Incompressible piezoviscous fluids: first steps, a long ways to go.

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Motivation,

- hydrodynamic lubrication
- Reynolds equation

incompressible piezoviscous fluids,

- viscosity depending on pressure and shear rate
- implicitely constituted models

challenges and results,

- inf–sup inequality and uniqueness of the pressure
- subclass of models: restricted growth of extra stress with pressure
- well-posed PDEs, convergence of FEM, pictures

and their limitations.

- subclass of models? limited range of pressures and shear rates
- relevance to hydrodynamic lubrication

Motivation,

- hydrodynamic lubrication
- Reynolds equation

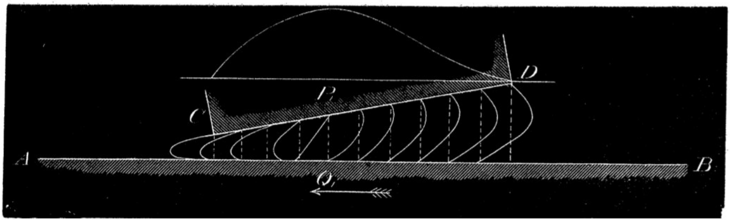
Motivation: hydrodynamic lubrication

IV. *On the Theory of Lubrication and its Application to Mr.* BEAUCHAMP TOWER'S *Experiments, including an Experimental Determination of the Viscosity of Olive Oil.*

By Professor OSBORNE REYNOLDS, *LL.D., F.R.S.*

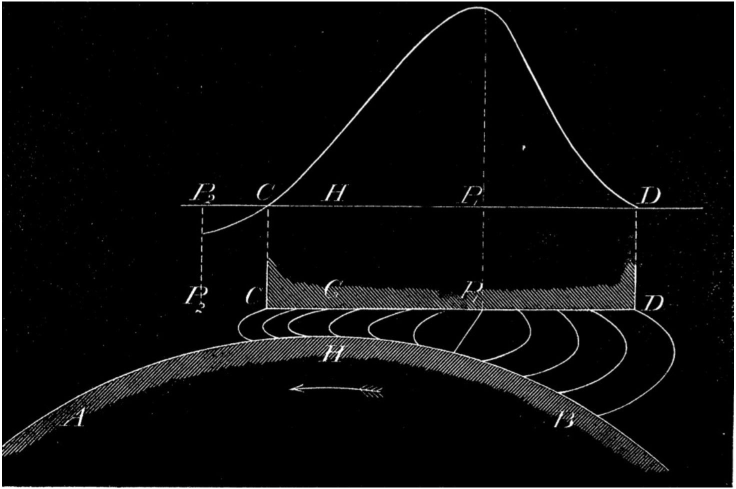
Received December 29, 1885,—Read February 11, 1886.

Fig. 9.

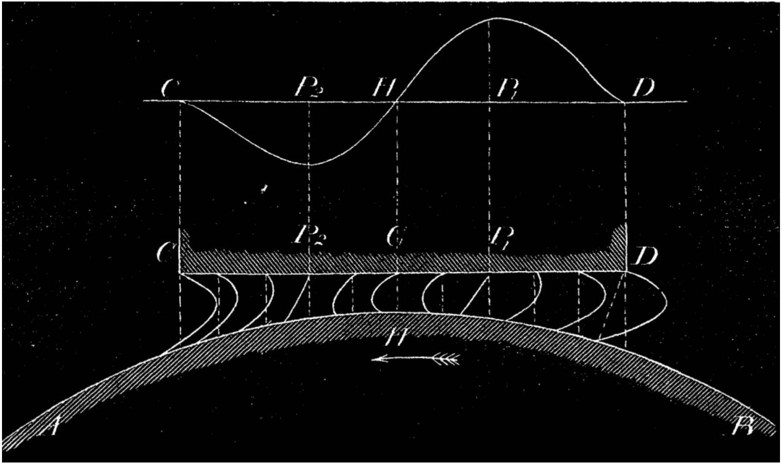


This is the explanation of continuous lubrication.

Motivation: hydrodynamic lubrication



Motivation: hydrodynamic lubrication



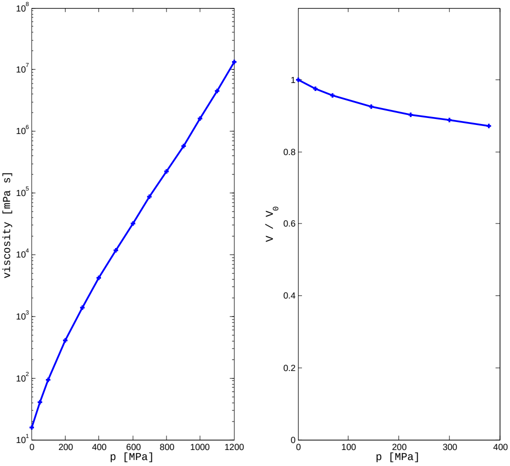
incompressible piezoviscous fluids,

- viscosity depending on pressure and shear rate
- implicitely constituted models

Viscosity at large pressure and shear rate

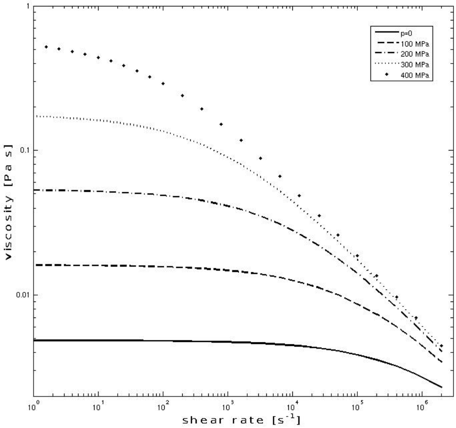
Viscosity and volume variation with pressure for *squalane*

("representing a low viscosity paraffinic mineral oil", S. Bair, *Tribology Letters*, 2006).



Viscosity at large pressure and shear rate

Viscosity for *SAE 10W/40 reference oil RL 88/1*, (partly) by Hutton, Jones, Bates, *SAE*, 1983.



Incompressible fluids with viscosity depending on pressure and shear rate

Mathematical formulation

inside $(0, T) \times \Omega$:

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= 0 \\ \partial_\tau \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \boldsymbol{S} &= -\nabla \pi + \boldsymbol{f}\,, \\ \boldsymbol{S} &= 2\,\nu(\pi, |\boldsymbol{D}(\boldsymbol{v})|^2)\,\boldsymbol{D}(\boldsymbol{v}) \end{aligned}$$

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- Cauchy stress tensor $\boldsymbol{T} = -\pi\boldsymbol{I} + 2\nu(\pi,|\boldsymbol{D}|^2)\boldsymbol{D}, \quad \operatorname{tr}\boldsymbol{D} = 0$
- π is the mean normal stress, $\pi = -\frac{1}{3}\operatorname{tr}\boldsymbol{T}$,
- implicitely constituted model

$$\boldsymbol{T} - \tfrac{1}{3}(\operatorname{tr}\boldsymbol{T})\boldsymbol{I} - 2\nu(-\tfrac{1}{3}\operatorname{tr}\boldsymbol{T},|\boldsymbol{D}|)\boldsymbol{D} = 0$$

see Rajagopal, *J. Fluid Mech.*, 2006 (and Málek, Rajagopal, 2006, 2007)

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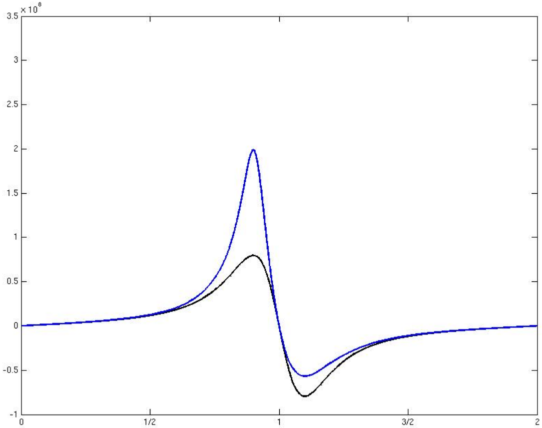
Viscosity formulas used in applications

$$\nu = \nu(\pi, |\boldsymbol{D}(\boldsymbol{v})|^2) = \begin{cases} \sim \ \exp(\alpha\pi), \\ \sim \ (1 + |\boldsymbol{D}(\boldsymbol{v})|^2)^{\frac{p-2}{2}}, \quad 1 < p < 2 \end{cases}$$

Motivation: hydrodynamic lubrication



challenges and results,

- inf–sup inequality and uniqueness of the pressure
- subclass of models: restricted growth of extra stress with pressure
- well-posed PDEs, convergence of FEM, pictures

Challenges

- the lubrication **works** and is used since before the invention of wheel
- the viscosity–pressure relation is present in the very basis of the theory of elastohydrodynamic lubrication
- so why should we bother with mathematical theory now?

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- lubrication is used everywhere (transportation, electricity production)
- any optimization can save energy consumption and prolongate the lifespan
- quantitative predictions are needed !

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[Bair, Gordon, 2006]:

" . . . there has been relatively little progress since the classic Newtonian solutions . . . toward relating film thickness and traction to the properties of individual liquid lubricants

and it not clear at this time that full numerical solutions can even be obtained for heavily loaded contacts using accurate models.

One central issue is the validity of Reynolds equation, derived under the isoviscous assumption. . . "

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- Rajagopal, Szeri, *Proc. R. Soc. Lond. A*, 2003

Challenges

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- so why should we bother with mathematical theory now?

[Babuška, Strouboulis, 2007 book]:

- It is necessary to realize that the FEM is a numerical method for constructing *approximate* solutions of a well defined mathematical problem. A necessary prerequisite is that the mathematical problem has 'good' properties: existence of solutions, possibly uniqueness, continuous dependence on input data.
- Mathematical formulation is a simplification of reality, and the existence of a physical solution which can be observed in an experiment, does not guarantee that the solution of the mathematical problem exists and has the expected properties following from physical intuition.

challenges and results,

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Incompressible fluids with viscosity depending on pressure and shear rate

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Problem well-posedness—first observations

$$\nu = \nu(\pi)$$

- M. Renardy, *Comm. Part. Diff. Eq.*, 1986.
- F. Gazzola, *Z. Angew. Math. Phys.*, 1997.
- F. Gazzola, P. Secchi, *Navier–Stokes eq.: th. and num. meth.* 1998.

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\boldsymbol{S} &= 2\,\nu(\pi, |\boldsymbol{D}(\boldsymbol{v})|^2)\,\boldsymbol{D}(\boldsymbol{v})
\end{aligned}
$$

Problem well-posedness—first positive results

$$
\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{D}} \sim (1 + |\boldsymbol{D}|^2)^{\frac{p-2}{2}} \qquad \left| \frac{\partial \boldsymbol{S}}{\partial \pi} \right| \leq \gamma_0\,(1 + |\boldsymbol{D}|^2)^{\frac{p-2}{4}} \qquad 1 < p < 2
$$

- Málek, Nečas, Rajagopal, *Arch. Rational Mech. Anal.*, 2002.
- Hron, Málek, Nečas, Rajagopal, *Math. Comput. Simulation*, 2003.
- Málek, Rajagopal, *Handbook of mathematical fluid dynamics*, 2007.

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on the boundary $(0, T) \times \partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_P$:

$$\begin{array}{lll} \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ and } -\boldsymbol{T}\boldsymbol{n} = \sigma\,\boldsymbol{v} & \text{on } \Gamma_N & \\ \boldsymbol{v} = \boldsymbol{v_D} & \text{on } \Gamma_D & \text{if } \Gamma_P = \emptyset, \\ -\boldsymbol{T}\boldsymbol{n} = \boldsymbol{b}(\boldsymbol{v}) & \text{on } \Gamma_P & \text{then } \int_{\Omega_0} \pi \,\mathrm{d}\boldsymbol{x} = 0 \end{array}$$

- Bulíček, Málek, Rajagopal, *Indiana Univ. Math. J.*, 2007
- Bulíček, Málek, Rajagopal, *SIAM J. Math. Anal.*, 2009

Incompressible fluids with viscosity depending on pressure and shear rate

Mathematical formulation

inside Ω:

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- Franta, Málek, Rajagopal, *Proc. Royal Soc. A*, 2005
- Lanzendörfer, *Nonlin. Anal.: Real World App.*, 2009

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- Stebel & Lanzendörfer, *Appl. Mat.–Czech.*, 2011
- Stebel & Lanzendörfer, *Math. Comput. Simulat.*, 2011
- Hirn, Stebel & Lanzendörfer, *IMA J. Num. Anal.*, 2012 (el.)

Basic a priori estimates

Weak formulation

$$(q, \operatorname{div} \boldsymbol{w})_\Omega = 0$$
$$([\nabla \boldsymbol{v}]\boldsymbol{v}, \boldsymbol{w})_\Omega + (\boldsymbol{S}(\pi, \boldsymbol{D}(\boldsymbol{v})), \boldsymbol{D}(\boldsymbol{w}))_\Omega - (\pi, \operatorname{div} \boldsymbol{w})_\Omega = (\boldsymbol{f}, \boldsymbol{w})_\Omega - (\boldsymbol{b}(\boldsymbol{v}), \boldsymbol{w})_{\Gamma_P}$$

Basic a priori estimates

Weak formulation

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Test eq. by solution

$$(\boldsymbol{S}(\pi, \boldsymbol{D}(\boldsymbol{v})), \boldsymbol{D}(\boldsymbol{v}))_\Omega \sim |\boldsymbol{D}(\boldsymbol{v})|^p \pm 1$$

$\Longrightarrow$

$$\|\boldsymbol{D}(\boldsymbol{v})\|_p \leq K \qquad \Longrightarrow \qquad \|\boldsymbol{v}\|_{1,p} + \|\boldsymbol{S}\|_{p'} \leq K$$

Basic a priori estimates

Weak formulation

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$$(\boldsymbol{S}(\pi, \boldsymbol{D}(\boldsymbol{v})), \boldsymbol{D}(\boldsymbol{w}))_\Omega - (\pi, \operatorname{div} \boldsymbol{w})_\Omega = (\boldsymbol{f}, \boldsymbol{w})_\Omega - (\boldsymbol{b} \quad , \boldsymbol{w})_{\Gamma_P}$$

Inf–sup inequality and the boundedness of $\partial_\pi \boldsymbol{S}$

$$0 < \beta \leq \inf_{q \in \mathrm{L}^{p'}_{b.c.}(\Omega)} \sup_{\boldsymbol{w} \in \mathbf{W}^{1,p}_{b.c.}(\Omega)} \frac{(q, \operatorname{div} \boldsymbol{w})_\Omega}{\|q\|_{p'} \|\boldsymbol{w}\|_{1,p}}$$

$\Longrightarrow$

$$\beta \|\pi\|_{p'} \leq \|\boldsymbol{S}(\pi, \boldsymbol{D}(\boldsymbol{v}))\|_{p'} + \|\boldsymbol{f} + \boldsymbol{b}\| \leq K$$

Basic a priori estimates

Weak formulation

$$(q, \operatorname{div} \boldsymbol{w})_\Omega = 0$$

$$(\boldsymbol{S}(\pi, \boldsymbol{D}(\boldsymbol{v})), \boldsymbol{D}(\boldsymbol{w}))_\Omega - (\pi, \operatorname{div} \boldsymbol{w})_\Omega = (\boldsymbol{f}, \boldsymbol{w})_\Omega - (\boldsymbol{b} \quad , \boldsymbol{w})_{\Gamma_P}$$

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Pressure uniquely determined by velocity?

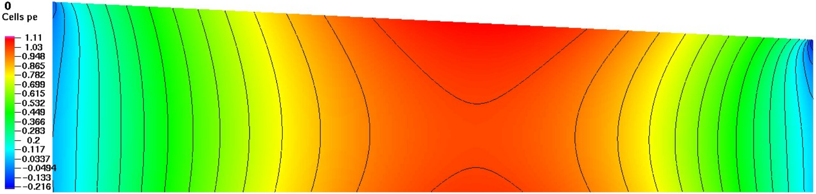
$$\beta \|\pi^1 - \pi^2\|_{p'} \leq \|\boldsymbol{S}(\pi^1, \boldsymbol{D}(\boldsymbol{v})) - \boldsymbol{S}(\pi^2, \boldsymbol{D}(\boldsymbol{v}))\|_{p'} \leq \left\| \int_{\pi^1}^{\pi^2} \frac{\partial \boldsymbol{S}(\pi, \boldsymbol{D}(\boldsymbol{v}))}{\partial \pi} \mathrm{d}\pi \right\|_{p'}$$

$$\leq \gamma_0 \|\pi^1 - \pi^2\|_{p'} \qquad \text{where} \qquad \left| \frac{\partial \boldsymbol{S}}{\partial \pi} \right| \leq \gamma_0$$

Motivation: hydrodynamic lubrication

Flow in a converging channel

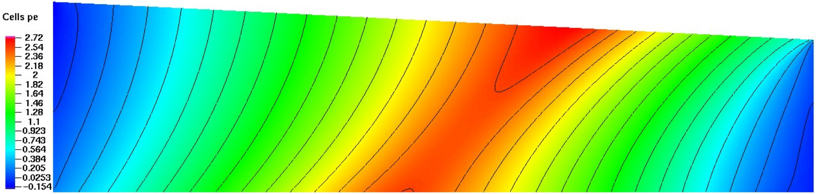
Newtonian model $\nu = \mathrm{const}$

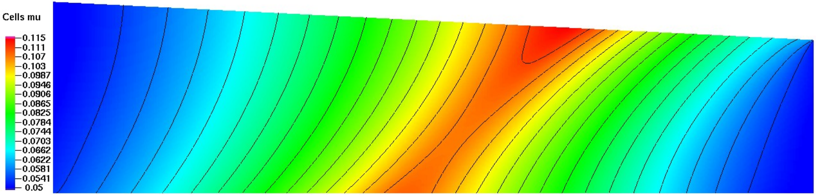


Motivation: hydrodynamic lubrication

Flow in a converging channel

Barus model $\nu = \exp(\alpha\pi)$, $\alpha = 0.306$

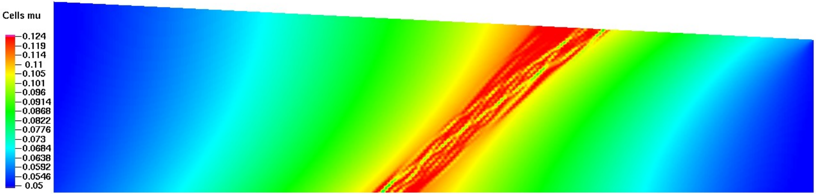




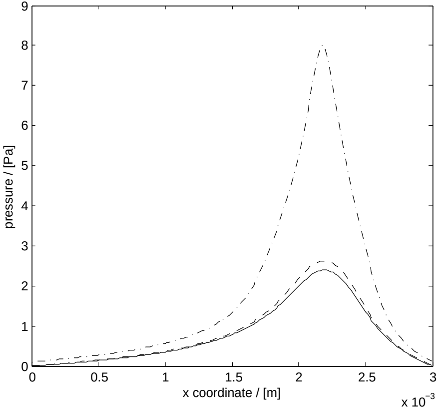
Motivation: hydrodynamic lubrication

Flow in a converging channel

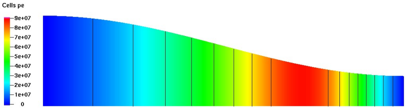
Barus model $\nu = \exp(\alpha\pi)$, $\alpha = 0.3061$



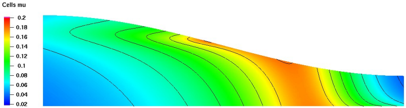
Sensitivity on boundary data



Pressure for $\pi(0) = \pi(L) = 0$ (full),
$1\ \mathrm{MPa}$ (dashed) a
$10\ \mathrm{MPa}$ (dash dotted).



pressure π



viskosity $\nu(\pi)$

Conclusion

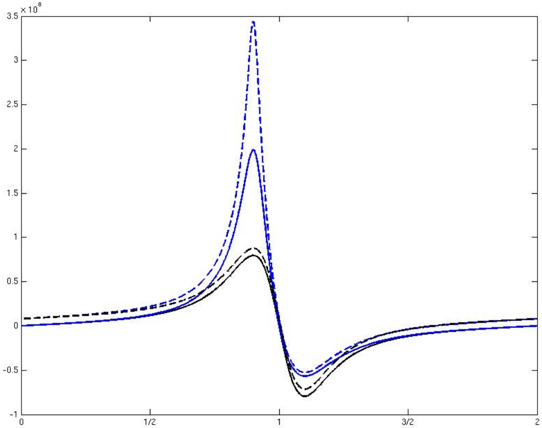
- mathematical theory being build succesfully for

$$\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{D}} \sim (1 + |\boldsymbol{D}|^2)^{\frac{p-2}{2}} \qquad \left|\frac{\partial \boldsymbol{S}}{\partial \pi}\right| \leq \gamma_0 (1 + |\boldsymbol{D}|^2)^{\frac{p-2}{4}} \qquad 1 < p < 2$$

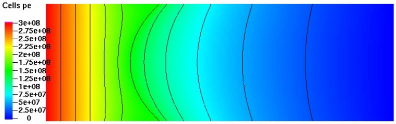
- for $|\partial \boldsymbol{S}/\partial \pi| > 1$ both the theory and the standard numerical approach fails
- is the model well-posed?

Thank you for your attention

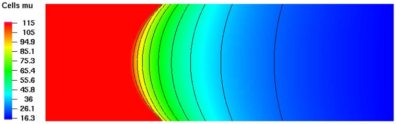
Motivation: hydrodynamic lubrication



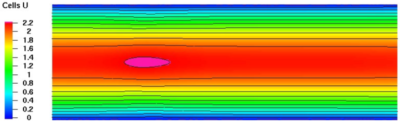
Poiseuille flow



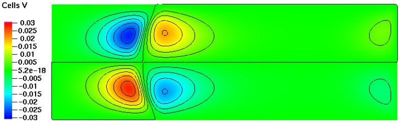
the pressure π



the viscosity $\nu(p)$



the velocity component $\boldsymbol{v} \cdot \boldsymbol{e}_x$



the velocity component $\boldsymbol{v} \cdot \boldsymbol{e}_y$