A constitutive model for non-reacting binary mixtures

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Double diffusive convection

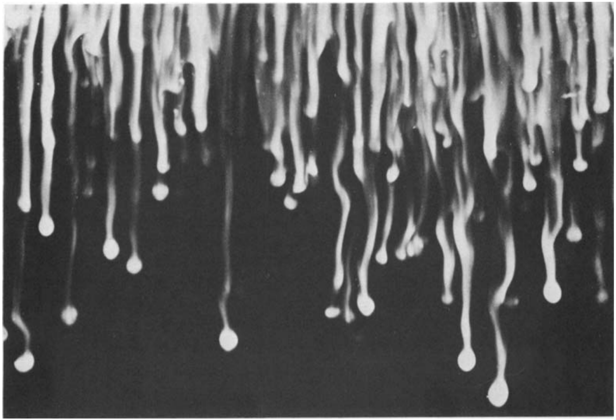

FIGURE 1. A field of salt fingers formed by setting up a stable temperature gradient and pouring a little salt solution on top. The downward-moving fingers were made visible by adding fluorescein to the salt and lighting through a slit from below.

Huppert & Turner (1981)

Mixture theory – Truesdell

Idea of co-occupancy:

v_{s}	velocity
v_{w}	velocity
ρ_{s}	density
ρ_{w}	density
θ_{s}, e_{s}	temperature/internal energy
θ_{w}, e_{w}	temperature/internal energy

Table: Unknowns in problems for binary mixtures.

Eckart(1940), Truesdell (1957), Samohýl (1987), Rajagopal & Tao (1995)

Governing equations

Governing equations for components - non-reacting binary mixture of non-polar constituents

$$
\begin{aligned}
\frac{\mathrm{d}_{\mathrm{s}}\rho_{\mathrm{s}}}{\mathrm{d}t} &= -\rho_{\mathrm{s}} \operatorname{div} \mathbf{v}_{\mathrm{s}}, \\
\frac{\mathrm{d}_{\mathrm{w}}\rho_{\mathrm{w}}}{\mathrm{d}t} &= -\rho_{\mathrm{w}} \operatorname{div} \mathbf{v}_{\mathrm{w}}, \\
\rho_{\mathrm{s}} \frac{\mathrm{d}_{\mathrm{s}}\mathbf{v}_{\mathrm{s}}}{\mathrm{d}t} &= \operatorname{div} \mathbb{T}_{\mathrm{s}} + \rho_{\mathrm{s}} \mathbf{b} + \boldsymbol{\pi}, \\
\rho_{\mathrm{w}} \frac{\mathrm{d}_{\mathrm{w}}\mathbf{v}_{\mathrm{w}}}{\mathrm{d}t} &= \operatorname{div} \mathbb{T}_{\mathrm{w}} + \rho_{\mathrm{w}} \mathbf{b} - \boldsymbol{\pi} \\
\rho_{\mathrm{s}} \frac{\mathrm{d}_{\mathrm{s}}}{\mathrm{dt}} \left(e_{\mathrm{s}} + \frac{1}{2} |\mathbf{v}_{\mathrm{s}}|^2 \right) &= \operatorname{div} (\mathbb{T}_{\mathrm{s}} \mathbf{v}_{\mathrm{s}}) - \operatorname{div} \mathbf{q}_{\mathrm{s}} + q_{\mathrm{s}} + e^i + \rho_{\mathrm{s}} \mathbf{b}_{\mathrm{s}} \bullet \mathbf{v}_{\mathrm{s}} \\
\rho_{\mathrm{w}} \frac{\mathrm{d}_{\mathrm{w}}}{\mathrm{dt}} \left(e_{\mathrm{w}} + \frac{1}{2} |\mathbf{v}_{\mathrm{w}}|^2 \right) &= \operatorname{div} (\mathbb{T}_{\mathrm{w}} \mathbf{v}_{\mathrm{w}}) - \operatorname{div} \mathbf{q}_{\mathrm{w}} + q_{\mathrm{w}} - e^i + \rho_{\mathrm{w}} \mathbf{b}_{\mathrm{w}} \bullet \mathbf{v}_{\mathrm{w}}
\end{aligned}
$$

Governing equations

Governing equations for components - non-reacting binary mixture of non-polar constituents

$$
\begin{aligned}
\frac{\mathrm{d}_{\mathrm{s}}\rho_{\mathrm{s}}}{\mathrm{d}t} &= -\rho_{\mathrm{s}}\,\mathrm{div}\,\mathbf{v}_{\mathrm{s}}, \\
\frac{\mathrm{d}_{\mathrm{w}}\rho_{\mathrm{w}}}{\mathrm{d}t} &= -\rho_{\mathrm{w}}\,\mathrm{div}\,\mathbf{v}_{\mathrm{w}}, \\
\rho_{\mathrm{s}}\frac{\mathrm{d}_{\mathrm{s}}\mathbf{v}_{\mathrm{s}}}{\mathrm{d}t} &= \mathrm{div}\,\mathbb{T}_{\mathrm{s}} + \rho_{\mathrm{s}}\mathbf{b} + \boldsymbol{\pi}, \\
\rho_{\mathrm{w}}\frac{\mathrm{d}_{\mathrm{w}}\mathbf{v}_{\mathrm{w}}}{\mathrm{d}t} &= \mathrm{div}\,\mathbb{T}_{\mathrm{w}} + \rho_{\mathrm{w}}\mathbf{b} - \boldsymbol{\pi} \\
\rho_{\mathrm{s}}\frac{\mathrm{d}_{\mathrm{s}}e_{\mathrm{s}}}{\mathrm{d}\mathrm{t}} &= \mathbb{T}_{\mathrm{s}} : \nabla\mathbf{v}_{\mathrm{s}} - \mathrm{div}\,\mathbf{q}_{\mathrm{s}} + q_{\mathrm{s}} + e^{i} - \boldsymbol{\pi}\bullet\mathbf{v}_{\mathrm{s}} + \rho_{\mathrm{s}}\mathbf{b}_{\mathrm{s}}\bullet\mathbf{v}_{\mathrm{s}} \\
\rho_{\mathrm{w}}\frac{\mathrm{d}_{\mathrm{w}}e_{\mathrm{w}}}{\mathrm{d}\mathrm{t}} &= \mathbb{T}_{\mathrm{w}} : \nabla\mathbf{v}_{\mathrm{w}} - \mathrm{div}\,\mathbf{q}_{\mathrm{w}} + q_{\mathrm{w}} - e^{i} + \boldsymbol{\pi}\bullet\mathbf{v}_{\mathrm{w}} + \rho_{\mathrm{w}}\mathbf{b}_{\mathrm{w}}\bullet\mathbf{v}_{\mathrm{w}}
\end{aligned}
$$

Single-component form

- We formulate balance laws for the mixture as a whole, rewrite them in a single-component form in terms of mixture properties and identify additional balances for complementary quantities .
- Mixture components properties $\rho_{\mathrm{s}},\ \rho_{\mathrm{w}},\ \mathbf{v}_{\mathrm{s}},\ \mathbf{v}_{\mathrm{w}},\ e_{\mathrm{s}},\ e_{\mathrm{w}}$
- Mixture properties

$$
\begin{aligned}
\rho &:= \rho_{\mathrm{s}} + \rho_{\mathrm{w}} \\
\rho\mathbf{v} &:= \rho_{\mathrm{s}}\mathbf{v}_{\mathrm{s}} + \rho_{\mathrm{w}}\mathbf{v}_{\mathrm{w}} \\
\rho e &:= \rho_{\mathrm{s}} e_{\mathrm{s}} + \rho_{\mathrm{w}} e_{\mathrm{w}} + \frac{1}{2}\rho_{\mathrm{s}}\mathbf{u}_{\mathrm{s}} \bullet \mathbf{u}_{\mathrm{s}} + \frac{1}{2}\rho_{\mathrm{w}}\mathbf{u}_{\mathrm{w}} \bullet \mathbf{u}_{\mathrm{w}}
\end{aligned}
$$

  where $\mathbf{u}_{\mathrm{s}} := \mathbf{v}_{\mathrm{s}} - \mathbf{v}$, $\mathbf{u}_{\mathrm{w}} := \mathbf{v}_{\mathrm{w}} - \mathbf{v}$.
- Complementary properties (only c and $\mathbf{J}_{\mathrm{s}}$, same temperature "$e_{\mathrm{s}} - e_{\mathrm{w}}$" not needed)

$$
\begin{aligned}
c &:= \frac{\rho_{\mathrm{s}}}{\rho} \\
\mathbf{J}_{\mathrm{s}} &:= \rho_{\mathrm{s}}(\mathbf{v}_{\mathrm{s}} - \mathbf{v})
\end{aligned}
$$

Governing equations in "new" variables

$$
\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} &= -\rho \operatorname{div} \mathbf{v} \\
\rho \frac{\mathrm{d}c}{\mathrm{d}t} &= -\operatorname{div} \mathbf{J}_{\mathrm{s}} \\
\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= \operatorname{div} \mathbb{T} + \rho \mathbf{b} \\
\frac{\mathrm{d}\mathbf{J}_{\mathrm{s}}}{\mathrm{d}t} &= -\mathbf{J}_{\mathrm{s}} \bullet (\nabla \mathbf{v} - (\operatorname{div} \mathbf{v})\, \mathbb{I}) - \operatorname{div} \left(\frac{1}{\rho} \left(\frac{1}{c} - \frac{1}{1-c} \right) \mathbf{J}_{\mathrm{s}} \otimes \mathbf{J}_{\mathrm{s}} \right) \\
&+ \operatorname{div} \left((1-c)\, \mathbb{T}_{\mathrm{s}} - c \mathbb{T}_{\mathrm{w}} \right) + \mathbb{T} \nabla c + \boldsymbol{\pi} \\
\rho \frac{\mathrm{d}e}{\mathrm{d}t} &= -\operatorname{div} \mathbf{q} + \mathbb{T} : \mathbb{D}(\mathbf{v}) + q
\end{aligned}
$$

Governing equations in "new" variables

$$
\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} &= -\rho \operatorname{div} \mathbf{v} \\
\rho \frac{\mathrm{d}c}{\mathrm{d}t} &= -\operatorname{div} \mathbf{J}_{\mathrm{s}} \\
\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= \operatorname{div} \mathbb{T} + \rho \mathbf{b} \\
\frac{\mathrm{d}\mathbf{J}_{\mathrm{s}}}{\mathrm{d}t} &= -\mathbf{J}_{\mathrm{s}} \bullet (\nabla \mathbf{v} - (\operatorname{div} \mathbf{v}) \mathbb{I}) - \operatorname{div} \left(\frac{1}{\rho} \left(\frac{1}{c} - \frac{1}{1-c} \right) \mathbf{J}_{\mathrm{s}} \otimes \mathbf{J}_{\mathrm{s}} \right) \\
&+ \operatorname{div} \left((1-c) \mathbb{T}_{\mathrm{s}} - c \mathbb{T}_{\mathrm{w}} \right) + \mathbb{T} \nabla c + \boldsymbol{\pi} \\
\rho \frac{\mathrm{d}e}{\mathrm{d}t} &= -\operatorname{div} \mathbf{q} + \mathbb{T} : \mathbb{D}(\mathbf{v}) + q
\end{aligned}
$$

with

$$
\mathbb{T} := \mathbb{T}_{\mathrm{s}} + \mathbb{T}_{\mathrm{w}} - \rho_{\mathrm{s}} \mathbf{u}_{\mathrm{s}} \otimes \mathbf{u}_{\mathrm{s}} - \rho_{\mathrm{w}} \mathbf{u}_{\mathrm{w}} \otimes \mathbf{u}_{\mathrm{w}}
$$

Governing equations in "new" variables

$$
\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} &= -\rho\,\mathrm{div}\,\mathbf{v} \\
\rho\frac{\mathrm{d}c}{\mathrm{d}t} &= -\,\mathrm{div}\,\mathbf{J}_\mathrm{s} \\
\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= \mathrm{div}\,\mathbb{T} + \rho\mathbf{b} \\
\frac{\mathrm{d}\mathbf{J}_\mathrm{s}}{\mathrm{d}t} &= -\mathbf{J}_\mathrm{s}\bullet(\nabla\mathbf{v} - (\mathrm{div}\,\mathbf{v})\,\mathbb{I}) - \mathrm{div}\left(\frac{1}{\rho}\left(\frac{1}{c} - \frac{1}{1-c}\right)\mathbf{J}_\mathrm{s}\otimes\mathbf{J}_\mathrm{s}\right) \\
&+ \mathrm{div}\left((1-c)\,\mathbb{T}_\mathrm{s} - c\mathbb{T}_\mathrm{w}\right) + \mathbb{T}\nabla c + \boldsymbol{\pi} \\
\rho\frac{\mathrm{d}e}{\mathrm{d}t} &= -\,\mathrm{div}\,\mathbf{q} + \mathbb{T} : \mathbb{D}(\mathbf{v}) + q
\end{aligned}
$$

with

$$
\mathbb{T} := \mathbb{T}_\mathrm{s} + \mathbb{T}_\mathrm{w} - \frac{1}{\rho c\,(1-c)}\mathbf{J}_\mathrm{s}\otimes\mathbf{J}_\mathrm{s}
$$

Governing equations in "new" variables

$$\rho \frac{\mathrm{d}e}{\mathrm{d}t} = -\operatorname{div}\mathbf{q} + \mathbb{T} : \mathbb{D}(\mathbf{v}) + q$$

where

$$\begin{aligned}
\mathbf{q} &:= \mathbf{J}_{\mathrm{e}} + \mathbf{J}_{\mathrm{e_k}} + \mathbf{J}_{\mathrm{q}} + \mathbf{J}_{\mathbb{T}} \\
\mathbf{J}_{\mathrm{e}} &:= \rho_{\mathrm{s}} e_{\mathrm{s}} \mathbf{u}_{\mathrm{s}} + \rho_{\mathrm{w}} e_{\mathrm{w}} \mathbf{u}_{\mathrm{w}} \\
\mathbf{J}_{\mathrm{e_k}} &:= \frac{1}{2}\rho_{\mathrm{s}} |\mathbf{u}_{\mathrm{s}}|^2 \mathbf{u}_{\mathrm{s}} + \frac{1}{2}\rho_{\mathrm{w}} |\mathbf{u}_{\mathrm{w}}|^2 \mathbf{u}_{\mathrm{w}} \\
\mathbf{J}_{\mathrm{q}} &:= \mathbf{q}_{\mathrm{s}} + \mathbf{q}_{\mathrm{w}} \\
\mathbf{J}_{\mathbb{T}} &:= -\mathbb{T}_{\mathrm{s}} \mathbf{u}_{\mathrm{s}} - \mathbb{T}_{\mathrm{w}} \mathbf{u}_{\mathrm{w}} \\
q &:= q_s + q_w
\end{aligned}$$

Governing equations in "new" variables

$$\rho \frac{\mathrm{d}e}{\mathrm{d}t} = -\operatorname{div}\mathbf{q} + \mathbb{T} : \mathbb{D}(\mathbf{v}) + q$$

where

$$\begin{aligned}
\mathbf{q} &:= \mathbf{J}_{\mathrm{e}} + \mathbf{J}_{\mathrm{e_k}} + \mathbf{J}_{\mathrm{q}} + \mathbf{J}_{\mathbb{T}} \\
\mathbf{J}_{\mathrm{e}} &:= \mathbf{J}_{\mathrm{s}}(e_{\mathrm{s}} - e_{\mathrm{w}}) \\
\mathbf{J}_{\mathrm{e_k}} &:= \frac{1}{2}\frac{1-2c}{\rho^2(c(1-c))^2}(\mathbf{J}_{\mathrm{s}} \bullet \mathbf{J}_{\mathrm{s}})\mathbf{J}_{\mathrm{s}} \\
\mathbf{J}_{\mathrm{q}} &:= \mathbf{q}_{\mathrm{s}} + \mathbf{q}_{\mathrm{w}} \\
\mathbf{J}_{\mathbb{T}} &:= -((1-c)\mathbb{T}_{\mathrm{s}} - c\mathbb{T}_{\mathrm{w}})\left(\frac{\mathbf{J}_{\mathrm{s}}}{\rho c(1-c)}\right) \\
q &:= q_s + q_w
\end{aligned}$$

Entropy balance

$$
\begin{aligned}
\rho_{\mathrm{s}} \frac{\mathrm{d}_{\mathrm{s}} \eta_{\mathrm{s}}}{\mathrm{d}t} + \operatorname{div} \mathbf{J}_{\eta_{\mathrm{s}}} &= \xi_{\mathrm{s}} + \zeta \\
\rho_{\mathrm{w}} \frac{\mathrm{d}_{\mathrm{w}} \eta_{\mathrm{w}}}{\mathrm{d}t} + \operatorname{div} \mathbf{J}_{\eta_{\mathrm{w}}} &= \xi_{\mathrm{w}} - \zeta
\end{aligned}
$$

$$
\boxed{\rho \frac{\mathrm{d}\eta}{\mathrm{d}t} + \operatorname{div} \mathbf{J}_{\eta} = \xi}
$$

provided we define

$$
\begin{aligned}
\rho\eta &:= \rho_{\mathrm{s}}\eta_{\mathrm{s}} + \rho_{\mathrm{w}}\eta_{\mathrm{w}} \\
\mathbf{J}_{\eta} &:= \mathbf{J}_{\eta_{\mathrm{s}}} + \mathbf{J}_{\eta_{\mathrm{w}}} + (\eta_{\mathrm{s}} - \eta_{\mathrm{w}})\mathbf{J}_{\mathrm{s}} \\
\xi &:= \xi_{\mathrm{s}} + \xi_{\mathrm{w}} \; .
\end{aligned}
$$

The "minimal" constitutive model

- From the energy and entropy balance

$$
\begin{aligned}
\rho e &= \rho_{\mathrm{s}} e_{\mathrm{s}} + \rho_{\mathrm{w}} e_{\mathrm{w}} + \frac{1}{2}\rho_{\mathrm{s}} \mathbf{u}_{\mathrm{s}} \bullet \mathbf{u}_{\mathrm{s}} + \frac{1}{2}\rho_{\mathrm{w}} \mathbf{u}_{\mathrm{w}} \bullet \mathbf{u}_{\mathrm{w}} \\
\rho \eta &:= \rho_{\mathrm{s}} \eta_{\mathrm{s}} + \rho_{\mathrm{w}} \eta_{\mathrm{w}}
\end{aligned}
$$

The "minimal" constitutive model

- From the energy and entropy balance

$$
\begin{aligned}
\rho e &= \rho c e_{\mathrm{s}} + \rho(1-c)e_{\mathrm{w}} + \frac{1}{2}\frac{|\mathbf{J}_{\mathrm{s}}|^2}{\rho c(1-c)} \\
\rho\eta &:= \rho c \eta_{\mathrm{s}} + \rho(1-c)\eta_{\mathrm{w}}
\end{aligned}
$$

The "minimal" constitutive model

- From the energy and entropy balance

$$\begin{aligned} \rho e &= \rho c e_{\mathrm{s}} + \rho(1-c) e_{\mathrm{w}} + \frac{1}{2}\frac{|\mathbf{J}_{\mathrm{s}}|^2}{\rho c(1-c)} \\ \rho\eta &:= \rho c \eta_{\mathrm{s}} + \rho(1-c)\eta_{\mathrm{w}} \end{aligned}$$

- Constitutive assumption:

$$\begin{aligned} \rho e(\rho, c, \eta, \mathbf{J}_{\mathrm{s}}) &= \rho c e_{\mathrm{s}}(\rho, c, \eta) + \rho(1-c) e_{\mathrm{w}}(\rho, c, \eta) + \frac{1}{2}\frac{|\mathbf{J}_{\mathrm{s}}|^2}{\rho c(1-c)} \\ \rho\eta(\rho, c, e) &:= \rho_{\mathrm{s}}\eta_{\mathrm{s}}(\rho, c, e) + \rho_{\mathrm{w}}\eta_{\mathrm{w}}(\rho, c, e) \end{aligned}$$

The "minimal" constitutive model - Helmholz potential

- Defining absolute temperature

$$\vartheta := \left.\frac{\partial \hat{e}}{\partial \eta}\right|_{c,\rho,\mathbf{J}_{\mathrm{s}}}$$

- Defining the Helmholz potential

$$\begin{aligned} \Psi_{\mathrm{s}} &:= e_{\mathrm{s}} - \vartheta \eta_{\mathrm{s}} \\ \Psi_{\mathrm{w}} &:= e_{\mathrm{w}} - \vartheta \eta_{\mathrm{w}} \\ \Psi &:= e - \vartheta \eta \end{aligned}$$

- our constitutive assumptions imply

$$\Psi(\rho, c, \vartheta, \mathbf{J}_{\mathrm{s}}) = c\Psi_{\mathrm{s}}(\rho, c, \vartheta) + (1-c)\Psi_{\mathrm{w}}(\rho, c, \vartheta) + \frac{1}{2}\frac{|\mathbf{J}_{\mathrm{s}}|^2}{\rho^2 c(1-c)}$$

- This is one of the results of theory of fluid mixtures of Müller (1968).

The "minimal" constitutive model

Using

$$\mu := \left(\frac{\partial \Psi}{\partial c} \bigg|_{\rho,\vartheta,\mathbf{J}_\mathrm{s}} \right) \qquad p := \rho^2 \left(\frac{\partial \Psi}{\partial \rho} \bigg|_{c,\vartheta,\mathbf{J}_\mathrm{s}} \right) \qquad \left(\frac{\partial \Psi}{\partial \vartheta} \bigg|_{\rho,c,\mathbf{J}_\mathrm{s}} \right) = -\eta$$

We may now express

$$\begin{aligned} \rho\vartheta \frac{\mathrm{d}\eta}{\mathrm{d}t} &= \rho \frac{\mathrm{d}e}{\mathrm{d}t} - \rho \left(\frac{\partial \Psi}{\partial \rho} \frac{\mathrm{d}\rho}{\mathrm{d}t} + \frac{\partial \Psi}{\partial c} \frac{\mathrm{d}c}{\mathrm{d}t} + \frac{\partial \Psi}{\partial \mathbf{J}_\mathrm{s}} \bullet \frac{\mathrm{d}\mathbf{J}_\mathrm{s}}{\mathrm{d}t} \right) \\ &= -\operatorname{div} \mathbf{q} + \mathbb{T} : \mathbb{D}(\mathbf{v}) + p \operatorname{div} \mathbf{v} + \mu \operatorname{div} \mathbf{J}_\mathrm{s} - \frac{\mathbf{J}_\mathrm{s}}{\rho c(1-c)} \bullet \frac{\mathrm{d}\mathbf{J}_\mathrm{s}}{\mathrm{d}t} \ , \end{aligned}$$

$$
\begin{aligned}
\rho\vartheta\frac{\mathrm{d}\eta}{\mathrm{d}t} &= \left(p + \frac{4}{3}\frac{|\mathbf{J}_\mathrm{s}|^2}{\rho c(1-c)} + \frac{1}{3}\mathrm{tr}\mathbb{T}\right)\mathrm{div}\,\mathbf{v} + \left(\mathbb{T}^\delta + \left(\frac{\mathbf{J}_\mathrm{s}\otimes\mathbf{J}_\mathrm{s}}{\rho c(1-c)}\right)^\delta\right) : \mathbb{D}^\delta(\mathbf{v}) \\
&+ \left(((1-c)\mathbb{T}_\mathrm{s} - c\mathbb{T}_\mathrm{w}) - \frac{\mathbf{J}_\mathrm{s}\otimes\mathbf{J}_\mathrm{s}}{\rho}\left(\frac{1}{c} - \frac{1}{1-c}\right)\right) : \nabla\left(\frac{\mathbf{J}_\mathrm{s}}{\rho c(1-c)}\right) \\
&- (\boldsymbol{\pi} + \rho c(1-c)\nabla\mu + \mathbb{T}\nabla c)\bullet\left(\frac{\mathbf{J}_\mathrm{s}}{\rho c(1-c)}\right) \\
&- \mathrm{div}\left(\mathbf{J}_\mathrm{q} + \mathbf{J}_\mathrm{e} - \mu\mathbf{J}_\mathrm{s} - \mathbf{J}_{\mathrm{e}_\mathrm{k}}\right)
\end{aligned}
$$

$$
\begin{aligned}
\rho\vartheta\frac{\mathrm{d}\eta}{\mathrm{d}t} &= \left(p + \frac{4}{3}\frac{|\mathbf{J}_{\mathrm{s}}|^2}{\rho c(1-c)} + \frac{1}{3}\mathrm{tr}\mathbb{T}\right)\operatorname{div}\mathbf{v} + \left(\mathbb{T}^{\delta} + \left(\frac{\mathbf{J}_{\mathrm{s}}\otimes\mathbf{J}_{\mathrm{s}}}{\rho c(1-c)}\right)^{\delta}\right) : \mathbb{D}^{\delta}(\mathbf{v}) \\
&+ \left(((1-c)\mathbb{T}_{\mathrm{s}} - c\mathbb{T}_{\mathrm{w}}) - \frac{\mathbf{J}_{\mathrm{s}}\otimes\mathbf{J}_{\mathrm{s}}}{\rho}\left(\frac{1}{c} - \frac{1}{1-c}\right)\right) : \nabla\left(\frac{\mathbf{J}_{\mathrm{s}}}{\rho c(1-c)}\right) \\
&- \left(\boldsymbol{\pi} + \rho c(1-c)\nabla\mu + \mathbb{T}\nabla c\right)\bullet\left(\frac{\mathbf{J}_{\mathrm{s}}}{\rho c(1-c)}\right) \\
&- \operatorname{div}\left(\mathbf{J}_{\mathrm{q}} + \mathbf{J}_{\mathrm{e}} - \mu\mathbf{J}_{\mathrm{s}} - \mathbf{J}_{\mathrm{e_k}}\right) \\
\nabla\mu &= \left(\left.\frac{\partial\mu}{\partial\rho}\right|_{c,\vartheta,\frac{\mathbf{J}_{\mathrm{s}}}{\rho c(1-c)}}\right)\nabla\rho + \left(\left.\frac{\partial\mu}{\partial c}\right|_{\rho,\vartheta,\frac{\mathbf{J}_{\mathrm{s}}}{\rho c(1-c)}}\right)\nabla c + \left(\left.\frac{\partial\mu}{\partial\vartheta}\right|_{c,\rho,\frac{\mathbf{J}_{\mathrm{s}}}{\rho c(1-c)}}\right)\nabla\vartheta \\
&+ \nabla\left(\frac{\mathbf{J}_{\mathrm{s}}}{\rho c(1-c)}\right)\bullet\left(\left.\frac{\partial\mu}{\partial\frac{\mathbf{J}_{\mathrm{s}}}{\rho c(1-c)}}\right|_{c,\rho,\vartheta}\right)
\end{aligned}
$$

$$
\begin{aligned}
\rho\vartheta\frac{\mathrm{d}\eta}{\mathrm{d}t} &= \left(p + \frac{4}{3}\frac{|\mathbf{J}_\mathrm{s}|^2}{\rho c(1-c)} + \frac{1}{3}\mathrm{tr}\mathbb{T}\right)\operatorname{div}\mathbf{v} + \left(\mathbb{T}^\delta + \left(\frac{\mathbf{J}_\mathrm{s}\otimes\mathbf{J}_\mathrm{s}}{\rho c(1-c)}\right)^\delta\right) : \mathbb{D}^\delta(\mathbf{v}) \\
&+ \left(((1-c)\mathbb{T}_\mathrm{s} - c\mathbb{T}_\mathrm{w}) - \frac{\mathbf{J}_\mathrm{s}\otimes\mathbf{J}_\mathrm{s}}{\rho}\left(\frac{1}{c} - \frac{1}{1-c}\right)\right) : \nabla\left(\frac{\mathbf{J}_\mathrm{s}}{\rho c(1-c)}\right) \\
&- (\boldsymbol{\pi} + \rho c(1-c)\nabla\mu + \mathbb{T}\nabla c)\bullet\left(\frac{\mathbf{J}_\mathrm{s}}{\rho c(1-c)}\right) \\
&- \operatorname{div}\left(\mathbf{J}_\mathrm{q} + \mathbf{J}_\mathrm{e} - \mu\mathbf{J}_\mathrm{s} - \mathbf{J}_{\mathrm{e_k}}\right) \\
\nabla\mu &= \left(\left.\frac{\partial\mu}{\partial\rho}\right|_{c,\vartheta,\frac{\mathbf{J}_\mathrm{s}}{\rho c(1-c)}}\right)\nabla\rho + \left(\left.\frac{\partial\mu}{\partial c}\right|_{\rho,\vartheta,\frac{\mathbf{J}_\mathrm{s}}{\rho c(1-c)}}\right)\nabla c + \left(\left.\frac{\partial\mu}{\partial\vartheta}\right|_{c,\rho,\frac{\mathbf{J}_\mathrm{s}}{\rho c(1-c)}}\right)\nabla\vartheta \\
&- \nabla\left(\frac{\mathbf{J}_\mathrm{s}}{\rho c(1-c)}\right)\bullet\frac{1-2c}{\rho c(1-c)}\mathbf{J}_\mathrm{s}
\end{aligned}
$$

Our model thus suggests to identify

$$
\begin{aligned}
\vartheta\xi \;=\;& \left(p + \frac{4}{3}\frac{|\mathbf{J}_{\mathrm{s}}|^2}{\rho c(1-c)} + \frac{1}{3}\mathrm{tr}\mathbb{T}\right)\operatorname{div}\mathbf{v} + \left(\mathbb{T}^{\delta} + \left(\frac{\mathbf{J}_{\mathrm{s}}\otimes\mathbf{J}_{\mathrm{s}}}{\rho c(1-c)}\right)^{\delta}\right) : \mathbb{D}^{\delta}(\mathbf{v}) \\
+\;& \left((1-c)\mathbb{T}_{\mathrm{s}} - c\mathbb{T}_{\mathrm{w}}\right) : \mathbb{D}\left(\frac{\mathbf{J}_{\mathrm{s}}}{\rho c(1-c)}\right) \\
-\;& \left(\boldsymbol{\pi} + \rho c(1-c)\left(\frac{\partial\mu}{\partial\rho}\nabla\rho + \frac{\partial\mu}{\partial c}\nabla c\right) + \mathbb{T}\nabla c\right)\bullet\left(\frac{\mathbf{J}_{\mathrm{s}}}{\rho c(1-c)}\right) \\
-\;& \left(\frac{(\mathbf{J}_{\mathrm{q}} + \mathbf{J}_{\mathrm{e}} - \mu\mathbf{J}_{\mathrm{s}} - \mathbf{J}_{\mathrm{e_k}})}{\vartheta}\right)\bullet\nabla\vartheta\,,
\end{aligned}
$$

and

$$
\mathbf{J}_{\eta} = \frac{\mathbf{J}_{\mathrm{q}} + \mathbf{J}_{\mathrm{e}} - \mu\mathbf{J}_{\mathrm{s}} - \mathbf{J}_{\mathrm{e_k}}}{\vartheta}\;.
$$

Our model thus suggests to identify

$$
\begin{aligned}
\vartheta\xi \;=\;& \left(p + \frac{4}{3}\frac{|\mathbf{J}_{\mathrm{s}}|^2}{\rho c(1-c)} + \frac{1}{3}\mathrm{tr}\mathbb{T}\right)\mathrm{div}\,\mathbf{v} + \left(\mathbb{T}_{\mathrm{s}} + \mathbb{T}_{\mathrm{w}}\right)^{\delta} : \mathbb{D}^{\delta}(\mathbf{v}) \\
+\;& \left((1-c)\mathbb{T}_{\mathrm{s}} - c\mathbb{T}_{\mathrm{w}}\right) : \mathbb{D}(\mathbf{u}_{\mathrm{s}} - \mathbf{u}_{\mathrm{w}}) \\
-\;& \left(\boldsymbol{\pi} + \rho c(1-c)\left(\frac{\partial\mu}{\partial\rho}\nabla\rho + \frac{\partial\mu}{\partial c}\nabla c\right) + \mathbb{T}\nabla c\right) \bullet (\mathbf{u}_{\mathrm{s}} - \mathbf{u}_{\mathrm{w}}) \\
-\;& \left(\frac{(\mathbf{J}_{\mathrm{q}} + \mathbf{J}_{\mathrm{e}} - \mu\mathbf{J}_{\mathrm{s}} - \mathbf{J}_{\mathrm{e_k}})}{\vartheta} + \rho c(1-c)\frac{\partial\mu}{\partial\vartheta}\frac{\mathbf{J}_{\mathrm{s}}}{\rho c(1-c)}\right) \bullet \nabla\vartheta\ ,
\end{aligned}
$$

and

$$
\mathbf{J}_{\eta} = \frac{\mathbf{J}_{\mathrm{q}} + \mathbf{J}_{\mathrm{e}} - \mu\mathbf{J}_{\mathrm{s}} - \mathbf{J}_{\mathrm{e_k}}}{\vartheta}\ .
$$

Our model thus suggests to identify

$$
\begin{aligned}
\vartheta\xi \quad = \quad & \left(p + \frac{4}{3}\frac{|\mathbf{J}_{\mathrm{s}}|^2}{\rho c(1-c)} + \frac{1}{3}\mathrm{tr}\mathbb{T}\right) \operatorname{div}\mathbf{v} + \left(\mathbb{T}_{\mathrm{s}} + \mathbb{T}_{\mathrm{w}}\right)^{\delta} : \mathbb{D}^{\delta}(\mathbf{v}) \\
+ \quad & \left((1-c)\mathbb{T}_{\mathrm{s}} - c\mathbb{T}_{\mathrm{w}}\right) : \mathbb{D}\left(\frac{\mathbf{J}_{\mathrm{s}}}{\rho c(1-c)}\right) \\
- \quad & \left(\boldsymbol{\pi} + \rho c(1-c)\left(\frac{\partial\mu}{\partial\rho}\nabla\rho + \frac{\partial\mu}{\partial c}\nabla c\right) + \mathbb{T}\nabla c\right) \bullet \left(\frac{\mathbf{J}_{\mathrm{s}}}{\rho c(1-c)}\right) \\
- \quad & \left(\frac{(\mathbf{J}_{\mathrm{q}} + \mathbf{J}_{\mathrm{e}} - \mu\mathbf{J}_{\mathrm{s}} - \mathbf{J}_{\mathrm{e_k}})}{\vartheta} + \rho c(1-c)\frac{\partial\mu}{\partial\vartheta}\frac{\mathbf{J}_{\mathrm{s}}}{\rho c(1-c)}\right) \bullet \nabla\vartheta \ ,
\end{aligned}
$$

and

$$
\mathbf{J}_{\eta} = \frac{\mathbf{J}_{\mathrm{q}} + \mathbf{J}_{\mathrm{e}} - \mu\mathbf{J}_{\mathrm{s}} - \mathbf{J}_{\mathrm{e_k}}}{\vartheta} \ .
$$

Our model thus suggests to identify

$$\begin{aligned}
\vartheta\xi &= \left(p + \frac{4}{3}\frac{|\mathbf{J}_s|^2}{\rho c(1-c)} + \frac{1}{3}\mathrm{tr}\mathbb{T}\right)\operatorname{div}\mathbf{v} + \left(\mathbb{T}_s + \mathbb{T}_w\right)^\delta : \mathbb{D}^\delta(\mathbf{v}) \\
&+ \left((1-c)\mathbb{T}_s - c\mathbb{T}_w\right) : \mathbb{D}\left(\frac{\mathbf{J}_s}{\rho c(1-c)}\right) \\
&- \left(\boldsymbol{\pi} + \rho c(1-c)\left(\frac{\partial\mu}{\partial\rho}\nabla\rho + \frac{\partial\mu}{\partial c}\nabla c + \frac{\partial\mu}{\partial\vartheta}\nabla\vartheta\right) + \mathbb{T}\nabla c\right) \bullet \left(\frac{\mathbf{J}_s}{\rho c(1-c)}\right) \\
&- \left(\frac{(\mathbf{J}_q + \mathbf{J}_e - \mu\mathbf{J}_s - \mathbf{J}_{e_k})}{\vartheta}\right) \bullet \nabla\vartheta\ ,
\end{aligned}$$

and

$$\mathbf{J}_\eta = \frac{\mathbf{J}_q + \mathbf{J}_e - \mu\mathbf{J}_s - \mathbf{J}_{e_k}}{\vartheta}\ .$$

Our model thus suggests to identify

$$
\begin{aligned}
\vartheta\xi \quad = \quad & \left(p + \frac{4}{3}\frac{|\mathbf{J}_\mathrm{s}|^2}{\rho c(1-c)} + \frac{1}{3}\mathrm{tr}\mathbb{T}\right)\mathrm{div}\,\mathbf{v} + \left(\mathbb{T}_\mathrm{s} + \mathbb{T}_\mathrm{w}\right)^\delta : \mathbb{D}^\delta(\mathbf{v}) \\
+ \quad & \left((1-c)\mathbb{T}_\mathrm{s} - c\mathbb{T}_\mathrm{w}\right) : \mathbb{D}\left(\frac{\mathbf{J}_\mathrm{s}}{\rho c(1-c)}\right) \\
- \quad & \left(\boldsymbol{\pi} + \rho c(1-c)\left(\frac{\partial \mu}{\partial \rho}\nabla\rho + \frac{\partial \mu}{\partial c}\nabla c + \alpha\frac{\partial \mu}{\partial \vartheta}\nabla\vartheta\right) + \mathbb{T}\nabla c\right) \bullet \left(\frac{\mathbf{J}_\mathrm{s}}{\rho c(1-c)}\right) \\
- \quad & \left(\frac{(\mathbf{J}_\mathrm{q} + \mathbf{J}_\mathrm{e} - \mu\mathbf{J}_\mathrm{s} - \mathbf{J}_{\mathrm{e}_\mathrm{k}})}{\vartheta} + (1-\alpha)\rho c(1-c)\frac{\partial \mu}{\partial \vartheta}\frac{\mathbf{J}_\mathrm{s}}{\rho c(1-c)}\right) \bullet \nabla\vartheta \,,
\end{aligned}
$$

and

$$
\mathbf{J}_\eta = \frac{\mathbf{J}_\mathrm{q} + \mathbf{J}_\mathrm{e} - \mu\mathbf{J}_\mathrm{s} - \mathbf{J}_{\mathrm{e}_\mathrm{k}}}{\vartheta} \,.
$$

- Let us postulate the rate of entropy production as

$$\begin{aligned}\vartheta\xi &= \frac{2\nu+3\lambda}{3}(\operatorname{div}\mathbf{v})^2 \\ &+ 2\nu\mathbb{D}^d(\mathbf{v}):\mathbb{D}^d(\mathbf{v}) + 2\tilde{\nu}\mathbb{D}\left(\frac{\mathbf{J}_\mathrm{s}}{\rho c(1-c)}\right):\mathbb{D}\left(\frac{\mathbf{J}_\mathrm{s}}{\rho c(1-c)}\right) \\ &+ k\left(\frac{\mathbf{J}_\mathrm{s}}{\rho c(1-c)}\right)\bullet\left(\frac{\mathbf{J}_\mathrm{s}}{\rho c(1-c)}\right) + \frac{\kappa}{\vartheta}\nabla\vartheta\bullet\nabla\vartheta \geq 0\end{aligned}$$

  for $\nu, \tilde{\nu} \geq 0, 2\nu+3\lambda \geq 0, k \geq 0, \kappa \geq 0$.

- Let us employ the postulate of maximization of rate of entropy production with respect to the thermodynamic affinities (see e.g. Rajagopal & Srinivasa (2004)) $\left(\operatorname{div}\mathbf{v}, \mathbb{D}^d(\mathbf{v}), \mathbb{D}\left(\frac{\mathbf{J}_\mathrm{s}}{\rho c(1-c)}\right), \frac{\mathbf{J}_\mathrm{s}}{\rho c(1-c)}, \nabla\vartheta\right)$

Final constitutive relations

$$
\begin{aligned}
p + \frac{4}{3}\frac{|\mathbf{J}_{\mathrm{s}}|^2}{\rho c(1-c)} + \frac{1}{3}\mathrm{tr}\mathbb{T} &= \frac{2\nu + 3\lambda}{3}\,\mathrm{div}\,\mathbf{v} \\
\mathbb{T}^{\delta} + \left(\frac{\mathbf{J}_{\mathrm{s}} \otimes \mathbf{J}_{\mathrm{s}}}{\rho c(1-c)}\right)^{\delta} &= 2\nu\mathbb{D}^{\delta}(\mathbf{v}) \\
((1-c)\mathbb{T}_{\mathrm{s}} - c\mathbb{T}_{\mathrm{w}}) &= 2\tilde{\nu}\mathbb{D}\left(\frac{\mathbf{J}_{\mathrm{s}}}{\rho c(1-c)}\right) \\
\boldsymbol{\pi} + \rho c(1-c)\left(\frac{\partial\mu}{\partial\rho}\nabla\rho + \frac{\partial\mu}{\partial c}\nabla c + \alpha\frac{\partial\mu}{\partial\vartheta}\nabla\vartheta\right) + \mathbb{T}\nabla c &= -k\mathbf{J}_{\mathrm{s}} \\
\frac{(\mathbf{J}_{\mathrm{q}} + \mathbf{J}_{\mathrm{e}} - \mu\mathbf{J}_{\mathrm{s}} - \mathbf{J}_{\mathrm{e_k}})}{\vartheta} + (1-\alpha)\rho c(1-c)\frac{\partial\mu}{\partial\vartheta}\frac{\mathbf{J}_{\mathrm{s}}}{\rho c(1-c)} &= -\frac{\kappa}{\vartheta}\nabla\vartheta
\end{aligned}
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\mathbb{T}_{\mathrm{s}}^{\delta} + \mathbb{T}_{\mathrm{w}}^{\delta} &= 2\nu\mathbb{D}^{\delta}(\mathbf{v}) \\
((1-c)\mathbb{T}_{\mathrm{s}} - c\mathbb{T}_{\mathrm{w}}) &= 2\tilde{\nu}\mathbb{D}\left(\mathbf{u}_{\mathrm{s}} - \mathbf{u}_{\mathrm{w}}\right) \\
\boldsymbol{\pi} + \rho c(1-c)\left(\frac{\partial\mu}{\partial\rho}\nabla\rho + \frac{\partial\mu}{\partial c}\nabla c + \alpha\frac{\partial\mu}{\partial\vartheta}\nabla\vartheta\right) + \mathbb{T}\nabla c &= -k\mathbf{J}_{\mathrm{s}} \\
\frac{(\mathbf{J}_{\mathrm{q}} + \mathbf{J}_{\mathrm{e}} - \mu\mathbf{J}_{\mathrm{s}} - \mathbf{J}_{\mathrm{e_k}})}{\vartheta} + (1-\alpha)\rho c(1-c)\frac{\partial\mu}{\partial\vartheta}\frac{\mathbf{J}_{\mathrm{s}}}{\rho c(1-c)} &= -\frac{\kappa}{\vartheta}\nabla\vartheta
\end{aligned}
$$

Heat and entropy fluxes

- Let us inspect the final form of heat flux $\mathbf{q}$ and entropy flux $\mathbf{J}_\eta$

$$\begin{aligned}
\mathbf{q} &= -\kappa\nabla\vartheta + \mathbf{J}_{\mathrm{e}} + \mathbf{J}_{\mathrm{e_k}} + \mathbf{J}_{\mathbb{T}} \\
&+ \left(c\frac{\partial e_{\mathrm{s}}}{\partial c} + (1-c)\frac{\partial e_{\mathrm{w}}}{\partial c} - \alpha\vartheta\frac{\partial}{\partial c}\left(c\eta_{\mathrm{s}} + (1-c)\eta_{\mathrm{w}}\right)\right)\mathbf{J}_{\mathrm{s}} \\
\mathbf{J}_\eta &= \frac{1}{\vartheta}\left(\mathbf{q} - 2\mathbf{J}_{\mathrm{e_k}} - \mathbf{J}_{\mathbb{T}} - \mu\mathbf{J}_{\mathrm{s}}\right) \\
&= -\frac{\kappa}{\vartheta}\nabla\vartheta - (1-\alpha)\frac{\partial\mu}{\partial\vartheta}\mathbf{J}_{\mathrm{s}}
\end{aligned}$$

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\mathbf{J}_\eta &= \frac{1}{\vartheta}\left(\mathbf{q} - 2\mathbf{J}_{\mathrm{e_k}} - \mathbf{J}_{\mathbb{T}} - \mu\mathbf{J}_{\mathrm{s}}\right) \\
&= -\frac{\kappa}{\vartheta}\nabla\vartheta + (1-\alpha)\frac{\partial}{\partial c}\left(c\eta_{\mathrm{s}} + (1-c)\eta_{\mathrm{w}}\right)\mathbf{J}_{\mathrm{s}}
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&= -\frac{\kappa}{\vartheta}\nabla\vartheta + (1-\alpha)\frac{\partial}{\partial c}\left(c\eta_{\mathrm{s}} + (1-c)\eta_{\mathrm{w}}\right)\mathbf{J}_{\mathrm{s}}
\end{aligned}$$

- Choice of α?: $\alpha = 0$

Heat and entropy fluxes

- Let us inspect the final form of heat flux $\mathbf{q}$ and entropy flux $\mathbf{J}_\eta$

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\mathbf{q} &= -\kappa\nabla\vartheta + \mathbf{J}_{\mathrm{e}} + \mathbf{J}_{\mathrm{e_k}} + \mathbf{J}_{\mathbb{T}} \\
&+ \left(c\frac{\partial e_{\mathrm{s}}}{\partial c} + (1-c)\frac{\partial e_{\mathrm{w}}}{\partial c}\right)\mathbf{J}_{\mathrm{s}} \\
\mathbf{J}_\eta &= \frac{1}{\vartheta}\left(\mathbf{q} - 2\mathbf{J}_{\mathrm{e_k}} - \mathbf{J}_{\mathbb{T}} - \mu\mathbf{J}_{\mathrm{s}}\right) \\
&= -\frac{\kappa}{\vartheta}\nabla\vartheta + (\eta_{\mathrm{s}} - \eta_{\mathrm{w}})\mathbf{J}_{\mathrm{s}} + \left(c\frac{\partial \eta_{\mathrm{s}}}{\partial c} + (1-c)\frac{\partial \eta_{\mathrm{w}}}{\partial c}\right)\mathbf{J}_{\mathrm{s}}
\end{aligned}$$

- Choice of α?: $\alpha = 0$
- satisfactory - no entropy contribution in heat flux $\mathbf{q}$ + correct diffusive term present in entropy flux $\mathbf{J}_\eta$

Heat and entropy fluxes

- Let us inspect the final form of heat flux $\mathbf{q}$ and entropy flux $\mathbf{J}_\eta$

$$
\begin{aligned}
\mathbf{q} &= -\kappa\nabla\vartheta + \mathbf{J}_{\mathrm{e}} + \mathbf{J}_{\mathrm{e_k}} + \mathbf{J}_{\mathbb{T}} \\
&+ \left(c\frac{\partial e_{\mathrm{s}}}{\partial c} + (1-c)\frac{\partial e_{\mathrm{w}}}{\partial c} \right) \mathbf{J}_{\mathrm{s}} \\
\mathbf{J}_\eta &= \frac{1}{\vartheta}\left(\mathbf{q} - 2\mathbf{J}_{\mathrm{e_k}} - \mathbf{J}_{\mathbb{T}} - \mu\mathbf{J}_{\mathrm{s}}\right) \\
&= -\frac{\kappa}{\vartheta}\nabla\vartheta + (\eta_{\mathrm{s}} - \eta_{\mathrm{w}})\mathbf{J}_{\mathrm{s}} + \left(c\frac{\partial \eta_{\mathrm{s}}}{\partial c} + (1-c)\frac{\partial \eta_{\mathrm{w}}}{\partial c} \right) \mathbf{J}_{\mathrm{s}}
\end{aligned}
$$

- Of particular interest is the quantity $\vartheta\mathbf{J}_\eta - \mathbf{q}$

Comparison $\vartheta \mathbf{J}_\eta - \mathbf{q}$

- Our result

$$\vartheta \mathbf{J}_\eta - \mathbf{q} = -\mathbf{J}_{\mathrm{e_k}} - \mathbf{J}_{\mathbb{T}} - \frac{\partial}{\partial c}(c\Psi_{\mathrm{s}} + (1-c)\Psi_{\mathrm{w}})\mathbf{J}_{\mathrm{s}}$$

- Fluid Mixture theory of Bowen (1976) (entropy flux postulated)

$$\vartheta \mathbf{J}_\eta - \mathbf{q} = -\mathbf{J}_{\mathrm{e_k}} - \mathbf{J}_{\mathbb{T}} - (\Psi_{\mathrm{s}} - \Psi_{\mathrm{w}})\mathbf{J}_{\mathrm{s}}$$

- Müller (1984) theory for mixtures of non-viscous fluids (entropy flux derived)

$$\vartheta \mathbf{J}_\eta - \mathbf{q} = -\mathbf{J}_{\mathrm{e_k}} - \mathbf{J}_{\mathbb{T}} - (\Psi_{\mathrm{s}} - \Psi_{\mathrm{w}})\mathbf{J}_{\mathrm{s}}$$

Evolution equation for $\mathbf{J}_\mathrm{s}$

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{J}_\mathrm{s}}{\mathrm{d}t} &= -\mathbf{J}_\mathrm{s} \bullet \left(\nabla\mathbf{v} - (\operatorname{div}\mathbf{v})\,\mathbb{I}\right) - \operatorname{div}\left(\frac{1}{\rho}\left(\frac{1}{c} - \frac{1}{1-c}\right)\mathbf{J}_\mathrm{s}\otimes\mathbf{J}_\mathrm{s}\right)\\
&+ \operatorname{div}\left((1-c)\,\mathbb{T}_\mathrm{s} - c\mathbb{T}_\mathrm{w}\right) + \mathbb{T}\nabla c + \boldsymbol{\pi}
\end{aligned}
$$

Evolution equation for $\mathbf{J}_\mathrm{s}$

$$\frac{\mathrm{d}\mathbf{J}_\mathrm{s}}{\mathrm{d}t} = -\mathbf{J}_\mathrm{s} \bullet (\nabla \mathbf{v} - (\operatorname{div} \mathbf{v})\,\mathbb{I}) - \operatorname{div}\left(\frac{1}{\rho}\left(\frac{1}{c} - \frac{1}{1-c}\right)\mathbf{J}_\mathrm{s} \otimes \mathbf{J}_\mathrm{s}\right) + \operatorname{div}\left((1-c)\,\mathbb{T}_\mathrm{s} - c\mathbb{T}_\mathrm{w}\right) + \mathbb{T}\nabla c + \boldsymbol{\pi}$$

Using the constitutive relations for $\boldsymbol{\pi} + \mathbb{T}\nabla c$ and $(1-c)\mathbb{T}_\mathrm{s} - c\mathbb{T}_\mathrm{w}$:

$$\frac{\mathrm{d}\mathbf{J}_\mathrm{s}}{\mathrm{d}t} = -\mathbf{J}_\mathrm{s} \bullet (\nabla \mathbf{v} - (\operatorname{div} \mathbf{v})\,\mathbb{I}) + \operatorname{div}\left(\frac{1}{\rho}\left(\frac{1}{c} - \frac{1}{1-c}\right)\mathbf{J}_\mathrm{s} \otimes \mathbf{J}_\mathrm{s}\right) + \operatorname{div}\left(2\tilde{\nu}\nabla_\mathrm{sym}\left(\frac{\mathbf{J}_\mathrm{s}}{\rho c(1-c)}\right)\right) - \rho c(1-c)\left(\frac{\partial \mu}{\partial \rho}\nabla\rho + \frac{\partial \mu}{\partial c}\nabla c\right) - k\mathbf{J}_\mathrm{s}$$

Evolution equation for $\mathbf{J}_{\mathrm{s}}$

$$\begin{aligned}\frac{\mathrm{d}\mathbf{J}_{\mathrm{s}}}{\mathrm{d}t} &= -\mathbf{J}_{\mathrm{s}} \bullet (\nabla \mathbf{v} - (\operatorname{div} \mathbf{v})\,\mathbb{I}) - \operatorname{div}\left(\frac{1}{\rho}\left(\frac{1}{c} - \frac{1}{1-c}\right)\mathbf{J}_{\mathrm{s}} \otimes \mathbf{J}_{\mathrm{s}}\right) \\ &+ \operatorname{div}\left((1-c)\,\mathbb{T}_{\mathrm{s}} - c\mathbb{T}_{\mathrm{w}}\right) + \mathbb{T}\nabla c + \boldsymbol{\pi}\end{aligned}$$

Using the constitutive relations for $\boldsymbol{\pi} + \mathbb{T}\nabla c$ and $(1-c)\mathbb{T}_{\mathrm{s}} - c\mathbb{T}_{\mathrm{w}}$:

$$\begin{aligned}\frac{\mathrm{d}\mathbf{J}_{\mathrm{s}}}{\mathrm{d}t} &= -\mathbf{J}_{\mathrm{s}} \bullet (\nabla \mathbf{v} - (\operatorname{div} \mathbf{v})\,\mathbb{I}) + \operatorname{div}\left(\frac{1}{\rho}\left(\frac{1}{c} - \frac{1}{1-c}\right)\mathbf{J}_{\mathrm{s}} \otimes \mathbf{J}_{\mathrm{s}}\right) \\ &+ \operatorname{div}\left(2\tilde{\nu}\nabla_{\mathrm{sym}}\left(\frac{\mathbf{J}_{\mathrm{s}}}{\rho c(1-c)}\right)\right) - \rho c(1-c)\left(\frac{\partial \mu}{\partial \rho}\nabla\rho + \frac{\partial \mu}{\partial c}\nabla c\right) - k\mathbf{J}_{\mathrm{s}}\end{aligned}$$

Provided fast relaxation $\frac{d\mathbf{J}_{\mathrm{s}}}{dt} \sim 0$ and neglecting non-linear terms in $\mathbf{v}$,$\mathbf{J}_{\mathrm{s}}$, and assuming $\tilde{\nu} \sim 0$, one obtains

$$k\mathbf{J}_{\mathrm{s}} \sim -\rho c(1-c)\left(\frac{\partial \mu}{\partial \rho}\nabla\rho + \frac{\partial \mu}{\partial c}\nabla c\right)$$

Conclusions

- Key features of the presented model: reformulation in traditional variables ρ, c, $\mathbf{v}$, ϑ and also $\mathbf{J}_{\mathrm{s}}$, for all of them evolution equations are available - equivalent to 2 mass balances, 2 momentum balances and single energy balance (class II mixtures (Hutter))
- Minimal scenario for Helmholz potential implied by balance eq.

$$\Psi(\rho, c, \vartheta, \mathbf{J}_{\mathrm{s}}) = c\Psi_{\mathrm{s}}(\rho, c, \vartheta) + (1-c)\Psi_{\mathrm{w}}(\rho, c, \vartheta) + \frac{1}{2}\frac{|\mathbf{J}_{\mathrm{s}}|^2}{\rho^2 c(1-c)}$$

  fits well the framework of earlier mixture theories.
- Choice of rate of entropy production - satisfaction of 2nd law of thermodynamics
- Maximization of r.e.p. $\rightarrow$ constitutive equations

Conclusions II

- Classical viscous-fluid relations for fluid part $\mathbb{T}_{\mathrm{s}} + \mathbb{T}_{\mathrm{w}} \sim \mathbb{D}(\mathbf{v})$
- Additional relation for $(1-c)\mathbb{T}_{\mathrm{s}} - c\mathbb{T}_{\mathrm{w}} \sim \mathbb{D}(\mathbf{u}_{\mathrm{s}} - \mathbf{u}_{\mathrm{w}})$
- Interaction force $\boldsymbol{\pi}$ contains all relevant (for diffusion) mechanisms.
- Energy and entropy flux consistent extensions of theories of Müller and Bowen.
- "Physical" limit of evolution eq. for diff. flux $\mathbf{J}_{\mathrm{s}}$ yields classical Fick's diffusion.

Thank you for your attention!