f dv \\ &= \frac{1}{4} \int f f_* r\left(\frac{f' f'_*}{f f_*} - 1\right) b dv dv_* d\omega \quad \text{with } r(z) = z\log(1+z) \end{aligned}$$

- the Darrozès-Guiraud information

$$E(f|M) = \frac{1}{\sqrt{2\pi}} \langle h(\mathrm{Mag}) \quad - h\left(\langle \mathrm{Mag} \rangle_{\partial\Omega}\right) \rangle_{\partial\Omega}$$

$$\text{with } \langle G \rangle_{\partial\Omega} \stackrel{\text{def}}{=} \int G M \sqrt{2\pi} (v \cdot n(x))_+ dv$$

▶ The relative entropy

The relative entropy bound

$$\iint Mh(\mathrm{Ma}g)dvdx \leq C\mathrm{Ma}^2$$

controls the size of the fluctuation.

- By Young's inequality

$$(1+|v|^2)g = O(1)_{L^\infty_t(L^1_{loc}(dx;L^1(Mdv)))}.$$

- Heuristically

$$h(z) \sim_{z\to 0} \frac{1}{2}z^2$$

so that we expect g to be almost in $L^\infty_t(L^2(dxMdv))$.

- We therefore define the **renormalized fluctuation**

$$\hat{g} = \frac{2}{\mathrm{Ma}}(\sqrt{1 + \mathrm{Ma}g} - 1)\,.$$

The functional inequality

$$\frac{1}{2}h(z) \geq (\sqrt{1+z} - 1)^2\,, \quad \forall z > -1$$

implies that

$$\hat{g} = O(1)_{L^\infty_t(L^2(dxMdv))}.$$

That refined a priori estimate will be used together with the identity

$$g = \hat{g} + \frac{1}{4}\mathrm{Ma}\hat{g}^2.$$

▶ The entropy dissipation

The bound on the entropy dissipation

$$\frac{1}{4}\int_0^t \int \int ff_* r\left(\frac{f'f'_*}{ff_*}-1\right) bdvdv_* d\omega dxds \leq C\mathrm{Ma}^3\mathrm{Kn}$$

controls some renormalized collision integral.

The functional inequality

$$(x-y)\log\frac{x}{y} \geq 4(\sqrt{x}-\sqrt{y})^2\,, \quad x,y>0$$

coupled with the Cauchy-Schwarz inequality, implies indeed

$$\begin{aligned}\hat{q} &= \frac{1}{\sqrt{\mathrm{Ma}^3\mathrm{Kn}}}\frac{1}{M}Q(\sqrt{Mf},\sqrt{Mf}) \\ &= O(1)_{L^2_{loc}(dt,L^2(M\nu^{-1}dvdx)}\end{aligned}$$

Remark : In order to control the relaxation process, we will further need estimates on the nonlinearity based on the continuity properties of Q and bounds on g.

▶ The Darrozès-Guiraud information

The bound on the boundary term

$$\int_0^t \int_{\partial\Omega} \langle h(\mathrm{Ma}g) - h\left(\langle \mathrm{Ma}g\rangle_{\partial\Omega}\right)\rangle_{\partial\Omega} d\sigma_x ds \leq C\frac{\mathrm{Ma}^3}{\alpha}$$

controls the variation of the trace in v.

By Taylor's formula (with cancellation of the first order), one indeed has

$$\begin{aligned}\hat\eta &= 2\sqrt{\frac{\alpha}{\mathrm{Ma}^3}}\left(\sqrt{1+\mathrm{Ma}g} - \sqrt{\langle 1+\mathrm{Ma}g\rangle_{\partial\Omega}}\right)\\ &= O(1)_{L^2_{loc}(dt, L^2(M(v\cdot n(x))_+ d\sigma_x dv))}\end{aligned}$$

Remark : In order to control the trace $g_{|\partial\Omega}$, we will further need estimates coming from the inside, on g and on $v \cdot \nabla_x g$.

Additional integrability in v coming from the relaxation

- Control of the relaxation

The fundamental identity

From the bilinearity of Q and the definition of $\hat{g}$, we have obviously

$$\begin{aligned} \mathcal{L}\hat{g} &= \frac{\mathrm{Ma}}{2}\mathcal{Q}(\hat{g},\hat{g}) - \frac{2}{\mathrm{Ma}}\frac{1}{M}Q(\sqrt{Mf},\sqrt{Mf}) \\ &= \frac{\mathrm{Ma}}{2}\mathcal{Q}(\hat{g},\hat{g}) - 2\sqrt{\mathrm{Ma}\mathrm{Kn}}\hat{q} \end{aligned}$$

For simplicity, we assume that ν is bounded from up and below.
Else we would have to use some truncated $\tilde{b}$, $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{Q}}$

Control of the quadratic term

By the continuity of $\mathcal{Q} : L^2(Mdv) \times L^2(M\nu dv) \to L^2(M\nu^{-1}dv)$
and the L^2 bound on $\hat{g}$, we get

$$\frac{\mathrm{Ma}}{2}\mathcal{Q}(\hat{g},\hat{g}) = O(\mathrm{Ma})_{L^\infty_t(L^1_x(L^2(Mdv))}$$

Control coming from the entropy dissipation
By the entropy dissipation bound,

$$2\sqrt{\mathrm{MaKn}}\hat{q} = O(\sqrt{\mathrm{MaKn}})_{L^2_{loc}(dt, L^2(dxMdv))}$$

The relaxation estimate
From the coercivity inequality for $\mathcal{L}$

$$\int g\mathcal{L}_M g(v)M(v)dv \geq C\|g - \Pi g\|^2_{L^2(M\nu dv)}\,.$$

we then deduce

$$\hat{g} - \Pi\hat{g} = O(\mathrm{Ma})_{L^\infty_t(L^1_x(L^2(Mdv))} + O(\sqrt{\mathrm{MaKn}})_{L^2_{loc}(dt, L^2(dxMdv))}$$

▶ Control of large velocities

By Young's inequality

$$\begin{aligned}(1+|v|^p)^2|\hat{g}|^2 &\leq \frac{\delta^2}{\mathrm{Ma}^2}|\mathrm{Ma}g|\frac{(1+|v|^p)^2}{\delta^2}\\ &\leq \frac{\delta^2}{\mathrm{Ma}^2}\left(h(\mathrm{Ma}g)+h^*\left(\frac{(1+|v|^p)^2}{\delta^2}\right)\right)\end{aligned}$$

Therefore, for any $\delta > 0$, $p < 1$, $q < +\infty$

$$(1+|v|^p)|\hat{g}| = O(\delta)_{L^\infty_t(L^2(Mdvdx))} + O\left(\frac{C_{\delta,q}}{\mathrm{Ma}}\right)_{L^\infty_{t,x}(L^q(Mdv))}$$

Remark : for $p = 1$ one can actually obtain a bound.

▶ Moments and equiintegrability in v

From the decomposition

$$\hat{g} = (\hat{g} - \Pi\hat{g}) + \Pi\hat{g}$$

we deduce that for $r < 2$, $q < +\infty$, $p < 1$

$$\begin{aligned}(1+|v|^p)^2|\hat{g}|^2 &= (1+|v|^{2p})\hat{g}\Pi\hat{g} + (1+|v|^{2p})(\hat{g}-\Pi\hat{g})\hat{g} \\ &= O(1)_{L^\infty_t(L^1_x(L^r(Mdv))} + (1+|v|^p)|\hat{g}-\Pi\hat{g}|O(\delta)_{L^\infty_t(L^2(Mdvdx))} \\ &\quad +(1+|v|^p)|\hat{g}-\Pi\hat{g}|O\left(\frac{C_{\delta,q}}{\mathrm{Ma}}\right)_{L^\infty_{t,x}(L^q(Mdv))}\end{aligned}$$

By the relaxation estimate, choosing δ sufficiently small, we get

$$(1+|v|^p)^2|\hat{g}|^2 = O(1)_{L^1_{loc}(dtdx,L^1(Mdv))} \text{ uniformly integrable in } v.$$

Additional integrability in x coming from the free transport

In viscous regime, we further use properties of the free-transport equation

$$\mathrm{Ma}\partial_t g + v \cdot \nabla_x g = S \tag{1}$$

- The free transport is the prototype of **hyperbolic operators**

$$g(t,x,v) = g_{in}(x - \mathrm{Ma}tv, v) + \int_0^t S(x - \mathrm{Ma}sv, v, t-s)ds$$

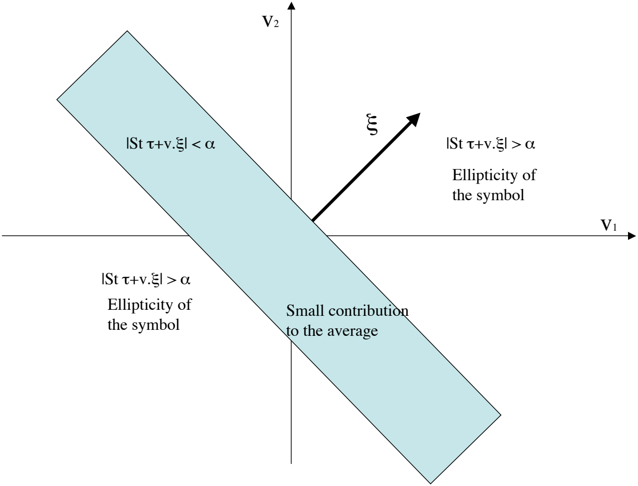
No regularizing effect on g. Propagation of singularities at finite speed.

- **Ellipticity of the symbol** outside from a small subset of $\mathbb{R}^3_v$

$$a(\tau, \xi, v) = i(\mathrm{Ma}\tau + v \cdot \xi)$$

Regularity in x of the averages $\int g\varphi(v)dv$ (moments).

Averaging properties



Theorem [L^2 averaging lemma] (Golse, Lions, Perthame, Sentis) :
Let $g \in L^2_{t,x,v}$ be the solution of the transport equation (1). Then, for all $\varphi \in L^\infty(\mathbb{R}^3_v)$

$$\left\| \int g\varphi(v)dv \right\|_{L^2(\mathbb{R}_t, H_x^{1/2})} \leq C_\varphi \|g\|^{1/2}_{L^2_{t,x,v}} \|S\|^{1/2}_{L^2_{t,x,v}}.$$

Sketch of the proof

- Take Fourier transform
- Split the integral into two contributions
- Estimate each contribution with the Cauchy-Schwarz inequality
- Optimize with respect to α

Can be extended to L^p spaces with $1 < p < \infty$.

Remark 1 : Because of concentration phenomena, velocity averaging fails in L^1 and L^∞ (as proved by the following counterexample).

Consider (S_n) bounded in $L^1_{t,x,v}$ such that

$$S_n \to \mathrm{St}\chi'(t)\delta_{x-\mathrm{Ma}^{-1}v_0 t} \otimes \delta_{v-v_0}$$

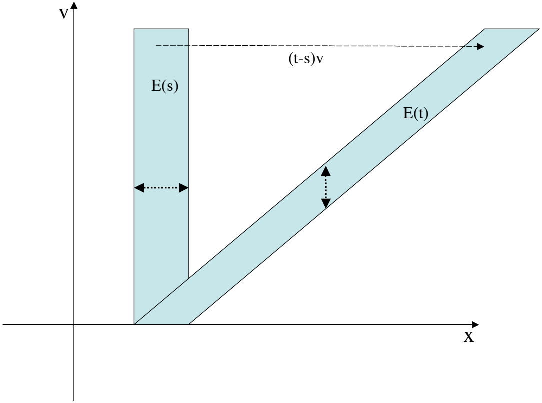
Let (f_n) be the corresponding solutions to (1). Then,

$$\int_{\mathbb{R}^3} f_n \varphi(v) dv \rightharpoonup \rho \text{ in } \mathcal{M}_{t,x},$$

$$\text{support}(\rho) \subset \mathbb{R} \times \mathbb{R}_+ v_0 \,.$$

Remark 2 : It is actually sufficient to control the concentration effects in v (non concentration in x will follow automatically).

- Mixing properties



A set of "small measure in x" becomes a set of "small measure in v"

Theorem [dispersion lemma] (Castella, Perthame) :
Let χ be the solution to

$$\partial_s \chi + \mathrm{Ma} \partial_t \chi + v \cdot \nabla_x \chi = 0.$$

Then, for all $(p,q) \in [1,+\infty]$ with $p \leq q$,

$$\forall s \in \mathbb{R}^*, \quad \|\chi(s)\|_{L^\infty_t(L^q_x(L^p_v))} \leq |s|^{-3\left(\frac{1}{p}-\frac{1}{q}\right)} \|\chi_{|s=0}\|_{L^\infty_t(L^p_x(L^q_v))}.$$

Sketch of the proof

- Start from the formula of characteristics
- Use the change of variables $v \mapsto x - vs$
- Conclude by interpolation with the conservation of mass

Coupled with Green's formula, and with a suitable choice of the parameter s, that gives the expected mixing property.

Combined with classical averaging results, it provides some criterion (equiintegrability in v) to get strong compactness of the moments in L^1.

▶ Control of the free transport

In viscous regime $\mathrm{Ma} \sim \mathrm{Kn}$, we can prove that

$$(\mathrm{Ma}\partial_t + v \cdot \nabla_x) \frac{\sqrt{f/M + \mathrm{Ma}^a} - 1}{\mathrm{Ma}}$$
$$= O(\mathrm{Ma}^{2-a/2})_{L^1(dtdxMdv)} + O(1)_{L^2(dtdx\nu^{-1}Mdv)} + O(\mathrm{Ma})_{L^1_{loc}(dtdx, L^2(\nu^{-1}Mdv))}$$

As the squareroot is not an admissible renormalization, we start from

$$(\mathrm{Ma}\partial_t + v \cdot \nabla_x) \frac{\sqrt{f/M + \mathrm{Ma}^a} - 1}{\mathrm{Ma}}$$
$$= \frac{1}{2\mathrm{Kn}\mathrm{Ma}} \frac{1}{\sqrt{f + \mathrm{Ma}^a M}\sqrt{M}} \iint \left(\sqrt{f'f'_*} - \sqrt{ff_*}\right)^2 b(v - v_*, \sigma) d\sigma dv_*$$
$$+ \frac{1}{\mathrm{Kn}\mathrm{Ma}} \frac{\sqrt{f}}{\sqrt{f + \mathrm{Ma}^a M}\sqrt{M}} \iint \left(\sqrt{f'f'_*} - \sqrt{ff_*}\right) \sqrt{f_*} b(v - v_*, \sigma) d\sigma dv_*$$

The L^2 bound on $\hat{q}$ (coming from the entropy dissipation) gives

$$\|Q^1\|_{L^1(dtdxMdv)} \leq \frac{1}{2} C_{in} \mathrm{Ma}^{2-a/2}.$$

The weighted L^2 bound on $\hat{g}$ implies

$$Q^2 = O\left(\sqrt{\frac{\mathrm{Ma}}{\mathrm{Kn}}}\right)_{L^2(dtdx\nu^{-1}Mdv)} + O\left(\mathrm{Ma}\sqrt{\frac{\mathrm{Ma}}{\mathrm{Kn}}}\right)_{L^1_{loc}(dtdx, L^2(\nu^{-1}Mdv))}.$$

Remark : In inviscid regime $\mathrm{Kn} << \mathrm{Ma}$, there is no bound on the transport, and consequently no a priori regularity estimate on the moments.

Combined with the comparison estimate

$$\left(\frac{\sqrt{f/M+\mathrm{Ma}^a}-1}{\mathrm{Ma}}\right)^2-\hat{g}^2= \quad O(\mathrm{Ma}^{a-1})_{L^2_{loc}(dtdx,L^2((1+|v|^p)Mdv))}$$

$$+O(\mathrm{Ma}^{a/2})_{L^2_{loc}(dtdx,L^1((1+|v|^p)Mdv))}.$$

it will provide the convenient control to get

- the equiintegrability with respect to x of

$$M\hat{g}^2(1+|v|^p)$$

- the spatial regularity of the moments

$$\int M\hat{g}\varphi(v)dv$$