Mechanical oscillators described by a system of differential-algebraic equations

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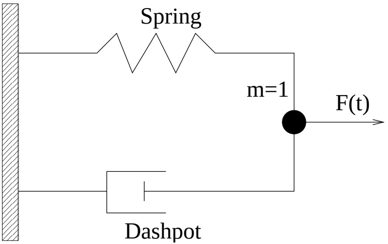
Problem

$$x'' + F_d + F_s = F(t)$$

x ....displacement
F_d ....dashpot force
F_s ....spring force
$F(t)$ .....external force



Constitutive relations

$$x'' + F_d + F_s = F(t)$$

"common" approach:

$$F_s = f(x) \qquad \text{(spring)}$$
$$F_d = g(x') \qquad \text{(dashpot)}$$

$$x'' + g(x') + f(x) = F(t)$$

apply the standard ODE theory

“Reversed” constitutive relations

IDEA:
what if we assume

$$x = f(F_s) \qquad \text{(spring)}$$
$$x' = g(F_d) \qquad \text{(dashpot)}$$

PHILOSOPHICALLY: kinematics (x and x') are a consequence, and hence a function of the forces (F_s and F_d).

$$\begin{aligned} x'' + F_d + F_s &= F(t) \\ x &= f(F_s) \\ x' &= g(F_d) \end{aligned}$$

differential-algebraic system of equations

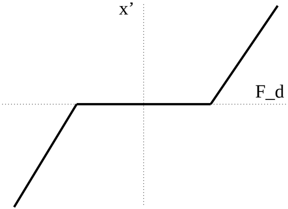
Implicit constitutive relations

For some materials, it is even reasonable to assume:

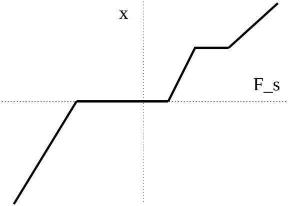
$$
\begin{aligned}
f(x, F_s) &= 0 \qquad \text{(spring)} \\
g(x', F_d) &= 0 \qquad \text{(dashpot)}
\end{aligned}
$$

That is to say, fully implicit constitutive relations.

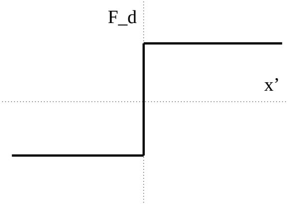
Examples



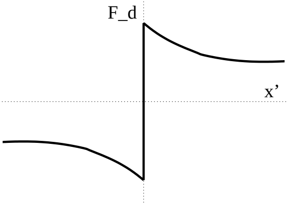
Bingham fluid



polymer response



Coulomb friction



. . . with relaxation

Mathematical results – an overview

① oscillators with reversed (monotone) constitutive relations

② oscillator with (generalized) Coulomb friction

③ problem: uniqueness for 2nd order ODE's

Oscillators with reversed constitutive relations

$$\begin{aligned} x'' + F_d + F_s &= F(t) \\ x &= f(F_s) \\ x' &= g(F_d) \end{aligned}$$

- f, g continuous, **non-decreasing**
- $|f(u)|, |g(u)| \sim |u| \qquad \text{for } |u| \to \infty$
- $F(t) \in L^2(0, T)$

THEOREM 1. There is at least one global solution.
Proof.

① approximation:

$$x = f_k(F_s) \qquad f_k = f + k^{-1}\,\mathrm{Id}$$
$$x' = g_k(F_d) \qquad g_k = g + k^{-1}\,\mathrm{Id}$$

② f_k, g_k invertible $\rightsquigarrow$

$$x'' + \underbrace{\{g_k\}_{-1}(x')}_{F_d} + \underbrace{\{f_k\}_{-1}(x)}_{F_s} = F(t)$$

③ coercivity of f, g $\quad\Longrightarrow\quad$ k-independent estimates

④ limit $k \to \infty$ (use monotonicity of f, g).

... uniqueness ... ?

x_1, x_2 ... solutions;
F_d^i, F_s^i, $i = 1, 2$ ... the corresponding forces.

$$(x_1 - x_2)'' + (F_d^1 - F_d^2) + (F_s^1 - F_s^2) = 0 \qquad / \cdot (x^1 - x^2)'$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{(x_1 - x_2)'\right\}^2 + \underbrace{(F_d^1 - F_d^2)(x^1 - x^2)'}_{\geq 0} + \underbrace{(F_s^1 - F_s^2)(x^1 - x^2)'}_{???} = 0$$

assume in addition:

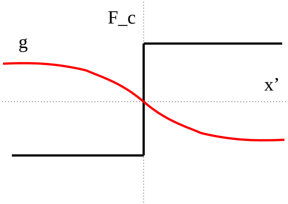
- structual properties of f
- $F(t) \equiv F_0$ ... **autonomous case**

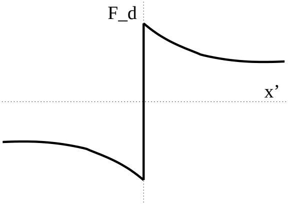
$\Longrightarrow$ **THEOREM 2.** Global (forward) uniqueness

Coulomb friction with relaxation

$$x'' + F_d + kx = F(t)$$
$$F_d = F_c + g(x')$$
$$(F_c, x') \in \mathcal{A}$$

F_c .... Coulomb-like friction force
$\mathcal{A}$ .... monotone graph
$g(\cdot)$ .... relaxation function





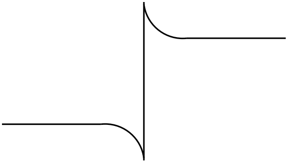
- g continuous, $|g(u)| \leq c(1 + |u|)$
- $\mathcal{A}$ maximal monotone, coercive

$\implies$ **THEOREM 1.** Global existence of solutions.

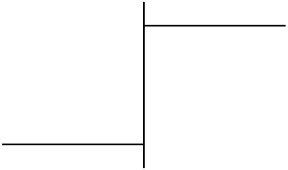
- moreover: g locally lipschitz

$\implies$ **THEOREM 2.** Global (forward) uniqueness.

examples of nonuniqueness:



(steep relaxation)



(non-monotone graph)

Simplification: uniqueness for ODE

motivation:

$$x'' + F_d + F_s = F(t)$$

$$\begin{array}{cc} \uparrow & \uparrow \\ x' & x \end{array}$$

① neglect F_s and x $\rightsquigarrow y' + f(y, t) = 0$ $(y = x')$

② neglect F_d and x' $\rightsquigarrow x'' + f(x, t) = 0$

Uniqueness for 1st order ODE ?

$$y' + f(y, t) = 0$$

- $f(\cdot, t)$ locally lipschitz: YES
- $f(\cdot, t)$ only Hölder: NO
- $f(\cdot, t)$ non-decreasing: YES (forward)

Uniqueness for 2nd order ODE ?

$$x'' + f(x, t) = 0$$

- $f(\cdot, t)$ locally lipschitz: YES
- $f(\cdot, t)$ only Hölder: NO
- $f(\cdot, t)$ non-decreasing: NO in general
  - linear counterexample: $x'' + Q(t)x = 0, \quad Q(t) \geq 0.$
  - autonomous problem: $\Longrightarrow$ **uniqueness**
  - "quasi-autonomous" case: $x'' + h(x) = f(t)$????

Thank you.